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The most symmetric non-singular plane curves of degree $n<8$

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0 Introduction

Throughout this paper k stands for the complex number field $\mathbf{C}$. A homogeneous polynomial $f(x,y,z)\in k[x,y,z]$ defines a plane algebraic curve $f=0$, or $C(f)$ in the projective plane $\mathbf{P}^{2}$. A non-singular matrix $A\in GL(3,k)$ defines a projectivity (A) sending a point P with the homogeneous coordinates (x) to a point $(A)P$ with the homogeneous coordinates $(x(^{t}A))$. Denote by $PGL(3,k)$ the group of projectivities in $\mathbf{P}^{2}$. Denote by $\mathrm{Aut}(f)$ the projective automorphism group of f, namely $\mathrm{Aut}(f)=\{(A)\in PGL(3,k);\ f_{A}$ is proportional to $f\}$, where $f_{A}(x,y,z)=f((x,y,z)(^{t}A^{-1}))$. When $C(f)$ is non-singular and of degree n, i.e. $\deg f=n$, then $C(f)$ is a compact Riemann surface of genus $g=(n-1)(n-2)/2$. In this case we can consider the holomorphic automorphism group $\mathrm{AUT}(f)$ of the Riemann surface $C(f)$. Clearly $\mathrm{Aut}(f)$ is a subgroup of $\mathrm{AUT}(f)$. If $\deg f\geq 4$ and $C(f)$ is non-singular, then $\mathrm{Aut}(f)=\mathrm{AUT}(f)$ [7, p.372], and $|\mathrm{AUT}(f)|\leq 84(g-1)$ [5]. Therefore $|\mathrm{Aut}(f)|$ is bounded above when $C(f)$ runs through *non-singular* plane curve of degree $n\geq 4$, n being fixed. As will be shown in the next section, the same is true for non-singular plane cubics.

Let an f in $k[x,y,z]$ be homogeneous. We call f singular or non-singular according as the curve $C(f)$ has a singualr point or not. A non-singular curve $C(f)$ of degree $n(n\geq 3)$ is the most symmetric, if it attains the maximum order of the projective automorphism groups for non-singular plane algebraic curves of degree $n(n\geq 3)$. We often identify the polynomial f and the curve $C(f)$.

Our main results are the following Theorems 1, 3, and 5. Theorem 2 is well known [3, pp.348–349].

Theorem 1 *Let f be a non-singular plane cubic.*
(1) $|\mathrm{Aut}(f)|\leq 54$.
(2) $|\mathrm{Aut}(f)|=54$ *if and only if f is projectively equivalent to* $x^{3}+y^{3}+z^{3}$.

Theorem 2 *Let f be a non-singular plane quartic.*
(1) $|\mathrm{Aut}(f)| \leq 168$.
(2) $|\mathrm{Aut}(f)| = 168$ *if and only if f is projectively equivalent to the Klein quartic* $x^3y + y^3z + z^3x$.

Theorem 3 *Let f be a non-singular plane quintic.*
(1) $|\mathrm{Aut}(f)| \leq 150$.
(2) $|\mathrm{Aut}(f)| = 150$ *if and only if f is projectively equivalent to* $x^5 + y^5 + z^5$.

Theorem 4 ([1]) *Let f be a non-singular plane sextic.*
(1) $|\mathrm{Aut}(f)| \leq 360$.
(2) $|\mathrm{Aut}(f)| = 360$ *if and only if f is projectively equivalent to the Wiman sextic* $10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6$.

Theorem 5 *Let f be a non-singular plane septic.*
(1) $|\mathrm{Aut}(f)| \leq 294$.
(2) $|\mathrm{Aut}(f)| = 294$ *if and only if f is projectively equivalent to* $x^7 + y^7 + z^7$.

Our definitions and notaions are as follows. Let $A,\ B \in GL(3,k)$, and $f \in k[x_1, x_2, x_3]$. We define $f_A \in k[x_1, x_2, x_3]$ as $f_A(x_1, x_2, x_3) = f([x_1, x_2, x_3]({}^tA^{-1}))$ so that $(f_A)_B = f_{BA}$. Let G be a subset of the group $PGL(3,k)$ of projectivities of the projective plane P^2. A homogeneous $f \in k[x,y,z]$ is called G- invariant, if $f_A \sim f$ for any $(A) \in G$. More generally, let H be an abstract group. By abuse of notation we call f is H-invariant, if there is a subgroup G of $PGL(3,k)$ such that 1) G and H are isomorphic, and 2) f is G-invariant. For a homogeneous $f \in k[x_1.x_2, x_3]$ Hess(f) denotes the Hessian of f: $\mathrm{Hess}(f) = \det[\frac{\partial^2}{\partial x_i \partial x_j} f]$. It is well known that, if f is non-singular, then the intersection $f \cap h$ coincides with the set of all flexes. It is also known that $\mathrm{Aut}(f) \subset \mathrm{Aut}(\mathrm{Hess}(f))$. Finally $E_3 = [e_1, e_2, e_3]$ denotes the unit matrix of $GL(3,k)$, where e_j stands for the j-th column of E_3. When two quantities a and b such as functions and matrices, $a \sim b$ means that a and b are proportional.

The cases of cubics, quintics, and septics are discussed in §1, §2, and §3 respectively. Proofs are not given in princile to make our report short.

1 Cubics

In this section we will prove Theorem 1. We begin with

Theorem 1. 1 ([8], [6]) *Let* $f = x^n + y^n + z^n (n \geq 3)$. *Then* $|\mathrm{Aut}(f)| = 6n^2$.

Theorem 1. 2 *Let f be a non-singular plane cubic.*
(1) $|\mathrm{Aut}(f)| \leq 54$.
(2) $|\mathrm{Aut}(f)| = 54$ *if and only if f is projectively equivalent to* $x^3 + y^3 + z^3$.

Proof. As is known, f has a flex P. Without loss of generality we may assume that $P=(0,1,0)$ and that the tangent there is z. Namely $f(x,1,z)=z+2z(ax+bz)+Ax^3+Bx^2z+Cxz^2+Dz^3$, or equivalently $f=y^2z+2yz(ax+bz)+Ax^3+Bx^2z+Cxz^2+Dz^3$. Substituting y for $y+ax+bz$, we get $f=y^2z+Ax^3+Bx^2z+Cxz^2+Dz^3$. So we may assume that $f=y^2z+x^3+Bx^2z+Cxz^2$. As can be seen easily, f is non-singular if and only if $C(B^2-4C)\neq 0$. Let $G_P=\{(A)\in \mathrm{Aut}(f);\ (A)P=P\}$, and assume $(A)\in G_P$. Since (A) fixes the tangent z at P as well, the rows of A take the form $[a_1,0,c_1]$, $[a_2,1,c_2]$, and $[0,0,c_3]$ respectively up to constant multiplication. Since $f_{A^{-1}}$ contains none of monomials of degree 1 with respect to y, $a_2=c_2=0$. Now $f_{A^{-1}}\sim f$, if and only if ${a_1}^3/c_3=1$, $3{a_1}^2c_1/c_3+B{a_1}^2=B$, $3a_1{c_1}^2/c_3+2a_1c_1B+a_1C/c_3=C$ and ${c_1}^3/c_3+{c_1}^2B+c_1c_3C=0$. From the first and the second equalities of these four equalities, we get $c_3={a_1}^3$ and $c_1=a_1(1-{a_1}^2)B/3$. So the third equality can be written as $({a_1}^4-1)(-B^2/3+C)=0$. If $C\neq B^2/3$, then $|G_P|\leq 4$. If $C=B^2/3$, then the fourth equality can be written as $(1-{a_1}^2)(1+{a_1}^2+{a_1}^4)B^3=0$. Note that y^2z+x^3 is singular. Hence, only when $C=B^2/3\neq 0$, f is non-singular and $|G_P|=6$. Since $|f\cap h|\leq 9$ by Bezout's theorem,

$$|\mathrm{Aut}(f)|/|G_P|=|\mathrm{Aut}(f)P|\leq 9.$$

So $|\mathrm{Aut}(f)|\leq 54$, and the equlity holds, if and only if $|G_P|=6$ and $|\mathrm{Aut}(f)P|=9$. We have shown that $|G_P|=6$ if and only if $C=B^2/3\neq 0$, namely $f=y^2z+x^3+Bx^2z+B^2xz^2/3$ with $B\neq 0$, which is projectively equivalent to $f'=y^2z+x^3+x^2x+xz^2/3$. Consequently, if there exists a non-singular cubic f with $|\mathrm{Aut}(f)|=54$, then f is projectively equivalent to f'. This means the uniqueness of non-singular cubics satisfying $|\mathrm{Aut}(f)|=54$. On the other hand there exists such a cubic by Theorem 1.1

2 Quintics

In this section we will specify the most symmetric non-singular quintics (Theorems 2.2 and 2.22).

Theorem 2. 1 (Hurwitz) *Denote by* $\mathrm{AUT}(C)$ *the holomorphic automorphism group of a compact Riemann surface* C *of genus* $g\geq 2$. *Let* $g'=g-1$. *The posssible values of the order* $d=|\mathrm{AUT}(C)|$ *are*

$$\begin{array}{llllllll} 84g', & 48g', & 40g', & 36g', & 30g', & \frac{132}{5}g', & 24g', & \frac{156}{7}g', \\ 21g', & 20g', & \frac{96}{5}g', & \frac{56}{3}g', & \frac{204}{11}g', & 18g' & & \text{or less.} \end{array}$$

Proof. The author of [5] cites values down to $36g'$. For our purposes, however, other possible values are necessary. The idea of the proof given below is entirely due to [5]. According to [5] there exist integers $\hat{g}\geq 0$, $s\geq 3$, and $m_1\geq m_2\geq \ldots \geq m_s\geq 2$ such that

$$2g'=d\{2(\hat{g}-1)+\sum_{j=1}^{s}(1-\frac{1}{m_j})\}.$$

If $\hat{g} \geq 2$, then $d \leq g'$. If $\hat{g} = 1$, then $d \leq 4g'$. Suppose $\hat{g} = 0$. Note that $2g' \geq d\{-2+s/2\}$. If $s \geq 5$, then $d \leq 4g'$. If $s = 4$, then $m_1 \geq 3$ so that $2g' \geq d\{-2+(1-1/3)+3/2\} = d/6$, namely $d \leq 12g'$. Assume $s = 3$.

Suppose $m_3 \geq 4$. Then $2g' \geq d(1-3/4) = d/4$, namely $d \leq 8g'$. Suppose $m_3 = 3$. Then $m_1 \geq 4$. If $m_1 \geq 5$, then $2g' \geq d(1-1/5-1/3-1/3) = 2d/15$, namely $d \leq 15g'$. If $m_1 = 4$ and $m_2 = 4$, then $2g' = d(1-1/2-1/3) = d/6$, namely $d = 12g'$. If $m_1 = 4$ and $m_2 = 3$, then $2g' = d(1-1/4-2/3) = d/12$, namely $d = 24g'$. Suppose $m_3 = 2$. Then $m_2 \geq 3$. If $m_2 \geq 6$, then $2g' \geq d(1-2/6-1/2) = d/6$, namely $d \leq 12g'$.
Let $m_2 = 5$. If $m_1 \geq 6$, then $2g' \geq d(1-1/6-1/5-1/2) = 2d/15$, namely $d \leq 15g'$.
If $m_1 = 5$, then $2g' = d(1-2/5-1/2) = d/10$, namely $d = 20g'$.
Let $m_2 = 4$. Then $m_1 \geq 5$. If $m_1 \geq 8$, then $2g' \geq d(1-1/8-1/4-1/2) = d/8$, namely $d \leq 16g'$.
If $m_1 = 7$, then $2g' = d(1-1/7-3/4) = 3d/28$, namely $d = 56g'/3$.
If $m_1 = 6$, then $2g' = d(1-1/6-3/4) = d/12$, namely $d = 24g'$.
If $m_1 = 5$, then $2g' = d(1-1/5-3/4) = d/20$, namely $d = 40g'$.
Let $m_2 = 3$. Then $m_1 \geq 7$. If $m_1 \geq 19$, then $2g' \geq d(1/6-1/19) = 13d/114$, namely $d \leq 228g'/13$.
If $m_1 = 18$, then $2g' = d(1/6-1/18) = 2d/18$, namely $d = 18g'$.
If $m_1 = 17$, then $2g' = d(1/6-1/17) = 11d/102$, namely $d = 204g'/11g'$.
If $m_1 = 16$, then $2g' = d(1/6-1/16) = 5d/48$, namely $d = 96g'/5$.
If $m_1 = 15$, then $2g' = d(1/6-1/15) = d/10$, namely $d = 20g'$.
If $m_1 = 14$, then $2g' = d(1/6-1/14) = 2d/21$, namely $d = 21g'$.
If $m_1 = 13$, then $2g' = d(1/6-1/13) = 7d/78$, namely $d = 156g'/7$.
If $m_1 = 12$, then $2g' = d(1/6-1/12) = d/12$, namely $d = 24g'$.
If $m_1 = 11$, then $2g' = d(1/6-1/11) = 5d/66$, namely $d = 132g'/5$.
If $m_1 = 10$, then $2g' = d(1/6-1/10) = d/15$, namely $d = 30g'$.
If $m_1 = 9$, then $2g' = d(1/6-1/9) = d/18$, namely $d = 36g'$.
If $m_1 = 8$, then $2g' = d(1/6-1/8) = d/24$, namely $d = 48g'$.
If $m_1 = 7$, then $2g' = d(1/6-1/7) = d/42$, namely $d = 84g'$.

Let f be a non-singular plane quintic, hence $C(f)$ is a compact Riemann surface of genus $g = 6$. From now on let $g' = g - 1 = 5$ throughout this section. Then possible values of $|\mathrm{Aut}(f)|$ are

$$84g' = 4 \cdot 3 \cdot 5 \cdot 7,\ 48g' = 16 \cdot 3 \cdot 5,\ 40g' = 8 \cdot 5^2,\ 36g' = 4 \cdot 3^2 \cdot 5,\ 30g' = 2 \cdot 3 \cdot 5^2 \text{ or less.}$$

We will prove the following theorem by showing that $|\mathrm{Aut}(f)|$ cannot be equal to none of $84g'$, $48g'$, $40g'$, and $36g'$.

Theorem 2. 2 *If f is a non-singular plane quintic, then* $|\mathrm{Aut}(f)| \leq 150$.

A proof of this theorem will be given after a series of lemmas and propositions.

Let ε be a primitive n-th root of $1 (n \geq 3)$. A cyclic subgroup G_n of order n in $PGL(3,k)$ is clearly conjugate to either $G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >$ or $G_{ij} =< (\mathrm{diag}[1,\varepsilon^i,\varepsilon^j]) >$

for some $1 \le i < j \le n-1$ satisfying the greatest common divisor $(i, j, n) = 1$.

Lemma 2. 3 *Let notations be as above. Suppose that $1 \le i < j \le n-1$, $1 \le i' < j' \le n-1$, and $(i, j, n) = (i', j', n) = 1$. Then G_{ij} is conjugate to $G_{i'j'}$ if and only if there exists an $1 \le m \le n-1$ with $(m, n) = 1$ and a permutation $\sigma \in S_3$ such that*

$$\mathrm{diag}[\varepsilon_{\sigma(1)}, \varepsilon_{\sigma(2)}, \varepsilon_{\sigma(3)}] \sim \mathrm{diag}[1, \varepsilon^{i'}, \varepsilon^{j'}],$$

where $[\varepsilon_1, \varepsilon_2, \varepsilon_3] = [1, \varepsilon^{im}, \varepsilon^{jm}]$.

Lemma 2. 4 *Let ε be a primitive 7-th root of 1. A subgroup G_7 of $PGL(3, k)$ is isomorphic to $\mathbf{Z}_7$ if and only if G_7 is conjugate to one of the following subgroups of $PGL(3, k)$: $G_{01} =< (\mathrm{diag}[1, 1, \varepsilon]) >$, $G_{12} =< (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >$, $G_{13} =< (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >$.*

Lemma 2. 5 *Let f_1, ..., f_n be non-zero homogeneous polynomials of the same degree such that $f_{jA} = \lambda_j f_j$ $(j = 1, 2, ..., n)$ for an $A \in GL(3, k)$ with mutually distinct λ_j. Then a linear combination $f = c_1 f_1 + ... + c_n f_n \ne 0$ satisfies $f_A = \lambda f$ for some $\lambda \in k$ if and only if $c_j \ne 0$ except for just one value of j.*

The following proposition implies that $|\mathrm{Aut}(f)| = 84g' = 4 \cdot 3 \cdot 5 \cdot 7$ is impossible for any non-singular quintic f.

Proposition 2. 6 *A $\mathbf{Z}_7$-invariant quintic has a singular point.*

Proof. Let ε be a primitive 7-th root of 1, and denote by $A_j (j = 1, 2, 3)$ the matrices $\mathrm{diag}[1, 1, \varepsilon]$, $\mathrm{diag}[1, \varepsilon, \varepsilon^2]$ and $\mathrm{diag}[1, \varepsilon, \varepsilon^3]$ respectively. Then a quintic satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \le n \le 6$ turns out to be singular. Indeed, let $f'(x, y, z)$ be a homogeneous polynomial of degree $d \ge 2$. Then $(1, 0, 0)$ is a singular point of $C(f)$, if and only if none of monomials x^d, $x^{d-1}y$ and $x^{d-1}z$ appears in f'. We summarize the values i such that $m_{A_j^{-1}} = \varepsilon^i m$ for each j and the special nine monomials m in the following table.

	x^5	x^4y	x^4z	y^5	y^4x	y^4z	z^5	z^4x	z^4y
(1)	0	0	1	0	0	1	5	4	4
(2)	0	1	2	5	4	6	3	1	2
(3)	0	1	3	5	4	0	1	5	6

From this table we can easily see that a quintic $C(f)$ satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \le n \le 6$ has a singular point $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$.

A finite group of order $48g'$ or $40g'$ contains a subgroup of order 8. Such a group is isomorphic to one of the following five groups [4, p.51–52]:

1) $\mathbf{Z}_8$
2) $\mathbf{Z}_2 \times \mathbf{Z}_4$
3) $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$
4) Q_8, which is generated by a and b such that $a^4 = 1$, $b^2 = a^2$, and $ba = a^{-1}b$.
5) D_8, which is generated by a and b such that $a^4 = 1$, $b^2 = 1$, and $ba = a^{-1}b$.

We may safely omit the proof of

Lemma 2. 7 *Let ε be a primitive 8-th root of 1. A subgroup G_8 of $PGL(3,k)$ is isomorphic to $\mathbf{Z}_8$, if and only if G_8 is conjugate to one of the following 4 subgroups of $PGL(3,k)$:*

$$G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >, \quad G_{12} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^2]) >,$$
$$G_{13} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^3]) >, \quad G_{14} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^4]) > .$$

Proposition 2. 8 *Let f be a $\mathbf{Z}_8$-invariant quintic.*
(1) *f is non-singular if and only if it is projectively equivalent to $f' = x^5+Bx^3z^2+xz^4+y^4z$ with $B^2-4\neq 0$.*
(2) $|\mathrm{Aut}(f')| \leq 148$.

Lemma 2. 9 *Let $p \neq 3$ be a prime and ε be a primitive p-th root of 1. Then a subgroup G of $PGL(3,k)$ is isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$ if and only if G is conjugate to $G_{p^2} =< (\mathrm{diag}[1,\varepsilon,1]), \ (\mathrm{diag}[1,1,\varepsilon]) >$.*

The following lemma is due to Hiroaki Taniguchi.

Lemma 2. 10 (Taniguchi) *Let p be a prime, let ε be a primitive p-th root of 1 and let G_{p^2} be as in Lemma 2.9. If $f(x,y,z)$ is a G_{p^2}-invariant homogeneous polynomial of degree d with $p \not| \, d$, then f is reducible.*

Proof. Let $A = \mathrm{diag}[1,\varepsilon,1]$, and $B = \mathrm{diag}[1,1,\varepsilon]$. Assume $f_A = \varepsilon^i f$ and $f_B = \varepsilon^j f$ for some $i,j \in \{0,1,...,p-1\}$. If $i>0$, then y divides f. Similarly if $j>0$, then z divides f. If $i=j=0$,then x dives f, becasue f is a linear combination of monomials $x^{d_1}y^{d_2}z^{d_3}$ with $d_2 \equiv d_3 \equiv 0 \bmod p$ so that $d_1 = n - d_2 - d_2 n \not\equiv 0 \bmod p$.

Proposition 2. 11 *A $\mathbf{Z}_2 \times \mathbf{Z}_4$-invariant quintic is singular.*

Proof. A $\mathbf{Z}_2 \times \mathbf{Z}_4$-invariant quintic is a $\mathbf{Z}_2 \times \mathbf{Z}_2$-invariant quintic. Such a quintic is reducible by Lemma 2.9 and Lemma 2.10.

Proposition 2. 12 *No subgroup of $PGL(3,k)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.*

Lemma 2. 13 *Let G_8 be a subgroup of $PGL(3,k)$.*
(1) *G_8 is isomorphic to Q_8 if and only if it is conjugate to*

$$< (\mathrm{diag}[1,\sqrt{-1},\sqrt{-1}^3]), \ ([e_1,e_3,e_2]\mathrm{diag}[1,\sqrt{-1},\sqrt{-1}]) > .$$

(2) *G_8 is isomorphic to D_8 if and only if it is conjugate to*

$$< (\mathrm{diag}[1,\sqrt{-1},\sqrt{-1}^3]), \ ([e_1,e_3,e_2]) > .$$

Proposition 2. 14 (1) *A Q_8-invariant quintic, if any, is singualr.*
(2) *A D_8-invariant quintic, if any, is singualr.*

A group of order $36g'$ contains a subgroup of order 9 by Sylow's theorem. Such a group is isomorphic to either $\mathbf{Z}_9$ or $\mathbf{Z}_3 \times \mathbf{Z}_3$ [4]. By Lemma 2.3 we get

Lemma 2. 15 *Let ε be a primitive* 9*-th root of* 1*. A subgroup G_9 of $PGL(3,k)$ is isomorphic to Z_9, if and only if it is conjugate to one of the following three subgroups:*

$$G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >, \ G_{12} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^2]) >, \ G_{13} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^3]) > .$$

Proposition 2. 16 *A Z_9-invariant quintic is singular.*

Lemma 2. 17 *Let ω be a primitive third root of* 1*. A subgroup G_9 of $PGL(3,k)$ is isomorphic to $Z_3 \times Z_3$ if and only if it is conjugate to one of the following two groups:*

$$G_{01} =< (\mathrm{diag}[1,1,\omega]), (\mathrm{diag}[1,\omega,1]) >, \quad G_{12} =< (\mathrm{diag}[1,\omega,\omega^2]), ([e_2,e_3,e_1]) > .$$

Proposition 2. 18 *A $Z_3 \times Z_3$-invariant quintic is singular.*

Proof of Theorem 2.2 Let f be a non-singular quintic, and let $d = |\mathrm{Aut}(f)|$. Recall that

$$84g' = 4 \cdot 3 \cdot 5 \cdot 7, \quad 48g' = 16 \cdot 3 \cdot 5, \quad 40g' = 8 \cdot 25, \quad 36g' = 4 \cdot 5 \cdot 9.$$

By Proposition 2.6 we get $d \neq 84g'$. The inequalities $d \neq 48g'$, $40g'$ follow from Propositions 2.8, 2.11, 2.12 and 2.14. Finally Propositions 2.16 and 2.18 imply $d \neq 36g'$.

We note that $30g' = 2 \cdot 3 \cdot 25$. A group of order 25 is isomorphic to Z_{25} or $Z_5 \times Z_5$ [4].

Lemma 2. 19 *Let ε be a primitive* 25*-th root of* 1*. A subgroup G_{25} of $PGL(3,k)$ is isomorphic to Z_{25} if and only if it is conjugate to one of the following subgroups:*

$$\begin{array}{lll} G_{01} =< (\mathrm{diag}[1,1,\varepsilon]) >, & G_{12} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^2]) >, & G_{13} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^3]) >, \\ G_{14} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^4]) >, & G_{15} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^5]) >, & G_{1,10} =< (\mathrm{diag}[1,\varepsilon,\varepsilon^{10}]) > . \end{array}$$

Proof. By Lemma 2.3 we can classify subgroups $G_{ij} =< (\mathrm{diag}[1,\varepsilon^i,\varepsilon^j]) > (1 \leq i < j \leq 24$ with the greatest common divisor $(i,j,5) = 1)$ up to conjugacy, using computer.

Proposition 2. 20 *A Z_{25}-invariant quintic is singualr.*

Proposition 2. 21 *A $Z_5 \times Z_5$-invariant non-singular quintic is projectively equivalent to $x^5 + y^5 + z^5$.*

Theorem 2. 22 *A non-singular quintic f satisfying $|\mathrm{Aut}(f)| = 150$ is projectively equivalent to $x^5 + y^5 + z^5$.*

Proof. Propositions 2.20 and 2.21 imply the theorem.

3 Septics

Let $g = 15$, the genus of non-singular plane septic(i.e. a curve of degree 7), and let $g' = g - 1 = 14$. By Theorem 1.1 $|\mathrm{Aut}(x^7 + y^7 + z^7)| = 21g'$. If f is a non-singular plane septic, then $|\mathrm{Aut}(f)|$ may take values

$$\begin{array}{llll} 84g' = 8 \cdot 3 \cdot 49, & 48g' = 32 \cdot 3 \cdot 7, & 40g' = 16 \cdot 5 \cdot 7, & 36g' = 8 \cdot 9 \cdot 7, \\ 30g' = 4 \cdot 3 \cdot 5 \cdot 7, & 24g' = 16 \cdot 3 \cdot 7, & \frac{156}{7}g' = 8 \cdot 3 \cdot 13, & 21g' = 2 \cdot 3 \cdot 49 \end{array}$$

or less by Theorem 2.1. The eight values above are multiples of 8 except for $30g'$ and $21g'$. As we remarked in §2, a group of order 8 is isomorphic to one of the following five groups: $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, Q_8 and D_8. No subgroup of $PGL(3, k)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by Proposition 2.12. As for a quintic we have following Propositions 3.1 and 3.2

Proposition 3. 1 *A $\mathbb{Z}_8$-invariant septic is singular.*

Proposition 3. 2 *A $\mathbb{Z}_2 \times \mathbb{Z}_4$-invariant septic is singular.*

Proposition 3. 3 (1) *A Q_8-invariant septic, if any, is singular.*
(2) *A D_8-invariant septic, if any, is singular.*

Theorem 3. 4 *The maximum value of $|\mathrm{Aut}(f)|$ for a non-singular septic f is equal to either $30g'$ or $21g'$.*

Proof. By Propositions 3.1, 3.2 and 3.3 the order $|\mathrm{Aut}(f)|$ does not belong to $\{84g', 48g', 40g', 36g', 30g', 24g', \frac{156}{7}g'\} \setminus \{30g'\}$. Meanwhile $|\mathrm{Aut}(x^7 + y^7 + z^7)| = 21g'$ by Theorem 1.1.

We will show that $|\mathrm{Aut}(f)| \neq 30g'$ for any non-singular septic. Note that $30g' = 4 \cdot 3 \cdot 5 \cdot 7$. As we notice in the proof of Proposition 3.2,

Proposition 3. 5 *A $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant septic is singular.*

Suppose that there exists a non-singular septic f' such that $|\mathrm{Aut}(f')| = 30g'$. Denote by G' the finite group $\mathrm{Aut}(f')$. By Proposition 3.5 Sylow 2-group of G' is isomorphic to $\mathbb{Z}_4$. So we can apply the following theorem to G'.

Theorem 3. 6 ([4, p.146]) *If the Sylow subgroups of a finite group G of order n are all cyclic, then it is generated by two elements a and b with defining relations:*

$$\begin{array}{l} a^i = 1, \;\; b^j = 1, \;\; b^{-1}ab = a^r, \\ ij = n, \\ \gcd(i, \; (r-1)j) = 1, \\ r^j \equiv 1 \bmod i. \end{array}$$

For our group G' of order $420 = 4 \cdot 3 \cdot 5 \cdot 7$, possible pairs of $\{i, j\}$ in Theorem 3.6 are the followings(note that $\gcd(i, j) = 1$ if $r > 1$):

$$\{1, 420\}, \{4, 105\}, \{3, 140\}, \{5, 84\}, \{7, 60\}, \{12, 35\}, \{20, 21\}, \{28, 15\}.$$

In particular G' has an element of order 10, 12 or 15.

Lemma 3. 7 *Let ε be a primitive 10-th root of 1. A subgroup G_{10} of $PGL(3, k)$ is isomorphic to Z_{10} if and only if G_{10} is conjugate to one of the following subgroups:*

$$< (\mathrm{diag}[1, 1, \varepsilon]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >,$$
$$< (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) > .$$

Proposition 3. 8 *A Z_{10}-invariant septic f is singular.*

Lemma 3. 9 *Let ε be a primitive 12-th root of 1. A subgroup G_{12} of $PGL(3, k)$ is isomorphic to Z_{12} if and only if G_{12} is conjugate to one of the following subgroups:*

$$< (\mathrm{diag}[1, 1, \varepsilon]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >,$$
$$< (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) > .$$

Proposition 3. 10 *If f is a Z_{12}-invaraiant non-singular septic, then $|\mathrm{Aut}(f)| \neq 30g' = 420$.*

Lemma 3. 11 *Let ε be a primitive 15-th root of 1. A subgroup G_{15} of $PGL(3, k)$ is isomorphic to Z_{15} if and only if it is conjugate to one of the following subgroups:*

$$< (\mathrm{diag}[1, 1, \varepsilon]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >,$$
$$< (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) > .$$

Proposition 3. 12 *A Z_{15}-invariant septic f is singular.*

Theorem 3. 13 $|\mathrm{Aut}(f)| \leq 21g' = 294$.

Proof. Propositions 3.8, 3.10, and 3.12 imply that $|\mathrm{Aut}(f)|$ cannot be equal to $30g'$. By Theorem 3.4 we get the desired inequality.

Finally we will show that non-singular septics f with $|\mathrm{Aut}(f)| = 21g' = 2 \cdot 3 \cdot 49$ are unique.

Lemma 3. 14 *Let ε be a primitive 49-th root of 1. A subgroup G_{49} of $PGL(3, k)$ is isomorphic to Z_{49}, if and only if it is conjugate to one of the following subgroups:*

$$< (\mathrm{diag}[1, 1, \varepsilon]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^2]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^3]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^4]) >,$$
$$< (\mathrm{diag}[1, \varepsilon, \varepsilon^5]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^6]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^7]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^{14}]) >,$$
$$< (\mathrm{diag}[1, \varepsilon, \varepsilon^{18}) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^{19}]) >, \quad < (\mathrm{diag}[1, \varepsilon, \varepsilon^{21}]) > .$$

Proof. In view of Lemma 2.3 we can classify subgroups $< (\mathrm{diag}[1, \varepsilon^i, \varepsilon^j]) >$ $(1 \leq i < j \leq 48)$ up to conjugacy, using computer.

Proposition 3. 15 *A* $\mathbb{Z}_{49}$*-invariant septic* f *is singular.*

Proposition 3. 16 *A* $\mathbb{Z}_7 \times \mathbb{Z}_7$*-invariant septic* f *is non-singular if and only if* f *is projectively equivalent to* $x^7 + y^7 + z^7$.

Proof. Let $A = \mathrm{diag}[1, 1, \varepsilon]$ and $B = \mathrm{diag}[1, \varepsilon, 1]$. By Lemma 2.9 a subgroup G of $PGL(3, k)$ is isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_7$, if and only if G is conjugate to $< (A), (B) >$. A septic f satisfying $f_{A^{-1}} = \varepsilon^i f$ and $f_{B^{-1}} = \varepsilon^j f$, if any, is a singular except for the case $i = j = 0$. In the exceptional case f is a linear combination of x^7, y^7 and z^7.

Theorem 3. 17 *A non-singular plane septic* f *with* $|\mathrm{Aut}(f)| = 21g' = 2 \cdot 9 \cdot 2$ *is projectively equivalent to* $x^7 + y^7 + z^7$.

Proof. The theorem is a trivial consequence of Propositions 3.15 and 3.16.

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