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GROUP TOPOLOGIES ON THE COMPLEX NUMBERS WHICH MAKE CERTAIN GEOMETRIC SEQUENCES CONVERGE

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ABSTRACT. Let $(\mathbb{C},+)$ be the additive group of complex numbers. First, we prove that for every $z \in \mathbb{C}$ with $|z| > 1$, there exists a metrizable group topology $\tau(z)$ on $(\mathbb{C},+)$ such that $\tau(z)$ is coarser than the Euclidean topology and the sequence $\{z^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{C},+,\tau(z))$. Second, let z be in $\mathbb{C}\setminus\mathbb{R}$ with $|z| > 1$, and for each $k \in N$, let $I_k'(z)$ be the set of all complex numbers of a form $\alpha_1 z^{k_1} + \alpha_2 z^{k_2} + \cdots + \alpha_n z^{k_n}$, where $\alpha_i \in \mathbb{Z}$, $k_i \in N$ $(i = 1, 2, \cdots, n)$, $k \le k_1 < k_2 < \cdots < k_n$ and $n \in N$. We prove that $\inf\{|w| : w \in I_k'(z) \setminus \{0\}\} \to \infty$ $(k \to \infty)$ if and only if z is an algebraic integer with degree 2. In this case, we can easily define a metrizable group topology τ on $(\mathbb{C},+)$ such that the sequence $\{z^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{C},+,\tau)$.

1. Let $(\mathbb{C},+)$ be the additive group of complex numbers and $(\mathbb{R},+)$ the subgroup of real numbers. Hattori asked the following problem in his lecture [2].

Problem. *For a real number r, does there exist a metrizable group topology $\tau(r)$ on $(\mathbb{R},+)$ such that $\tau(r)$ is coarser than the usual topology and the sequence $\{r^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{R},+,\tau(r))$?*

Obviously, the answer is positive for all real number r with $|r| < 1$ and is negative for $r = 1$. Hattori [1] showed that the answer is positive for $r = 2$ and his proof can apply to all integers r with $|r| \ge 2$ (see Section 3 below). The problem, however, has been still unsolved for general $r > 1$. The purpose of this paper is to settle the problem by proving the result stated in the abstract.

Throughout the paper, let $\mathbb{Z}$ denote the set of integers and N the set of positive integers. As usual, we write $-A = \{-z : z \in A\}$, $A + B = \{w + z : w \in A, z \in B\}$ and $w + A = \{w + z : z \in A\}$ for $A, B \subseteq \mathbb{C}$ and $w \in \mathbb{C}$.

The following lemma was proved in the paper [3, Lemma 1].

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Lemma 1. *Let $z \in \mathbb{C}$ with $|z| > 1$ and assume that there exists a family $\{I_k : k \in N\}$ of subsets of $\mathbb{C}$ satisfying the following conditions* (1)–(5) :

(1) $\forall k \in N \ (0 \in I_k)$,
(2) $\forall k \in N \ (-I_k = I_k$ *and* $I_{k+1} + I_{k+1} \subseteq I_k)$,
(3) $\forall k \in N \ \forall w \in I_k \ \exists m \in N \ (w + I_m \subseteq I_k)$,
(4) $\inf\{|w| : w \in I_k \setminus \{0\}\} \to +\infty \ (k \to \infty)$, *and*
(5) $\forall k \in N \ \exists m \in N \ \forall n \in N \ (m \leq n \Rightarrow z^n \in I_k)$.

Then, there exists a metrizable group topology $\tau(z)$ on $(\mathbb{C}, +)$ such that $\tau(z)$ is coarser than the Euclidean topology and the sequnence $\{z^n : n \in N\}$ converges to 0 *in the topological group $(\mathbb{C}, +, \tau(z))$.*

2. In this section, we prove that for every $z \in \mathbb{C}$ with $|z| > 1$, there exists a metrizable group topology $\tau(z)$ on $(\mathbb{C}, +)$ such that $\tau(z)$ is coarser than the Euclidean topology and the sequence $\{z^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{C}, +, \tau(z))$.

Lemma 2. *Let $z \in \mathbb{C}$ with $|z| > 1$. For each positive integers $\varepsilon, \delta, n \in N$, there exists $p = p(\varepsilon, \delta, n) \in N$ satisfying the following condition* (6) :

(6) *For each $m \leq n$, $a_i \in \mathbb{Z}$ with $|a_i| \leq \delta$ and $k_i \in N$ $(i = 1, 2, \cdots, m)$, if $k_1 < k_2 < \cdots < k_m$, $p \leq k_m$ and $a_j z^{k_j} + a_{j+1} z^{k_{j+1}} + \cdots + a_m z^{k_m} \neq 0$ for each $j \in \{1, 2, \cdots, m\}$, then $|a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_m z^{k_m}| \geq \varepsilon$.*

Proof. To prove this by induction on n, we first consider the case of $n = 1$. For each $\varepsilon \in N$, there is $p \in N$ such that $|z|^p \geq \varepsilon$. If $p \leq k_1$ and $a_1 \in \mathbb{Z}$ with $a_1 \neq 0$, then $|a_1 z^{k_1}| \geq |z|^{k_1} \geq |z|^p \geq \varepsilon$. Thus, p satisfies (6) for each $\delta \in N$ and $n = 1$. Next, we assume that the existence of $p(\varepsilon, \delta, n-1)$ has been proved for all $\varepsilon \in N$ and all $\delta \in N$. We now fix $\varepsilon \in N$ and $\delta \in N$ and show that there exists $p(\varepsilon, \delta, n)$. By inductive hypothesis, we have $p' = p(\varepsilon + \delta, \delta, n-1)$. Let S be the set of all complex numbers u which can be written as a form

$$u = b_1 + b_2 z^{\ell_1} + \cdots + b_m z^{\ell_{m-1}},$$

where $m \leq n$, $b_i \in \mathbb{Z}$, $|b_i| \leq \delta$ $(i = 1, 2, \cdots, m)$, $\ell_i \in N$ $(i = 1, \cdots, m-1)$ and $\ell_1 < \cdots < \ell_{m-1} < p'$. Since S is a finite set, we have $s = \min\{|u| : u \in S, u \neq 0\} > 0$. Choose $p'' \in N$ such that $|z|^{p''} \cdot s \geq \varepsilon$, and define $p = p' + p''$. We show that p satisfies the condition (6). Let $m \leq n$, $a_i \in \mathbb{Z}$ with $|a_i| \leq \delta$ and $k_i \in N$ $(i = 1, 2, \cdots, m)$, and suppose that $k_1 < k_2 < \cdots < k_m$, $p \leq k_m$ and

$$a_j z^{k_j} + a_{j+1} z^{k_{j+1}} + \cdots + a_m z^{k_m} \neq 0 \text{ for each } j \in \{1, 2, \cdots, m\}. \tag{7}$$

Let $w = a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_m z^{k_m}$. To show that $|w| \geq \varepsilon$, we write $w = z^{k_1}(a_1 + u)$, where $u = a_2 z^{k_2 - k_1} + \cdots + a_m z^{k_m - k_1}$. Note that $u \neq 0$ and $a_1 + u \neq 0$ by (7). We distinguish two cases: If $k_m - k_1 < p'$, then $a_1 + u \in S$ and $k_1 > k_m - p' \geq p''$, because $k_m \geq p = p' + p''$. Thus, $|w| = |z|^{k_1} \cdot |a_1 + u| > |z|^{p''} \cdot s \geq \varepsilon$. If $k_m - k_1 \geq p'$, then $|u| \geq \varepsilon + \delta$ by the definition of p' and (7). Since $|a_1| \leq \delta$, it follows that $|a_1 + u| \geq \varepsilon$. Hence, $|w| = |z|^{k_1} \cdot |a_1 + u| \geq |z|^{k_1} \cdot \varepsilon \geq \varepsilon$. □

We now prove the main theorem announced in the abstract. For a set A, $\#(A)$ denotes the cardinality of A.

Theorem 1. *For every $z \in \mathbb{C}$ with $|z| > 1$, there exists a metrizable group topology $\tau(z)$ on $(\mathbb{C}, +)$ such that $\tau(z)$ is coarser than the Euclidean topology and the sequence $\{z^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{C}, +, \tau(z))$.*

Proof. Let $p_1 = 1$ and define $p_j = \max\{p_{j-1}, p(j, 2^j, 2(2^j - 1))\}$ for each $j \in N$ with $j \geq 2$. Let $N_j = \{k \in N : p_j \leq k < p_{j+1}\}$ for each $j \in N$. For each $k \in N$, let I_k be the set of all complex numbers w which can be written as a form

$$w = a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_n z^{k_n}$$

such that

(8) $a_i \in \mathbb{Z}$, $k_i \in N$ $(j = 1, 2, \cdots, n)$, $k_1 < k_2 < \cdots < k_n$ and $n \in N$,
(9) $\{k_1, k_2, \cdots, k_n\} \subseteq \bigcup_{j \geq k} N_j$,
(10) $\forall j \in N$ $(\#(\{k_1, k_2, \cdots, k_n\} \cap N_j) \leq 2^{j-k+1})$, and
(11) $\forall j \in N$ $\forall i \in \{1, 2, \cdots, n\}$ $(k_i \in N_j \Rightarrow |a_i| \leq 2^{j-k+1})$.

It suffices to show that the family $\mathbb{I} = \{I_k : k \in N\}$ satisfies (1)–(5) in Lemma 1. It is not difficult to prove that $\mathbb{I}$ satisfies (1), (2) and (5). To prove that $\mathbb{I}$ satisfies (3) and (4), let $k \in N$ and let $w \in I_k$. Then, w can be written as a form $w = a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_n z^{k_n}$ satisfying (8)–(11). Choose $s \in N$ with $s > \max\{j \in N : \{k_1, k_2, \cdots, k_n\} \cap N_j \neq \emptyset\}$. Then, $w + I_{s+1} \subseteq I_k$, which means that $\mathbb{I}$ satisfies (3). Finally, we show that $|w| \geq k$ if $w \neq 0$. Let $m = \min\{\ell \in N : \ell \leq n \text{ and } w = a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_\ell z^{k_\ell}\}$. Then,

$$w = a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_m z^{k_m}$$

and $a_j z^{k_j} + a_{j+1} z^{k_{j+1}} + \cdots + a_m z^{k_m} \neq 0$ for each $j \in \{1, 2, \cdots, m\}$. Let $t = \max\{j \in N : \{k_1, k_2, \cdots, k_m\} \cap N_j \neq \emptyset\}$; then $t \geq k$ by (9). Since $k_m \in N_t$, $k_m \geq p_t \geq p(t, 2^t, 2(2^t - 1))$. By (9) and (10),

$$m \leq \sum_{i=k}^{t} 2^{i-k+1} = 2(2^{t-k+1} - 1) \leq 2(2^t - 1).$$

Moreover, by (11), $|a_i| \leq 2^{t-k+1} \leq 2^t$ for each $i = 1, 2, \cdots, m$. Hence, it follows from Lemma 2 that $|w| \geq t \geq k$. Now, we have proved that $\inf\{|w| : w \in I_k \backslash \{0\}\} \geq k$, which implies that $\mathbb{I}$ satisfies (4). □

Remark 1. Hattori kindly informed the authors that the space $(\mathbb{C}, \tau(z))$ is not a Baire space. In fact, the set $U_n = \{z \in \mathbb{C} : |z| > n\}$ is dense and open in $(\mathbb{C}, \tau(z))$ for each $n \in N$, but $\bigcap_{n \in N} U_n = \emptyset$. Hence, the space $(\mathbb{C}, \tau(z))$ cannot be completely metrizable.

The following corollary, which settles Hattori's problem, immediately follows from the above theorem.

Corollary 3. *For every $r \in \mathbb{R}$ with $|r| > 1$, there exists a metrizable group topology $\tau(r)$ on $(\mathbb{R}, +)$ such that $\tau(r)$ is coarser than the usual topology and the sequence $\{r^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{R}, +, \tau(r))$.*

3. Let $z \in \mathbb{C}$ with $|z| > 1$. For each $k \in N$, define $I'_k(z)$ to be the set of all complex numbers w which can be written as a form

$$w = \alpha_1 z^{k_1} + \alpha_2 z^{k_2} + \cdots + \alpha_n z^{k_n},$$

where $\alpha_i \in \mathbb{Z}$, $k_i \in N$ $(i = 1, 2, \cdots, n)$, $k \leq k_1 < k_2 < \cdots < k_n$ and $n \in N$. Then, it is natural to ask if the family $\mathbb{I}'(z) = \{I'_k(z) : k \in N\}$ satisfies (1)–(5) in Lemma 1. However, the answer is negative; more precisely, $\mathbb{I}'(z)$ always satisfies (1)–(3) and (5), but it does not necessarily satisfy (4). In particular, if $\mathbb{I}'(z)$ satisfies (4) then the topology obtained by simply taking the family $\mathcal{B}(z) = \{u + U_k : u \in \mathbb{C}, k \in N\}$ as a base, where $U_k = \bigcup_{w \in I'_k(z)} \{u \in \mathbb{C} : |u - w| < 1/2^k\}$ for each $k \in N$, is called the *simple topology* induced by z and is denoted by $\tau'(z)$.

First, we show that for a real number r with $|r| > 1$, $\mathbb{I}'(r)$ satisfies (4) if and only if $r \in \mathbb{Z}$. If $r \in \mathbb{Z}$, then $I'_k(r)$ is no other than the set of all integral multiples of r^k, i.e., $I'_k(r) = \{\alpha r^k : \alpha \in \mathbb{Z}\}$, for each $k \in N$. This implies that $\inf\{|w| : w \in I'_k(r), w \neq 0\}) = r^k \to +\infty$, and hence, $\mathbb{I}'(r)$ satisfies (4). This is essentially Hattori's proof in [1] that the problem has the positive answer for $r = 2$. Conversely, the following fact shows that $\mathbb{I}'(r)$ does not satisfy (4) if $r \in \mathbb{R} \setminus \mathbb{Z}$ and $|r| > 1$.

Fact. *Let $r \in \mathbb{R} \setminus \mathbb{Z}$ with $|r| > 1$. Then, the set $I'_k(r)$ defined above is dense in $\mathbb{R}$ for each $k \in N$.*

Proof. Since every integral multiple of an element of $I'_k(r)$ is in $I'_k(r)$, it suffices to show that

$$\inf\{|w| : w \in I'_k(r), w \neq 0\} = 0 \tag{12}$$

for each $k \in N$. We distinguish two cases. First, we assume that r is a rational number, i.e., $r = a/b$ for some $a, b \in \mathbb{Z}$. We may assume that the fraction a/b is irreducible. To prove (12), let $\varepsilon > 0$. Then, we can find even numbers $m, n \in N$ such that $k \leq m < n$ and $a^m/b^n < \varepsilon$. Since a/b is irreducible, b^{n-m} and a^{n-m} are mutually prime, which implies that there are $\alpha, \beta \in \mathbb{Z}$ such that $\alpha b^{n-m} + \beta a^{n-m} = 1$. Now, we have $0 < \alpha r^m + \beta r^n < \varepsilon$, because

$$\alpha r^m + \beta r^n = \alpha \left(\frac{a}{b}\right)^m + \beta \left(\frac{a}{b}\right)^n = \left(\frac{a^m}{b^n}\right)(\alpha b^{n-m} + \beta a^{n-m}).$$

Since $\alpha r^m + \beta r^n \in I'_k(r) \setminus \{0\}$, (12) is proved.

Next, we assume that r is an irrational number. Let $\varepsilon > 0$. Choose $m \in N$ with $|r|^k/m < \varepsilon$ and let $M = \{1, 2, \cdots, m+1\}$. Consider the set $A = \{ir - \lfloor ir \rfloor : i \in M\}$, where $\lfloor ir \rfloor$ is the greatest integer not greater than ir. Since r is irrational, $ir - \lfloor ir \rfloor \neq jr - \lfloor jr \rfloor$ for $i \neq j$. Hence, A contains $m + 1$ many distinct elements between 0 and 1. This means that

$$0 < |(ir - \lfloor ir \rfloor) - (jr - \lfloor jr \rfloor)| < 1/m$$

for some $i, j \in M$ with $i \neq j$. Let $\alpha = i - j$ and $\beta = \lfloor ir \rfloor - \lfloor jr \rfloor$. Then, $\alpha, \beta \in \mathbb{Z}$ and $0 < |\alpha r - \beta| < 1/m$. Hence, $0 < |\alpha r^{k+1} - \beta r^k| = |r|^k |\alpha r - \beta| < |r|^k/m < \varepsilon$. Since $\alpha r^{k+1} - \beta r^k \in I'_k(r) \setminus \{0\}$, we have (12). □

Second, we determine a complex number z such that $\mathbb{I}'(z)$ satisfy (4) by proving the following thorem:

Theorem 2. *Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. Then, $\mathbb{I}'(z)$ satisfies* (4) *if and only if z is an algebraic integer with degree* 2, *i.e., $z^2 + \alpha z + \beta = 0$ for some $\alpha, \beta \in \mathbb{Z}$.*

To prove Theorem 2, we need some notations and a lemma. As usual, let $\mathbb{Z}[x]$ denote the set of all polynomials with integral coefficients and $r\mathbb{Z} = \{rn : n \in \mathbb{Z}\}$ for each $r \in \mathbb{R}$. Further, let $\mathbb{Z}_0[x]$ be the subset of $\mathbb{Z}[x]$ consisting of all polynomials such that the coefficient of the term with the maximum degree is 1. For a set A, $\sharp A$ denotes the cardinality of A.

Lemma 4. *Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. Assume that z is not an algebraic integer with degree* 2. *Then, there exists $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$.*

(For the proof, see [4, Lemma 2, P.130–131].)

Let $z \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic integer with degree 2. Then, z is contained in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{m})$, where m is a negative square free integer. As is well known, the ring $\mathfrak{a}_K$ of algebraic integers in K is a lattice, i.e., a free $\mathbb{Z}$-module of rank 2 whose basis are 1 and u, where $u = (1 + \sqrt{m})/2$ if $m \equiv 1$ (mod 4) and $u = \sqrt{m}$ if $m \equiv 2$ or 3 (mod 4).

Proof of Theorem 2. Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$. If z is an algebraic integer with degree 2, then $f(z) \in \mathfrak{a}_K$ for each $f(x) \in \mathbb{Z}[x]$, where $\mathfrak{a}_K$ is defined as above. Since $\mathfrak{a}_K$ is a lattice, we have $\alpha = \min\{|f(z)| : f(z) \neq 0, f(x) \in \mathbb{Z}[x]\} > 0$. For each $w \in I'_k(z) \setminus \{0\}$, w can be written as $w = z^k f(z)$ for some $f(x) \in \mathbb{Z}[x]$, and thus,

$$|w| = |z|^k |f(z)| \geq |z|^k \alpha.$$

Hence, $\inf\{|w| : w \in I'_k(z) \setminus \{0\}\} = |z|^k \alpha$, which implies that $\mathbb{I}'(z)$ satisfies (4). Conversely, assume that z is not an algebraic integer with degree 2. By Lemma 4, there is $f(x) \in \mathbb{Z}[x]$ such that $0 < |f(z)| < 1$. Let $k \in N$ be fixed. Then, $z^k f(z)^n \in I'_k(z) \setminus \{0\}$ for each $n \in N$. Since $|z^k f(z)^n| = |z|^k |f(z)|^n \to 0$ $(n \to \infty)$, we have

$$\inf\{|w| : w \in I'_k(z) \setminus \{0\}\} = 0.$$

Hence, $\mathbb{I}'(z)$ fails to satisfy (4), which completes the proof. □

Corollary 5. *Assume that either $z \in \mathbb{Z}$ or z is an imaginary algebraic integer with degree* 2, *and that $|z| > 1$. Then, there exists a metrizable group topology τ on $(\mathbb{C}, +)$ such that τ is coarser than the Euclidean topology and the sequence $\{\alpha^n z^n : n \in N\}$ coverges to* 0 *in the topological group $(\mathbb{C}, +, \tau)$ for each $\alpha \in \mathbb{Z}$.*

Proof. By Theorem 2, $\mathbb{I}'(z)$ satisfies (4). Hence, the simple topology $\tau'(z)$ induced by z is a required topology; infact, $\{\alpha^n z^n : n \in N\}$ converges to 0 in $(\mathbb{C}, +, \tau'(z))$ for each $\alpha \in \mathbb{Z}$, because $\alpha^n z^n \in I'_k(z)$ whenever $n \geq k$, for every $k, n \in N$. □

Remark 2. It is open whether, for every two $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ and $|z_i| > 1$ $(i = 1, 2)$, there is a metrizable group topology τ on $(\mathbb{C}, +)$ such that τ is coarser than the Euclidean topology and both $\{z_1^n : n \in N\}$ and $\{z_2^n : n \in N\}$ converge to 0 in $(\mathbb{C}, +, \tau)$. In particular, the following question asked by Hattori [2] still remains open: Does there exist a metrizable group topology τ on $(\mathbb{R}, +)$ such that τ is coarser than the Euclidean topology and both $\{2^n : n \in N\}$ and $\{3^n : n \in N\}$ converge to 0 in the topological group $(\mathbb{R}, +, \tau)$?

Remark 3. Theorem 2 enables us to construct the simple topology $\tau'(z)$ by a geometrical method. To show this, let $z \in \mathbb{C} \setminus \mathbb{R}$ be a complex number, with $|z| > 1$, such that $\mathbb{I}'(z)$ satisfies (4). Then, z is an algebraic integer with degree 2 by Theorem 2. Let $\mathfrak{a}_K$ be the same as the one defined before the proof of Theorem 2. Let $k \in N$ be fixed for a while. Since $I'_k(z)$ is a subgroup of $\mathfrak{a}_K$, $I'_k(z)$ is also a lattice, and hence, the quotient topological group $T_k = \mathbb{C}/I'_k(z)$ is homeomorphic to the torus. Let $h_k : \mathbb{C} \to T_k$ be the natural homomorphism. If we define $h_k : \mathbb{C} \to T_k$ for each $k \in N$, then we have a continuous homomorphism

$$h : \mathbb{C} \to T = \prod_{k \in N} T_k$$

such that $h_k = \pi_k \circ h$ for each $k \in N$, where $\pi_k : T \to T_k$ is the projection. Let $\rho(z)$ be the relative topology on $h[\mathbb{C}]$ induced by the product topology on T. Since $z^n \in I'_k(z)$ for each $k \leq n$, the sequence $\{h(z^n) : n \in N\}$ converges to $h(0)$ with respect to the topology $\rho(z)$. Now, observe that condition (4) implies that h is a monomorphism. Moreover, it is not difficult to see that the map $h : (\mathbb{C}, \tau'(z)) \to (h[\mathbb{C}], \rho(z))$ is a homeomorphism. Hence, we can consider that $\rho(z) = \tau'(z)$.

For an integer $r \in \mathbb{Z}$, $I'_k(r)$ coincides with the set of all integral multiples of r^k, i.e., $I'_k(r) = r^k\mathbb{Z}$ for each $k \in N$. If $|r| > 1$, then the topology $\tau'_{\mathbb{R}}(r)$ on $\mathbb{R}$ generated by a base $\{s + V_k : s \in \mathbb{R}, k \in N\}$, where $V_k = \bigcup_{n \in \mathbb{Z}} \{x \in \mathbb{R} : |x - r^k n| < 1/2^k\}$, is also a metrizable group topology on $\mathbb{R}$ such that $\tau'_{\mathbb{R}}(r)$ is coarser than the Euclidean topology and the sequence $\{r^n : n \in N\}$ converges to 0 in the topological group $(\mathbb{R}, +, \tau'_{\mathbb{R}}(r))$. The topology $\tau'_{\mathbb{R}}(r)$ was first studied by Hattori [1] for $r = 2$. Similarly to the above, $\tau'_{\mathbb{R}}(r)$ is obtained as a relative topology induced by the product topology on the product of countably many circles $\{\mathbb{R}/r^k\mathbb{Z} : k \in N\}$.

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