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A Stefan Problem with Memory and Nonlinear Boundary Condition

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Abstract. This note is devoted to the study of a Stefan problem with memory that includes a third type boundary condition associated with a maximal monotone nonlinearity. The corresponding initial-boundary value problem can be formulated as a Cauchy problem for an abstract doubly nonlinear integrodifferential equation which belongs to a class already analyzed by the authors in a recent paper [2]. A slight variation of the abstract theory developed in [2] is then applied to deduce the existence of a solution to our Stefan problem.

1. Introduction

Let us consider a two-phase material which occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ, at any time $t \in [0,T]$, $T > 0$ being fixed. This system is characterized by a pair of state variables, namely the (relative) temperature ϑ and the phase proportion χ. We assume that the evolution of the pair (ϑ, χ) is governed by the following energy balance equation (see [7, 8, 9] and references therein)

$$\partial_t(\vartheta + \chi + \varphi * \vartheta + \psi * \chi) - \Delta(\vartheta + k * \vartheta) = g \quad \text{in } Q := \Omega \times (0,T) \tag{1.1}$$

coupled with the condition

$$\chi \in \mathcal{H}(\vartheta) \quad \text{in } Q \tag{1.2}$$

relating χ to ϑ. Here, Δ is the usual Laplace operator acting on the space variables, $\partial_t = \partial/\partial t$, and $*$ denotes the convolution product with respect to time over $(0,t)$, that is, for instance,

$$(\varphi * \vartheta)(\cdot, t) = \int_0^t \varphi(t-s)\vartheta(\cdot, s)ds, \qquad t \in [0,T].$$

In addition, $\mathcal{H}$ stands for the Heaviside graph ($\mathcal{H}(r) = 0$ if $r < 0$, $\mathcal{H}(0) = [0,1]$, $\mathcal{H}(r) = 1$ if $r > 0$) and the memory kernels φ, ψ, $k : (0,T) \to \mathbb{R}$ are given along with the function $g : Q \to \mathbb{R}$.

Initial and boundary value problems for the system (1.1)-(1.2) have been investigated in several papers (see [4, 6, 7, 9], cf. also [1, 5, 11] for related problems). Nevertheless, in all the mentioned literature, (1.1)-(1.2) is complemented with variational boundary conditions, that turn out to be linear with respect to ϑ and/or the outward normal derivative $\partial_\nu \vartheta$. On the contrary, in this note we prove the existence of solutions to an initial-boundary value problem for (1.1)-(1.2) characterized by a nonlinear boundary condition. To be more precise, we supply the system with

$$\partial_\nu(\vartheta + k * \vartheta) + \alpha(\vartheta) \ni h \quad \text{on } \Sigma := \Gamma \times (0,T) \tag{1.3}$$

$$(\vartheta + \chi)(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{1.4}$$

where $\alpha : \mathbb{R} \to 2^{\mathbb{R}}$ denotes a maximal monotone graph, and the functions $h : \Sigma \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are known.

Problem (1.1)-(1.4) contains two monotone nonlinearities represented by the maximal monotone graphs $\mathcal{H}$ and α. In Section 3, we consider an extended version of (1.1)-(1.4) in which the kernels φ and ψ are allowed to depend on the space variables too, and where the term $-k * \Delta\vartheta$ is replaced by a rather general second order linear convolution operator acting on ϑ. Moreover, we let the right hand side g of (1.1) incorporate an additional nonlinearity in order to represent not only a measurable function of (x,t) but a Lipschitz continuous function of ϑ as well. Then we show that the resulting problem can be reformulated as a Cauchy problem for a doubly nonlinear integrodifferential evolution equation.

The abstract formulation we obtain essentially reduces to a particular case of a class of evolution equations studied in [2]. In that paper, two existence results are proved by means of a semi-implicit time discretization procedure. Here, in Section 2, we state a slight generalization of the main theorem of [2], whose proof can be achieved by performing simple changes in the original one. This result applies to the abstract equation

$$(M\vartheta)' + A\vartheta + B * \vartheta \ni f + F(M\vartheta) + G(\vartheta) \quad \text{in } V', \text{ a.e. in } (0,T) \tag{1.5}$$

where V' is meant to be the dual space of $V = H^1(\Omega)$ in the framework of (1.1)-(1.4). We also point out that M takes the place of $\mathcal{I} + \mathcal{H}$ ($\mathcal{I}$ being the identity mapping) and is maximal monotone from $H = L^2(\Omega)$ to the same space H (identified with its dual space). The other maximal monotone operator is A which works from V to V' and collects the contributions of $-\Delta\vartheta$ and $\alpha(\vartheta)$ from (1.1) and (1.3), while B is a function from $[0,T]$ into the space of linear bounded operators from V to V'. On the other hand, f maps $(0,T)$ into V' and F, G are *causal* (cf. Section 2 for a precise definition) Lipschitz continuous operators on $L^2(0,T;H)$. In addition, F is required to be linear and it is naturally applied to the same selection of $M(\vartheta)$ appearing on the left hand side of (1.5).

The existence of a solution to the Cauchy problem for (1.5) is established in the next section. Afterwards, the abstract result is used in Section 3 to deduce the existence of weak solutions to the above mentioned generalized version of (1.1)-(1.4).

2. Abstract result

On account of [2, Sect. 2], we introduce the hypotheses on the data of the Cauchy problem associated with (1.5).

(A1) Let V and W be reflexive real Banach spaces and let H denote a real Hilbert space which is identified with its dual. We assume that

$$V \hookrightarrow W \hookrightarrow H \hookrightarrow W' \hookrightarrow V'$$

with dense and continuous injections, the first and the last embeddings being also compact.

(A2) M is a maximal monotone operator from H to H that is linearly bounded, namely,

$$\exists\, C_1 > 0 \ : \quad \|w\|_H \le C_1\,(1 + \|v\|_H) \qquad \forall v \in H, \ \forall w \in M(v) \tag{2.1}$$

and M^{-1} is Lipschitz continuous, i.e.,

$$\begin{aligned} \exists\, C_2 > 0 \ : \quad & C_2 \|v_1 - v_2\|_H^2 \le (w_1 - w_2, v_1 - v_2) \\ & \forall v_1, v_2 \in H, \ \forall w_1 \in M(v_1), \ \forall w_2 \in M(v_2) \end{aligned} \tag{2.2}$$

where $(\cdot\,,\cdot)$ stands for the scalar product in H.

(A3) A is a maximal monotone and bounded operator from V to V' such that $A = A_1 + A_2$, where A_i coincides with the subdifferential ∂J_i of a convex and lower semicontinuous function $J_i : V \to \mathbb{R}$, for $i = 1, 2$. Furthermore, A_1 is linear, A_2 is bounded from V to W', and $J := J_1 + J_2$ satisfies

$$\frac{1}{2}\|v\|_H^2 + J(v) \ge C_3 \|v\|_V^p - C_4 \qquad \forall v \in V \tag{2.3}$$

for some constants $p \ge 2$, $C_3 > 0$, $C_4 \ge 0$.

(A4) $B \in W^{1,1}(0,T; \mathcal{L}(V,V'))$, where $\mathcal{L}(V,V')$ stands for the Banach space of all the linear and continuous operators from V to V'.

(A5) $F, G : L^2(0,T;H) \to L^2(0,T;H)$ are two Lipschitz continuous operators that are *causal* in the sense that

$$\begin{aligned} &\text{if } v_1, v_2 \in L^2(0,T;H), \ t \in (0,T), \text{ and } v_1 = v_2 \text{ a.e. in } (0,t), \\ &\quad \text{then } F(v_1) = F(v_2), \ G(v_1) = G(v_2) \text{ a.e. in } (0,t). \end{aligned}$$

Moreover, F is linear.

(A6) $f \in L^2(0,T;H) + W^{1,1}(0,T;V')$.

(A7) $u_0 \in H, \qquad \vartheta_0 := M^{-1}(u_0) \in V, \qquad J(\vartheta_0) < +\infty.$

Here is the precise formulation of the Cauchy problem.

Problem (P) *Find* $\vartheta \in L^\infty(0,T;V)$ *and two auxiliary functions*

$$u \in W^{1,2}(0,T;V') \cap L^\infty(0,T;H), \qquad \xi \in L^\infty(0,T;V') \tag{2.4}$$

such that

$$u' + \xi + B * \vartheta = f + F(u) + G(\vartheta) \quad \text{in } V', \text{ a.e. in } (0,T) \tag{2.5}$$
$$u(t) \in M(\vartheta(t)) \quad \text{for a.a. } t \in (0,T) \tag{2.6}$$
$$\xi(t) \in A(\vartheta(t)) \quad \text{for a.a. } t \in (0,T) \tag{2.7}$$
$$u(0) = u_0 \quad \text{in } V'. \tag{2.8}$$

The existence of a solution to **(P)** is ensured by

Theorem 2.1 *Let (A1)-(A7) hold. Then there exists at least one solution* (ϑ, u, ξ) *to Problem* **(P)***, with the additional property that* $\vartheta \in W^{1,2}(0,T;H)$.

A comparison between our Problem **(P)** and its counterpart in [2] shows that the term

$$(B * \vartheta)(t) = \int_0^t B(t-s)\vartheta(s)ds, \qquad t \in [0,T]$$

is now used in place of the original one, which is $k * B\vartheta$ for a kernel k in $W^{1,1}(0,T)$ and some operator $B \in \mathcal{L}(V,V')$ (in fact, $k*B$ is a special case of $B*$, cf. (A4)). However, a careful examination of the proof of Theorem 2.1 in [2] reveals that the procedure devised there also works in the present case. Basically, the main change concerns the proof of [2, Lemma 3.6], where one has to deduce [2, ineq. (3.19)]. This can be done by taking into account that [2, ineq. (3.25)] still follows from [2, ineq. (3.23)] in our current setup.

Remark 2.2 Regarding (A3), we note that the subdifferential ∂J coincides with the sum $\partial J_1 + \partial J_2 = A$ and that the functions J, J_1, and J_2 are all continuous from V to $\mathbb{R}$ (cf. Remarks 2.3 and 2.4 in [2]).

3. Application

Here we consider a generalization of the Stefan problem (1.1)-(1.2) and provide a weak formulation of it in accordance with Problem **(P)**. Then, the existence of solutions can be demonstrated by applying Theorem 2.1 (see [2, Sect. 5] for other possible applications of the abstract result).

Throughout this section, Ω will denote a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ and the notation for Γ, Q, Σ is the same as in the Introduction. As usual, the variable in $\Omega \cup \Gamma$ is indicated by $x = (x_1, \ldots, x_N)$ and ∂_{x_j} simply replaces $\partial/\partial x_j$, $j = 1, \ldots, N$.

We start by setting the (formal) Stefan problem for the unknowns $\vartheta : Q \to \mathbb{R}$ and $\chi : Q \to [0,1]$ which have to satisfy

$$\partial_t(\vartheta + \chi + \varphi(x,\cdot) * \vartheta + \psi(x,\cdot) * \chi) + \mathcal{A}\vartheta + \mathcal{B} * \vartheta = g(x,t,\vartheta) \quad \text{in } Q \tag{3.1}$$

$$\chi \in \mathcal{H}(\vartheta) \quad \text{in } Q \tag{3.2}$$

$$\partial_{\nu(\mathcal{A}+\mathcal{B}*)}\vartheta + \alpha(\vartheta) \ni h(x,t) \quad \text{on } \Sigma \tag{3.3}$$

$$(\vartheta + \chi)|_{t=0} = u_0 \quad \text{in } \Omega \tag{3.4}$$

in a suitable sense, where φ, $\psi : Q \to \mathbb{R}$ and $g : Q \times \mathbb{R} \to \mathbb{R}$ are prescribed. Moreover, $\mathcal{A}$ is the linear second order differential operator

$$(\mathcal{A}v)(x) := -\sum_{j,m=1}^{N} \partial_{x_j}(a_{jm}(x)\partial_{x_m}v(x)), \qquad x \in \Omega \tag{3.5}$$

and $\mathcal{B} * \vartheta$ is defined by

$$(\mathcal{B} * v)(x,t) := -\sum_{j,m=1}^{N} \partial_{x_j} \int_0^t (b_{jm}(x,t-s)\partial_{x_m}v(x,s))ds, \qquad (x,t) \in Q. \tag{3.6}$$

Here the coefficients a_{jm} and b_{jm} are measurable functions from Ω and Q, respectively, to $\mathbb{R}$. Note that both $\mathcal{A}$ and $\mathcal{B}$ are in divergence form. Besides, $\partial_{\nu(\mathcal{A}+\mathcal{B}*)}$ denotes the conormal derivative related to the operator $\mathcal{A} + \mathcal{B}*$ (see below for details), while $h : \Sigma \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are given data.

Let us introduce now the assumptions that will enable us to reformulate (3.1)-(3.4) as **(P)**.

(B1) φ, $\psi \in W^{1,1}(0,T;L^\infty(\Omega))$.

(B2) g is a Carathéodory function satisfying $g(\cdot,\cdot,0) \in L^2(Q)$ and

$$|g(t,x,z_1) - g(t,x,z_2)| \le c_1|z_1 - z_2| \quad \text{for a.a. } (x,t) \in Q, \ \forall z_1, z_2 \in \mathbb{R}.$$

for some positive constant c_1.

(B3) $a_{jm} = a_{mj} \in L^\infty(\Omega)$ and $b_{jm} \in W^{1,1}(0,T;L^\infty(\Omega))$ for $j,m = 1,\ldots,N$. In addition, there exists a constant $c_2 > 0$ such that

$$\sum_{j,m=1}^{N} a_{jm}(x)y_jy_m \ge c_2|y|^2 \quad \forall y = (y_1,\ldots,y_N) \in \mathbb{R}^N, \text{ for a.a. } x \in \Omega. \tag{3.7}$$

Also, setting

$$a(v,w) := \sum_{j,m=1}^{N} \int_\Omega a_{jm}v_{x_j}w_{x_m} \qquad \forall v,w \in H^1(\Omega)$$

and associating with any $v \in L^2(0,T;H^1(\Omega))$ the element $\beta * v \in C^0([0,T];H^1(\Omega)')$ specified by

$$_{H^1(\Omega)'}\langle(\beta * v)(t), w\rangle_{H^1(\Omega)} := \sum_{j,m=1}^{N} \int_\Omega \left(b_{jm} * v_{x_j}\right)(\cdot,t)\, w_{x_m}$$
$$\forall w \in H^1(\Omega), \ \forall t \in [0,T], \tag{3.8}$$

we point out that the conormal derivative $\partial_{\nu(\mathcal{A}+\mathcal{B}*)}$ is then defined for all $v \in L^2(0,T;H^1(\Omega))$ such that $(\mathcal{A}+\mathcal{B}*)v \in L^2(0,T;L^2(\Omega))$ by

$$
\begin{aligned}
&{}_{L^2(0,T;H^{-1/2}(\Gamma))}\langle \partial_{\nu(\mathcal{A}+\mathcal{B}*)}v, w\rangle_{L^2(0,T;H^{1/2}(\Gamma))} \\
&:= \int_0^T \left(a(v(\cdot,t),w(\cdot,t)) + {}_{H^1(\Omega)'}\langle(\beta * v)(t),, w(\cdot,t)\rangle_{H^1(\Omega)}\right) dt \\
&\quad - \int_0^T\int_\Omega w(\mathcal{A}+\mathcal{B}*)v \qquad \forall w \in L^2(0,T;H^1(\Omega)). \qquad (3.9)
\end{aligned}
$$

(B4) $\quad \alpha = \partial\phi$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a convex potential satisfying

$$\phi(z) \le c_3\left(|z|^2+1\right) \qquad \forall z \in \mathbb{R}$$

for some positive constant c_3.

(B5) $\quad h \in W^{1,1}(0,T;L^2(\Gamma))$, $u_0 \in L^2(\Omega)$, and $\vartheta_0 = (\mathcal{I}+\mathcal{H})^{-1}(u_0) \in H^1(\Omega)$.

Therefore, on account of (B1)-(B5), we can now state a weak formulation of the Stefan problem (3.1)-(3.4). For the sake of convenience, in the sequel we denote by $< \cdot,\cdot >$ the duality pairing between $H^1(\Omega)'$ and $H^1(\Omega)$.

Problem (S) *Find* $\vartheta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ *and the auxiliary functions*

$$\chi \in L^\infty(Q), \qquad \eta \in L^\infty(0,T;L^2(\Gamma))$$

which satisfy

$$\vartheta + \chi \in W^{1,1}(0,T;H^1(\Omega)') \qquad (3.10)$$

$$
\begin{aligned}
&< \partial_t(\vartheta + \chi + \varphi * \vartheta + \psi * \chi), v > + a(\vartheta, v) + < \beta * \vartheta, v > + \int_\Gamma \eta v \\
&\quad = (g(\cdot,\cdot,\vartheta), v) + \int_\Gamma hv \qquad \forall v \in H^1(\Omega), \ \text{a.e. in } (0,T) \qquad (3.11)
\end{aligned}
$$

$$\chi \in \mathcal{H}(\vartheta) \quad \text{a.e. in } Q \qquad (3.12)$$

$$\eta \in \alpha(\vartheta) \quad \text{a.e. on } \Sigma \qquad (3.13)$$

$$(\vartheta + \chi)(0) = u_0 \quad \text{in } H^1(\Omega)'. \qquad (3.14)$$

Our main result is

Theorem 3.1 *Let (B1)-(B5) hold. Then Problem* **(S)** *admits a solution.*

Remark 3.2 It is worth noting that Theorem 3.1 can be viewed as a generalization of [10, Prop. 2.4]. Moreover, making a comparison between Problem **(S)** and (3.1)-(3.4), we observe that equation (3.1) does not hold in $L^2(Q)$ and, especially, the boundary condition (3.3) cannot be recovered in the sense of traces in $L^2(0,T;H^{-1/2}(\Gamma))$ (contrary to the example developed in [2, Subsect. 5.1]). However, choosing $v \in H_0^1(\Omega)$ as a test function in (3.11), it is straightforward to deduce

$$
\begin{aligned}
a(\vartheta, v) + < \beta * \vartheta, v > &= - < \partial_t(\vartheta + \chi + \varphi * \vartheta + \psi * \chi) - g(\cdot,\cdot,\vartheta), v > \\
&\qquad \forall v \in H_0^1(\Omega), \ \text{a.e. in } (0,T).
\end{aligned}
$$

Then, integrating in time over $(0,t)$, $t \in (0,T]$, and recalling (B3), (3.5), and (3.6), we obtain with the help of (3.14)

$$((\mathcal{A}+\mathcal{B}*)(1*\vartheta))(t) = -(\vartheta+\chi+\varphi*\vartheta+\psi*\chi)(\cdot,t)+u_0+\int_0^t g(\cdot,s,\vartheta(\cdot,s))ds$$
$$\text{in } H^{-1}(\Omega), \text{ for a.a. } t\in(0,T) \qquad (3.15)$$

where $(1*\vartheta)(\cdot,t) = \int_0^t \vartheta(\cdot,s)ds$. Note that the right hand side of (3.15) belongs to $L^2(Q)$. Hence, we have that $(\mathcal{A}+\mathcal{B}*)(1*\vartheta) \in L^2(Q)$ and, in view of (3.9), the *integrated* boundary condition

$$\partial_{\nu(\mathcal{A}+\mathcal{B}*)}(1*\vartheta)+1*\eta \ni 1*h$$

(cf. (3.13) as well) holds in the sense of traces in $L^2(0,T;H^{-1/2}(\Gamma))$. At this point, we could also argue that equation (3.1) makes sense, e.g., in $W^{-1,2}(0,T;H^{-1}(\Omega))$.

Proof of Theorem 3.1. It suffices to show that Problem **(S)** can be put in the abstract framework of **(P)**. Then, the existence will follow from Theorem 2.1. Hence, let $V = H^1(\Omega)$, $H = L^2(\Omega)$, and introduce the new variable

$$u = \vartheta + \chi. \qquad (3.16)$$

Note that, owing to (B1), the relations (3.11)-(3.12) can be rewritten in the form

$$<\partial_t u, v> + a(\vartheta,v) + \int_\Gamma \eta v + <\beta*\vartheta, v> = <f,v> + (F(u)+G(\vartheta),v)$$
$$\forall v \in V', \text{ a.e. in } (0,T)$$
$$u \in (\mathcal{I}+\mathcal{H})(\vartheta) \quad \text{a.e. in } Q$$

where

$$<f(t),v> = \int_\Gamma h(\cdot,t)v \qquad (3.17)$$

for any $v\in V$ and almost any $t\in[0,T]$. Here, we have set

$$F(u)(x,t) = -\psi(x,0)u(x,t) - (\partial_t\psi*u)(x,t) \qquad (3.18)$$
$$G(\vartheta)(x,t) = g(x,t,\vartheta(x,t)) + (\psi-\varphi)(x,0)\vartheta(x,t) + (\partial_t(\psi-\varphi)*\vartheta)(x,t) \qquad (3.19)$$

for almost all $(x,t)\in Q$. Using (B1)-(B2) and Young's inequality for convolution products, it is not difficult to check that F and G are Lipschitz continuous and causal operators from $L^2(0,T;H)$ to itself, whence (A5) is fulfilled.

On the other hand, the maximal monotone operator M defined by

$$Mv = (\mathcal{I}+\mathcal{H})(v), \quad v\in H \qquad (3.20)$$

clearly satisfies (A2) and, in particular, (2.1)-(2.2). Next, let us take $W = H^{3/4}(\Omega)$, so that (A1) holds, and specify the functions

$$J_1(v) = \frac{1}{2}a(v,v), \quad J_2(v) = \int_\Gamma \phi(v), \qquad v\in V. \qquad (3.21)$$

In view of (B3), the quadratic form a is continuous and symmetric. Therefore $A_1 = \partial J_1$ is a linear and bounded operator from V to V' which is given by

$$< A_1(v), w > = a(v, w) \quad \forall v,\, w \in V. \tag{3.22}$$

As far as $A_2 = \partial J_2$ is concerned, we can invoke, for instance, [2, Lemmas 5.1 and 5.2] and verify that

$$w \in A_2(z) \text{ if and only if } \langle w, v\rangle = \int_\Gamma \omega v \quad \forall v \in V,$$
$$\text{for some } \omega \in L^2(\Gamma) \text{ such that } \omega \in \partial\phi(z) \text{ a.e. in } \Gamma. \tag{3.23}$$

In addition, from (B4) it follows that (see, e.g., [2, Lemma 5.2]) there exists a positive constant C_5, depending only on c_3 and the surface measure of Γ, such that

$$|< w, v >| \le C_5 \left(1 + \|z|_\Gamma\|_{L^2(\Gamma)}\right) \|v|_\Gamma\|_{L^2(\Gamma)} \quad \forall z,\, v \in V,\ \forall w \in A_2(z). \tag{3.24}$$

Since the trace operator $v \mapsto v|_\Gamma$ is continuous from W to $L^2(\Gamma)$, by (3.24) we deduce that $A_2 = \partial J_2$ maps bounded sets of V into bounded sets of the dual space of W. Then, in order to conclude the verification of (A3), it remains to check (2.3). Note, however, that (2.3) is a direct consequence of (3.21), (3.7), and the fact that ϕ is bounded from below by an affine function (see, e.g., [3, Prop. 2.1, p. 51]). Hence, by recalling that $A = A_1 + A_2$, it turns out that assumption (A3) is completely satisfied.

Next, we introduce the operator

$$< B(t)v, w > = \sum_{j,m=1}^{N} \int_\Omega b_{jm}(\cdot, t) v_{x_j} w_{x_m} \quad \forall v, w \in V,\ \forall t \in [0, T]. \tag{3.25}$$

and use (B3) to infer that B fulfills (A4). Moreover, on account of (3.8), it is clear that

$$\text{the image of } v \in L^2(0, T; V) \text{ under } (B*) \text{ is } \beta * v \in L^2(0, T; V').$$

Finally, we observe that (B4), (B5), (3.17), (3.20), and (3.21) entail the validity of (A6) and (A7).

In conclusion, thanks to (3.16)-(3.23) and (3.25), we deduce that Problem **(S)** can be equivalently set as Problem **(P)**. Indeed, the solution component ξ in **(P)** satisfies $\xi = A_1\vartheta + \xi_2$ for some $\xi_2 \in A_2(\vartheta)$ almost everywhere in $(0, T)$, and η in **(S)** is exactly the boundary function corresponding to ξ_2 in (3.23). Thus, the $L^\infty(0, T; L^2(\Gamma))$ regularity of η follows from (2.4) and (3.24). Note also that $\chi \in L^\infty(Q)$ comes directly from (3.12), which actually implies that $0 \le \chi \le 1$ almost everywhere in Q. Then, Theorem 2.1 enables us to conclude the proof. □

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