数理解析研究所講究録
1104巻 1999年 151-152

CUSP OPENINGS IN COMPLEX HYPERBOLIC GEOMETRY

SADAYOSHI KOJIMA (小島定吉)

The space of marked n distinct points on the complex projective line $\mathbf{CP}^1$ up to projective transformations is called a configuration space and we denote it by $\mathcal{Q}$. It admits a structure of a complex manifold of dimension $n-3$, and has a long history for attracting many mathematicians, though we focused in the talk only on ones related with complex hyperbolic geometry.

Deligne and Mostow construct a family of equivariant maps of the universal cover of $\mathcal{Q}$ to the $(n-3)$-dimensional complex projective space with respect to the action of $\pi_1(\mathcal{Q})$ and the projective transformations in [3]. It is parameterized by the exponents of an integral representation of a several variable analogue of the hypergeometric function. The main focus of their paper is to discuss when the holonomy representation, which is shown to lie in $\mathrm{PU}(1,n-3) \subset \mathrm{PGL}_{n-2}(\mathbf{C})$ is discrete, and to find many complex hyperbolic lattices.

On the other hand, Thurston provides a different construction of complex hyperbolic structures on $\mathcal{Q}$ in [13] based on euclidean cone structures on $\mathbf{CP}^1$, each of which is assigned to a configuration via a generalized Schwarz-Christofell correspondence. It is parameterized by the cone angles. His approach re-discovers complex hyperbolic lattices found by Deligne and Mostow. Strictly speaking, Thurston constructed structures not on $\mathcal{Q}$ but rather on the quotient of $\mathcal{Q}$ by the action of remarking cone points with the same cone angles, and in fact he found more lattices.

Although the discovery of lattices has been emphasized as a common part of their papers, they both actually constructed the rooted families of incomplete complex hyperbolic structures on $\mathcal{Q}$ which provide lattices in particular cases. The first purpose of the talk was to confirm that their underlying families of complex hyperbolic structures on $\mathcal{Q}$ are the same.

Deligne and Mostow studied the family from a viewpoint of Mumford's compactification in [10]. On the other hand, Thurston studied their completion from a viewpoint of cone manifolds. However, neither papers have deformation theoretic viewpoints, though Kapovich and Millson pointed out such aspect in relation with the study of mechanical linkages in [5, 6]. The second purpose of the talk was to review their families as the deformations of complex hyperbolic cone structures on $\mathcal{Q}$ for small n, in view of the deformation theory for real hyperbolic cone 3-manifolds developed by [12, 2, 11, 4, 7, 1]. The study stays in very primitive stage still, but a few small, and we believe suggestive, observations in contrast with [9, 14] were presented.

We will discuss the details in [8].

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Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ohokayama, Meguro, Tokyo 152-8552 Japan

E-mail address: `sadayosi@is.titech.ac.jp`