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An Estimate on the Heat Kernel of Magnetic Schrödinger Operators and Uniformly Elliptic Operators with Non-negative Potentials

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Abstract

In this paper we show an estimate of the heat kernel to the Schrödinger operator with magnetic fields and to uniformly elliptic operators with non-negative potentials which belongs to the reverse Hölder class. We also give a weighted smoothing estimates for the semigroup generated by the operators above.

1 Introduction and Main Results

We consider the uniformly elliptic operator $L_E = -\nabla(A(x)\nabla)+V(x)$ with certain non-negative potential V and the Schrödinger operator $L_M = (i^{-1}\nabla - a(x))^2 + V(x)$ with a magnetic field $a(x) = (a_1(x), \dots, a_n(x))$, $n \geq 2$. We use the notation L_J for $J = E$ or $J = M$. The purpose of this paper is to give an estimate of the fundamental solution (or heat kernel) $\Gamma_J(x, t; y, s)$ to

$$(\partial_t + L_J)u(x, t) = 0, \quad (x, t) \in \mathbf{R}^n \times (0, \infty), \quad (1)$$

namely $\Gamma_J(x, t; y, s)$ satisfies

$$(\partial_t + L_J)\Gamma_J(x, t; y, s) = 0, \quad x \in \mathbf{R}^n, \quad t > s, \quad (2)$$

$$\lim_{t \rightarrow s} \Gamma_J(x, t; y, s) = \delta(x - y). \quad (3)$$

For the elliptic operator L_E , we assume the following conditions for $A(x) = (a_{ij}(x))$.

ASSUMPTION (A.1): $a_{ij}(x)$ is a real-valued measurable function and satisfies $a_{ij}(x) = a_{ji}(x)$ for every $i, j = 1, \dots, n$ and $x \in \mathbf{R}^n$.

ASSUMPTION (A.2): There exists a constant $\lambda > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi^i\xi^j \leq \lambda^{-1}|\xi|^2, \quad \xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n. \quad (4)$$

To state our assumptions on V and a , we prepare some notations. We say $U \in (RH)_\infty$ if $U \in L_{loc}^\infty(\mathbf{R}^n)$ and satisfies

$$\sup_{y \in B(x,r)} |U(y)| \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| dy, \quad (5)$$

and say $U \in (RH)_q$ if $U \in L_{loc}^q(\mathbf{R}^n)$ and satisfies

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)|^q dy \right)^{1/q} \leq C \frac{1}{|B(x,r)|} \int_{B(x,r)} |U(y)| dy, \quad (6)$$

for some constant C and for every $x \in \mathbf{R}^n$ and $r > 0$, respectively. We can define the function $m(x, U)$ for $U \in (RH)_q$ with $q > n/2$ as follows:

$$\frac{1}{m(x, U)} = \sup\{r > 0; \frac{r^2}{|B(x,r)|} \int_{B(x,r)} U(y) dy \leq 1\}. \quad (7)$$

We note that if there exist positive constants K_1 and K_2 such that $K_1 U_1(x) \leq U_2(x) \leq K_2 U_1(x)$, then it is easy to see that there exist positive constants K'_1 and K'_2 such that

$$K'_1 m(x, U_1) \leq m(x, U_2) \leq K'_2 m(x, U_1).$$

When $n \geq 3$, since it is known $U \in (RH)_{n/2}$ actually belongs to $(RH)_{n/2+\epsilon}$ for some $\epsilon > 0$, $m(x, U)$ can be defined for $U \in (RH)_{n/2}$ ([Sh1]). For other properties of the class $(RH)_q$, see, e.g., [KS]. We denote by $B(x) = (B_{jk}(x))$ the magnetic field defined by $B_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)$. We use the notation $m_J(x)$:

$$m_E(x) = m(x, V), \quad m_M(x) = m(x, |B| + V)$$

for the operator L_J , $J = E$ or M , respectively. We assume the following conditions for $V(x)$ and $a(x) = (a_1(x), \dots, a_n(x))$.

ASSUMPTION(V, a, B): For each $j = 1, \dots, n$, $a_j(x)$ is a real-valued $C^1(\mathbf{R}^n)$ -function, V is non-negative.

(i) For $n \geq 3$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_{n/2}, \quad |\nabla B(x)| \leq Cm(x, V + |B|)^3.$$

(ii) For $n = 2$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_q, \quad |\nabla B(x)| \leq Cm(x, V + |B|)^3$$

for some $q > 1$.

Remark 1 For $n = 2$, we may assume the condition (ii') instead of (ii) by employing Lemma 1 (b).

(ii') $V \in L_{loc}^\infty(\mathbf{R}^2)$, $B(x) \geq 0$ and that $m_J(x)$ satisfies

$$C_1 \frac{m_J(x)}{(1 + |x - y|m_J(x))^{k_0/(k_0+1)}} \leq m_J(y) \leq C_2(1 + |x - y|m_J(x))^{k_0} m_J(x) \quad (8)$$

for some positive constants C_1, C_2, k_0 and for every $x, y \in \mathbf{R}^2$, where $m_E(x) = \sqrt{V(x)}$ and $m_M(x) = \sqrt{V(x) + B(x)}$.

We remark that it is known that $m_J(x)$ satisfies (8) under the assumption (V, a, B) for $n \geq 3$ ([Sh1]) and even for $n = 2$ in the same way. We also note that if $|B| + V \in (RH)_\infty$, then it is easy to see that $|B(x)| + V(x) \leq Cm(x, |B| + V)^2$ holds. For example, the condition $|B| + V \in (RH)_\infty$ is satisfied for any $a_j(x) = Q_j(x)$, $V(x) = |P(x)|^\alpha$, where $P(x)$ and $Q_j(x)$, $j = 1, \dots, n$, are polynomials and α is a positive constant. In this case, under the assumption (V, a, B) (i) or (ii), we see that there exists a positive constant m_0 such that $m_J(x) \geq m_0$, although in general we cannot say $|B| + V$ is strictly positive for inhomogeneous polynomials. To state our main result, we introduce the notation:

$$\Gamma_{C_0}(x, t; y, s) = \frac{1}{(t - s)^{n/2}} \exp(-C_0 \frac{|x - y|^2}{t - s})$$

for some positive constant C_0 .

Theorem 1 (a) Suppose $A(x)$ and $V(x)$ satisfy the assumptions (A.1), (A.2) and $(V, 0, 0)$. Then, there exist positive constants α_0 and C_j ($j = 0, 1, 2$) such that

$$(0 \leq) \Gamma_E(x, t; y, s) \leq C_1 \exp\left(-C_2(1 + m_E(x)(t - s)^{1/2})^{\alpha_0/2}\right) \Gamma_{C_0}(x, t; y, s) \quad (9)$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

(b) Suppose $V(x)$ and $a(x)$ satisfy the assumption (V, a, B) . Then, there exist positive constants α_0 and C_j ($j = 0, 1, 2$) such that

$$|\Gamma_M(x, t; y, s)| \leq C_1 \exp\left(-C_2(1 + m_M(x)(t - s)^{1/2})^{\alpha_0/2}\right) \Gamma_{C_0}(x, t; y, s) \quad (10)$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

The number α_0 is actually defined by $\alpha_0 = 2/(k_0 + 1)$, where k_0 is the constant in (8). The exponent $\alpha_0/2$ would not be sharp. If we restrict for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, the following sharp estimate is known ([Ma], [Er1,2] for $n \geq 3$ and [LT] for $n = 2$):

$$|\Gamma_M(x, t; y, s)| \leq D_1 \exp(-D_2 B_0 t) \Gamma_{D_0}(x, t; y, s).$$

More detail informations on the constants D_j ($j = 0, 1, 2$) can be seen in those papers. By using the parabolic distance:

$$d_P((x, t), (y, s)) = \max(|x - y|, |t - s|^{1/2}),$$

we have the following decay estimate.

Corollary 1 (a) Under the same assumptions as in Theorem 1, there exist positive constants C_j ($j = 1, 2$) and C_0 such that

$$|\Gamma_J(x, t; y, s)| \leq C_1 \exp\left(-C_2(1 + m_J(x)d_P((x, t), (y, s)))^{2\alpha_0/(\alpha_0+4)}\right) \Gamma_{C_0}(x, t; y, s)$$

for $J = E$ and M , for every $x, y \in \mathbf{R}^n$ and $t > s > 0$.

(b) Under the same assumptions as in Theorem 1, for each $k > 0$ there exist positive constants C_k and C_0 such that

$$|\Gamma_J(x, t; y, s)| \leq \frac{C_k}{(1 + m_J(x)d_P((x, t), (y, s)))^k} \Gamma_{C_0}(x, t; y, s)$$

for $J = E$ and M .

Remark 2 Actually we can show the estimate in Theorem 1 for the operators $L_E = -\nabla(A(x,t)\nabla) + V(x,t)$ with time-dependent coefficients, if we assume the uniform ellipticity (4) of $A(x,t)$ and the existence of constants $C_j, j = 1, 2$, such that $C_1U(x) \leq V(x,t) \leq C_2U(x)$ and U satisfies the condition $(U, 0, 0)$. For the magnetic Schrödinger operator $L_M = (i^{-1}\nabla - a(x,t))^2 + V(x,t)$, the estimate in Theorem 1 still holds, if there exists positive constants $C_j, j = 1, \dots, 5$, such that $C_1U(x) \leq V(x,t) \leq C_2U(x)$, $C_3|B'(x)| \leq |B(x,t)| \leq C_4|B'(x)|$, and $|\nabla B(x,t)| \leq C_5m(x, |B'| + U)^3$, where $a(x,t)$ is C^1 and $B_{jk}(x,t) = \partial_j a(x,t) - \partial_k a_j(x,t)$ and $U(x)$ and $B'(x)$ satisfy the ASSUMPTION (U, a, B') (except $|\nabla B'(x)| \leq Cm_J(x)^3 (= Cm(x, |B'| + U)^3)$), and if the upper bound :

$$|\Gamma_M(x, t; y, s)| \leq C\Gamma_{C_0}(x, t; y, s)$$

holds for some constants C and C_0 .

Remark 3 In particular, Corollary 1 (b) yields

$$\begin{aligned} |\Gamma_J(x, t; y, s)| &\leq \frac{C_k}{(1 + m_J(x)|x - y|)^k(1 + m_J(x)|t - s|)^k} \Gamma_{C_0}(x, t; y, s) \\ &\leq \frac{C_k}{(1 + m_J(x)|x - y|)^k} \Gamma_{C_0}(x, t; y, s) \end{aligned} \quad (11)$$

for $J = E$ or M . Let $n \geq 3$. Then this implies

$$|\Gamma_J(x, y) \equiv \int_s^{+\infty} \Gamma_J(x, t; y, s) dt| \leq \frac{C_k}{(1 + m_J(x)|x - y|)^k |x - y|^{n-2}}$$

where $\Gamma_J(x, y)$ is the fundamental solution to $L_J u = 0$. This estimate for the elliptic operator was proved by Shen [Sh1,2]. Thus, Corollary 1 (b) is a generalization of his estimate.

Remark 4 Recently we are informed by Z. Shen that he obtained the following shape estimate [Sh3] for the elliptic operators: under the assumption $V \in (RH)_{n/2}$ for $n \geq 3$ and $V \in (RH)_q$ with $q > 1$ for $n = 2$,

$$C_1 \exp(-C_2 d(x, y)) |x - y|^{2-n} \leq \Gamma_E(x, y) \leq C_3 \exp(-C_4 d(x, y)) |x - y|^{2-n}$$

holds for some positive constants C_j ($j = 1, 2, 3, 4$), where $d(x, y)$ is defined by

$$d(x, y) = \inf_{\gamma} \int_0^1 m(\gamma(t), V) \left| \left(\frac{d\gamma}{dt} \right) (t) \right| dt.$$

Here the infimum is taken over all curves γ such that $\gamma(0) = x$ and $\gamma(1) = y$. Moreover, he gave the following estimate:

$$C_1(1 + m(x)|x - y|^{\alpha_0/2}) \leq d(x, y) \leq C_2(1 + m(x)|x - y|^{\beta_0})$$

for some positive constants $C_j (j = 1, 2)$ and β_0 . In particular, it follows

$$\Gamma_E(x, y) \leq C_5 \exp(-C_6(1 + m_E(x)|x - y|^{\alpha_0/2})|x - y|^{2-n})$$

for some positive constants C_5 and C_6 . We remark that this decay estimate also can be shown for the fundamental solution $\Gamma_M(x, y)$ to L_M in a similar way. On the other hand, it follows from Corollary 1 (a) a somewhat weaker decay estimate:

$$|\Gamma_J(x, y)| \leq C \exp(-C(1 + m_J(x)|x - y|^{2\alpha_0/(\alpha_0+4)})|x - y|^{2-n})$$

for $J = E$ or M . We do not know whether his sharp estimate can be generalized to heat kernel estimates or not.

We denote by e^{-tL_J} the semigroup generated by L_J . Here we also denote by L_J the self-adjoint operator determined from the form associated with L_J (see, e.g., [Si], [LS]). We obtain the following weighted smoothing estimate by using Corollary 1 (b).

Theorem 2 Assume the same assumptions as in Theorem 1. Let $J = E$ or M . Suppose $1 < p \leq q \leq +\infty$ and $1/p - 1/q < 1$ and put $\gamma = n(1/p - 1/q)$. Then for each $l \in [0, (n - \gamma)/2]$ there exists a constant C_l such that

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^q(\mathbf{R}^n)} \leq \frac{C_l}{t^{l+(\gamma/2)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0. \quad (12)$$

Corollary 2 Suppose the additional condition $|B| + V \in (RH)_\infty$. Then we have the following estimates:

$$\|(|B| + V)^l e^{-tL_J} f\|_{L^p(\mathbf{R}^n)} \leq \frac{C_l}{t^l} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0 \quad (13)$$

holds for $1 < p < +\infty$ and $l \in [0, n/2]$, and

$$\|(|B| + V)^l e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C_l}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0 \quad (14)$$

holds for $1 \leq p < +\infty$ and $l \in [0, n/(2p)]$. Here $1/p' = 1 - 1/p$ and C_l is a constant depending on l and p .

Corollary 2 is an easy consequence of Theorem 2 by using the inequality $(|B| + V)(x) \leq C m_J(x)^2$. Note that (14) for the case $l = 0$ is a classical result.

Theorem 1 yields a weighted smoothing estimate with an exponential decay in time.

Theorem 3 *Assume the same assumptions as in Theorem 1 and the additional assumption $m_J(x) \geq m_0 > 0$.*

(a) *Let $1 \leq p < +\infty$ and $l \in [0, n/(2p')]$. Then we have*

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq C \exp(-C(1 + m_0 t^{1/2})^{\frac{\alpha_0}{2}}) \frac{1}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0.$$

(b) *Let $1 \leq p \leq 2$ and $l \in [0, n/(2p')]$. Then we have*

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq C \exp(-C(1 + m_0^2 t)) \frac{1}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0.$$

Especially, for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, Theorem 3 (b) yields an exponential decay estimate in time:

$$\|e^{-tL_M} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C_1}{t^{n/2}} \exp(-C_2 B_0 t) \|f\|_{L^1(\mathbf{R}^n)}, \quad t > 0 \quad (15)$$

for some positive constant C_1 and C_2 , which is known (see, e.g., [Ma], [Er1,2], [Ue], [LT]). Indeed, in this case $m_M(x) \sim \sqrt{B_0}$ holds. Note that Theorem 3 (a) gives weaker decay rate $e^{-C\sqrt{B_0}t}$, since $k_0 = 0$ and $\alpha_0 = 2$. We also emphasize that Theorem 3 can be applied to any polynomial like magnetic field $B(x)$ which may be zero somewhere.

Definition 1 *We say $u(x, t)$ is a complex-valued weak solution to*

$$(\partial_t + L_M)u = 0 \quad \text{in } Q_r(x_0, t_0),$$

if $u \in L^\infty((t_0 - r^2, t_0); L^2(B(x_0, r); \mathbf{C})) \cap L^2((t_0 - r^2, t_0); H^1(B(x_0, r); \mathbf{C}))$ and satisfies

$$\begin{aligned} & \int_{B(x_0, r)} u(x, t) \overline{\phi(x, t)} dx - \int_{t_0 - r^2}^t \int_{B(x_0, r)} u(x, s) \partial_s \overline{\phi(x, s)} dx ds \\ & + \int_{t_0 - r^2}^t \int_{B(x_0, r)} \sum_{j=1}^n D_j^a u(x, s) \overline{D_j^a \phi(x, s)} dx ds \\ & + \int_{t_0 - r^2}^t \int_{B(x_0, r)} V(x) u(x, s) \overline{\phi(x, s)} dx ds = 0 \end{aligned} \quad (16)$$

for every $\phi \in \mathcal{C} \equiv \{\phi \in L^2((t_0 - r^2, t_0); H^1(B(x_0, r); \mathbf{C})); \partial_s \phi \in L^2((t_0 - r^2, t_0); L^2(B(x_0, r); \mathbf{C})), \phi(x, t_0 - r^2) = 0\}$, where $\bar{\phi}$ is the complex conjugate of ϕ .

Here, we used the notation $D_j^a = i^{-1} \partial_{x_j} - a_j(x)$ and

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbf{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0\}.$$

A real-valued weak solution u to $(\partial_t + L_E)u = 0$ in $Q_r(x_0, t_0)$ can be defined in a similar way. Our proof of Theorem 1 is based on the following subsolution estimate.

Theorem 4 *Let $u(x, t)$ be a weak solution to $\partial_t u + L_J u = 0$ in $Q_{2r}(x_0, t_0)$. Then there exists positive constants $C_j, j = 1, 2$, such that*

$$\sup_{(x,t) \in Q_{r/2}(x_0, t_0)} |u(x, t)| \leq C_1 \exp\left(-C_2(1 + rm_J(x_0))^{\alpha_0/2}\right) \left(\frac{1}{r^{n+2}} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt\right)^{1/2}. \quad (17)$$

Throughout this paper, we use the following notation: $D = i^{-1} \nabla - a$,

$$B(x_0, r) = \{y \in \mathbf{R}^n; |y - x_0| < r\}, \quad \langle A \nabla u, \nabla u \rangle = \sum_{j,k=1}^n a_{jk} \partial_{x_j} u \partial_{x_k} u,$$

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbf{R}^n \times (0, +\infty); |x - x_0| < r, t_0 - r^2 < t < t_0\}.$$

2 Proof of Theorem 4

We use the following inequalities.

Lemma 1 (a) ([Sh2]) *Suppose $n \geq 2$ and $V(x)$ and $a(x)$ satisfy the condition (V, a, B) . Then there exists a constant C_0 such that*

$$\int m(x, |B| + V)^2 |u|^2 dx \leq C_0 \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx$$

for $u \in C_0^\infty(\mathbf{R}^n; \mathbf{C})$.

(b) ([AHS]) *Suppose $n = 2, V \geq 0, V \in L_{loc}^\infty(\mathbf{R}^2), a \in C^1(\mathbf{R}^2)$, and $B(x) \geq 0$. Then the inequality*

$$\int (B(x) + V(x)) |u|^2 dx \leq \int |(i^{-1} \nabla - a(x))u|^2 + V(x)|u|^2 dx$$

holds for $u \in C_0^\infty(\mathbf{R}^n; \mathbf{C})$.

We also prepare the following Caccioppoli-type inequality.

Lemma 2 *Let $0 < \sigma < 1$. Let u be a weak solution to $(\partial_s + L_J)u = 0$ in $Q_{2r}(x_0, t_0)$ for $J = E$ or $J = M$. Then there exists a constant C such that*

$$\begin{aligned} \sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, \sigma r)} |u(x, t)|^2 dx + \int \int_{Q_{\sigma r}(x_0, t_0)} |(i^{-1}\nabla - a)u|^2 + V|u|^2 dx ds \\ \leq \frac{C}{(1 - \sigma)^2 r^2} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt. \end{aligned}$$

PROOF: Although the proof is standard, we give it here for the sake of completeness. We show the estimate for a weak solution u to $(\partial_t + L_E)u = 0$ in $Q_{2r}(x_0, t_0)$. Since we can show the estimate for a weak solution to $(\partial_t + L_M)u = 0$ in the similar way, we just mention some modifications we need at the end of this proof. Take functions $\chi(x) \in C_0^\infty(B(x_0, r))$ and $\eta(t) \in C^\infty(\mathbf{R}^1)$ satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) \equiv 1$ on $B(x_0, \sigma r)$ and $|\nabla \chi(x)| \leq C/(1 - \sigma)r$, and $0 \leq \eta(t) \leq 1$, $\eta(t) \equiv 1$ on $t \geq t_0 - (\sigma r)^2$, $\eta(t) \equiv 0$ on $t \leq t_0 - r^2$, $|\partial_t \eta(t)| \leq C/r^2(1 - \sigma^2)$. For the sake of simplicity, we also assume $\partial_t u \in L^2(Q_{2r}(x_0, t_0))$. Actually, we can remove this additional assumption by using the argument as in [AS]. Fix $t \in [t_0 - (\sigma r)^2, t_0]$. Multiplying $\eta^2(t)\chi^2(x)u(x, t)$ to the equation and integrating over $B(x_0, r) \times [t_0 - r^2, t]$, we have

$$\begin{aligned} & \frac{1}{2} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 dx \\ & + \int_{t_0 - r^2}^t \int_{B(x_0, r)} \langle A(x) \nabla u(x, s), \nabla u(x, s) \rangle \eta(s)^2 \chi(x)^2 dx ds \\ & + \int_{t_0 - r^2}^t \int_{B(x_0, r)} V(x) u(x, s)^2 \eta(s)^2 \chi(x)^2 dx ds \\ & = \int_{t_0 - r^2}^t \int_{B(x_0, r)} u(x, s)^2 \chi(x)^2 \eta(s) \partial_s \eta(s) dx ds \\ & - \int_{t_0 - r^2}^t \int_{B(x_0, r)} \langle A(x) \nabla u(x, s), \nabla(\chi^2(x)) \rangle \eta(s)^2 u(x, s) dx ds. \end{aligned} \quad (18)$$

Because of the ellipticity of $A(x)$ and the positivity of V , we obtain by (18)

$$\begin{aligned} & \sup_{t_0 - (\sigma r)^2 \leq t \leq t_0} \int_{B(x_0, r)} u(x, t)^2 \chi(x)^2 dx \\ & \leq \int \int_{Q_r(x_0, t_0)} u^2 |\partial_s \eta| dx ds \end{aligned} \quad (19)$$

$$\begin{aligned}
& + \int \int_{Q_r(x_0, t_0)} |\nabla u| |u| \eta^2 \chi |\nabla \chi| dx ds \\
& \leq \frac{C}{(1-\sigma)} \left\{ \frac{1}{r^2} \int \int_{Q_r(x_0, t_0)} u^2 dx ds + \int \int_{Q_r(x_0, t_0)} \chi^2 \eta^2 |\nabla u|^2 dx ds \right\}.
\end{aligned}$$

By using (18) again, we have

$$\begin{aligned}
& \lambda \int \int_{Q_r(x_0, t_0)} |\nabla u|^2 \chi^2 \eta^2 dx ds + \int \int_{Q_r(x_0, t_0)} V u^2 \chi^2 \eta^2 dx ds \\
& \leq \int \int_{Q_r(x_0, t_0)} \langle A \nabla u, \nabla u \rangle \partial_k u \chi^2 \eta^2 dx ds + \int \int_{Q_r(x_0, t_0)} V u^2 \chi^2 \eta^2 dx ds \\
& \leq \frac{C}{(1-\sigma)r^2} \int \int_{Q_r(x_0, t_0)} u^2 dx ds + \int \int_{Q_r(x_0, t_0)} |\nabla u| |\nabla \chi| \chi \eta^2 |u| dx ds \\
& \leq \frac{C}{(1-\sigma)r^2} \int \int_{Q_r(x_0, t_0)} u^2 dx ds + \frac{\lambda}{2} \int \int_{Q_r(x_0, t_0)} |\nabla u|^2 \chi^2 \eta^2 dx ds. \quad (20)
\end{aligned}$$

It follows

$$\begin{aligned}
& \frac{\lambda}{2} \int \int_{Q_r(x_0, t_0)} |\nabla u|^2 \chi^2 \eta^2 dx ds + \int \int_{Q_r(x_0, t_0)} V u^2 \chi^2 \eta^2 dx ds \\
& \leq \frac{C}{(1-\sigma)^2 r^2} \int \int_{Q_r(x_0, t_0)} u^2 dx ds. \quad (21)
\end{aligned}$$

(19) and (21) yield the desired result. For L_M , we can prove in a similar way by noting the following identities:

$$D_j^a(u\chi) = (D_j^a u)\chi + u(i^{-1}\nabla\chi), \quad \int D_j^a u \bar{v} dx = \int u \overline{D_j^a v} dx.$$

□

Proof of Theorem 3: Let $k \in \mathbf{N}$ and define p_j ($j = 1, 2, \dots, k+1$) by $p_j = 2/3 + ((j-1)/k)(1 - (2/3))$. Let $\chi_j(x) \in C_0^\infty(B(x_0, p_j r))$ and $\eta_j(t) \in C^\infty(\mathbf{R})$ be the functions satisfying $0 \leq \chi_j \leq 1$, $\chi_j(x) \equiv 1$ on $B(x_0, p_{j-1}r)$, $|\nabla \chi_j(x)| \leq Ck/r$, and $0 \leq \eta_j \leq 1$, $\eta_j(t) \equiv 1$ on $t \geq t_0 - (p_{j-1}r)^2$, $\eta_j(t) \equiv 0$ on $t \leq t_0 - (p_j r)^2$, $|\nabla \eta_j(t)| \leq Ck/r^2$. By Lemma 2 (see also (21)), we have

$$\begin{aligned}
& \int \int_{Q_{p_{j+1}r}(x_0, t_0)} \left(|(i^{-1}\nabla - a)u|^2 \chi_{j+1}^2 \eta_{j+1}^2 + V|u|^2 \chi_{j+1}^2 \eta_{j+1}^2 \right) dx ds \\
& \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} |u|^2 dx ds.
\end{aligned}$$

We write just $\chi = \chi_{j+1}$ and $\eta = \eta_{j+1}$, for simplicity. Since $|(i^{-1}\nabla - a)(u\eta\chi)|^2 \leq 2|(i^{-1}\nabla - a)u|^2\chi^2\eta^2 + 2u^2|\nabla\chi|^2\eta^2$, it follows that

$$\begin{aligned} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} \left(|(i^{-1}\nabla - a)(\eta\chi u)|^2\chi^2\eta^2 + V|u|^2\chi^2\eta^2 \right) dx ds \\ \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} |u|^2 dx ds \end{aligned}$$

for $j = 1, \dots, k$. By using Lemma 1, we obtain

$$\int_{t_0 - (p_{j+1}r)^2}^{t_0} \left(\int_{B(x_0, p_{j+1}r)} m_J(x)^2 |\eta\chi u|^2 dx \right) dt \leq \frac{Ck^2}{r^2} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} |u|^2 dx ds.$$

By using $m_J(x) \geq C(1 + p_{j+1}rm_J(x_0))^{-k_0/(1+k_0)}m_J(x_0)$ on $|x - x_0| < p_{j+1}r$ and noting $2/3 \leq p_{j+1} \leq 1$ (see (8) and the remark after that), we have

$$\begin{aligned} \int \int_{Q_{p_j r}(x_0, t_0)} |u|^2 dx dt &\leq \int_{t_0 - (p_{j+1}r)^2}^{t_0} \left(\int_{B(x_0, p_{j+1}r)} |\eta\chi u|^2 \right) dx dt \\ &\leq \frac{Ck^2}{r^2 m_J(x_0)^2} (1 + rm_J(x_0))^{2k_0/(k_0+1)} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} |u|^2 dx dt. \\ &\leq \frac{Ck^2}{(1 + rm_J(x_0))^{2/(k_0+1)}} \int \int_{Q_{p_{j+1}r}(x_0, t_0)} |u|^2 dx dt \end{aligned} \quad (22)$$

for each $j = 1, 2, \dots, k$. Here we used a trival inequality $\int \int_{Q_{p_j r}(x_0, t_0)} (\dots) dx dt \leq \int \int_{Q_{p_{j+1}r}(x_0, t_0)} (\dots) dx dt$ for the case $rm_J(x_0) \leq 1$. By this procedure, we can obtain the following: there exists a constant C such that for every $k \in \mathbf{N}$

$$\int \int_{Q_{2r/3}(x_0, t_0)} |u|^2 dx dt \leq \frac{C^k(k^2)^k}{(1 + rm_J(x_0))^{k\alpha_0}} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt, \quad (23)$$

where $\alpha_0 = 2/(k_0 + 1)$. Since $V(x) \geq 0$, the well-known subsolution estimate (see, e.g., [AS]) yields

$$\sup_{Q_{r/2}(x_0, t_0)} |u| \leq C \left(\frac{1}{r^{n+2}} \int \int_{Q_{2r/3}(x_0, t_0)} |u|^2 dx dt \right)^{1/2} \quad (24)$$

for some constant C . For the magnetic Schrödinger operator case, we have used Kato's inequality. Combining (23) and (24), we arrive at

$$\sup_{Q_{r/2}(x_0, t_0)} |u| \leq C \frac{C^{k/2}k^k}{(1 + rm_J(x_0))^{k\alpha_0/2}} \left(\frac{1}{r^{n+2}} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt \right)^{1/2} \quad (25)$$

for every $k \in \mathbf{N}$. Note that, by Stirling's formula $k^k \sim e^k k! (1/\sqrt{2\pi k})$ as $k \rightarrow \infty$, there exists a constant C_0 such that $k^k \leq C_0 e^k k!$ for $k \geq 1$. Multiplying $\epsilon^k/k!$ and taking the summation, we obtain

$$\begin{aligned} & \left(\sup_{Q_{r/2}(x_0, t_0)} |u| \right) \sum_{k=1}^{\infty} \frac{(\epsilon(1 + rm_J(x_0))^{\alpha_0/2})^k}{k!} \\ & \leq CC_0 \sum_{k=1}^{\infty} (\epsilon e \sqrt{C})^k \left(\frac{1}{r^{n+2}} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt \right)^{1/2}. \end{aligned}$$

Take $\epsilon > 0$ so that $\epsilon e \sqrt{C} < 1$. Then we have

$$\sup_{Q_{r/2}(x_0, t_0)} |u| \leq C \exp(-\epsilon(1 + rm_J(x_0))^{\alpha_0/2}) \left(\frac{1}{r^{n+2}} \int \int_{Q_r(x_0, t_0)} |u|^2 dx dt \right)^{1/2}.$$

This complete the proof. \square

3 Proof of Theorem 1

To show Theorem 1 we prove the following proposition.

Proposition 1 *Under the assumptions as in Theorem 1, there exist positive constants C_1 and C_2 such that*

$$|\Gamma_J(x, t; y, s)| \leq C_1 \exp(-C_2(1 + m_J(x)|t - s|^{1/2})^{\alpha_0/2}) \frac{1}{(t - s)^{n/2}} \quad (26)$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

PROOF: Assume $t - s \geq 2|y - x|^2$. Take $r^2 = |t - s|/8$. Then $u(z, u) = \Gamma_J(z, u; y, s)$ satisfies $(\partial_t + L_J)u(z, u) = 0$ in $Q_{2r}(x, t)$. Hence, by applying Theorem 4 to $u(z, u)$, we obtain

$$\begin{aligned} |\Gamma_J(x, t; y, s)| & \leq \sup_{Q_{r/2}(x, t)} |u| \\ & \leq C \exp(-C(1 + m_J(x)|t - s|^{1/2})^{\alpha_0/2}) \left(\frac{1}{r^{n+2}} \int \int_{Q_r(x, t)} |\Gamma(z, u; y, s)|^2 dz du \right)^{1/2}. \end{aligned}$$

By using the maximum principle for L_E and the diamagnetic inequality (see, e.g., [AS], [LS], [AHS]) for L_M , we have

$$|\Gamma_J(z, u; y, s)| \leq \frac{C}{(u - s)^{n/2}} \exp\left(-C \frac{|z - y|^2}{(u - s)}\right) \quad (27)$$

for some constant $C = C(n, \lambda)$. Since $t - s \geq u - s \geq 7r^2 \geq (7/8)(t - s)$ on $(z, u) \in Q_r(x, t)$, it is easy to see

$$\left(\frac{1}{r^{n+2}} \int \int_{Q_r(x, t)} |\Gamma_J(z, u; y, s)|^2 dz du \right)^{1/2} \leq \frac{C}{(t - s)^{n/2}}.$$

This yields the desired estimate. \square

Proof of Theorem 1: The positivity of $\Gamma_E(x, t; y, s)$ is a consequence of $V \geq 0$ and the maximum principle. Hence Proposition 1 and (27) imply

$$|\Gamma_J(x, t; y, s)|^2 \leq C \exp(-C(1 + |t - s|^{1/2} m_J(x))^{\alpha_0/2}) \frac{1}{(t - s)^n} \exp\left(-C \frac{|y - x|^2}{(t - s)}\right)$$

for some constant C . This concludes the desired estimate. \square

Proof of Corollary 1: Let $f(t) = (m_J(x)t^{1/2})^{\alpha_0/2} + |x - y|^2/t$ for $t > 0$. The, an easy computation shows that

$$\inf_{t>0} f(t) \geq C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0+4)}$$

for some positive constant C . Thus, we obtain

$$\begin{aligned} |\Gamma_J(x, t; y, s)| &\leq C \frac{1}{(t - s)^{n/2}} \exp(-Cf(t - s)) \exp\left(-\frac{C|x - y|^2}{t}\right) \\ &\times \exp(-C(m_J(x)(t - s)^{1/2})^{\alpha_0/2}) \\ &\leq C\Gamma_{C_0}(x, t; y, s) \exp(-C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0+4)}) \\ &\times \exp(-C(m_J(x)t^{1/2})^{\alpha_0/2}). \end{aligned}$$

This proves the part (a) since $2\alpha_0/(\alpha_0 + 4) \leq \alpha_0/2$. The part (b) is an easy consequence of the part (a). \square

4 Proof of Theorem 2, 3

To show Theorem 2, we prove the following inequality.

Theorem 5 *Let $\gamma \in [0, n)$. Then there exists a constant C such that*

$$|m_J(x)^{2l}(e^{-tL_J} f)(x)| \leq \frac{C}{t^{l+(\gamma/2)}} (M_\gamma |f|)(x) \quad (28)$$

holds for every $0 < l \leq (n - \gamma)/2$. Here $M_\gamma f$ is the fractional maximal function defined by

$$(M_\gamma f)(x) = \sup_{x \in B} \frac{1}{|B|^{1-\gamma/n}} \int_B |f| dy,$$

where the supremum is taken all balls B containing x .

Theorem 2 is a consequence of Theorem 5 and the following lemma (see, e.g., [St]).

Lemma 3 Let $0 \leq \gamma < n$. There exists a constant C such that

$$\|M_\gamma f\|_q \leq C \|f\|_p$$

for $1 < p \leq q \leq +\infty$ and $1/q = 1/p - \gamma/n$.

Proof of Theorem 5: Let $r = 1/m_J(x)$. By Corollary 1 (b) we have

$$\begin{aligned} & |m_J(x)^{2l} (e^{-tL_J} f)(x)| \\ & \leq C m_J(x)^{2l} \int \frac{|f(y)|}{(1 + m_J(x)|x - y|)^{ktn/2}} \exp\left(-\frac{C|x - y|^2}{t}\right) dy \\ & \leq \frac{C}{r^{2l} t^{n/2}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r < |x-y| \leq 2^j r\}} \frac{|f(y)|}{(1 + 2^{j-1})^k} \exp\left(-\frac{C(2^j r)^2}{t}\right) dy. \end{aligned} \quad (29)$$

By the assumption on l , we take $\alpha \geq 0$ such that $2\alpha = n - \gamma - 2l$. Put $C_\alpha = \sup_{s>0} s^\alpha e^{-s} < +\infty$ for $\alpha \geq 0$. Then the right hand side of (29) is dominated by

$$\begin{aligned} & C_\alpha \frac{C}{t^{n/2}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r < |x-y| \leq 2^j r\}} \frac{1}{r^{2l} (1 + 2^{j-1})^k} \left(\frac{C(2^{j-1}r)^2}{t}\right)^{-\alpha} |f(y)| dy \\ & \leq \frac{C_\alpha C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1 + 2^{j-1})^k (2^{j-1})^{2\alpha}} \left(\frac{1}{(2^j r)^{n-\gamma}} \int_{\{|x-y| \leq 2^j r\}} |f(y)| dy\right). \\ & \leq \frac{C_\alpha C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1 + 2^{j-1})^k (2^{j-1})^{2\alpha}} (M_\gamma |f|)(x). \end{aligned} \quad (30)$$

Now, since $n - \gamma - 2\alpha = 2l > 0$, by taking $k > 2l$ we have

$$\sum_{j=1}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1 + 2^{j-1})^k (2^{j-1})^{2\alpha}} \leq \sum_{j=1}^{+\infty} \frac{C}{2^{j(k-2l)}} < +\infty,$$

and

$$\sum_{j=-\infty}^0 \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^{j-1})^{2\alpha}} \leq \sum_{j=-\infty}^0 C(2^j)^{2l} < +\infty.$$

Thus, we obtain the desired result. \square

Proof of Theorem 3: First, the estimate for the case $l = 0$ and $p = 1$ is classical except the exponential factor in time. Under the assumption, by Corollary 1 (a) we have

$$\begin{aligned} |\Gamma_J(x, t; y, s)| &\leq C\Gamma_{C_0}(x, t; y, s) \exp(-C(1+m_J(x)|x-y|)^{2\alpha_0/(\alpha_0+4)}) \\ &\times \exp(-C(1+m_0 t^{1/2})^{\alpha_0/2}) \end{aligned} \quad (31)$$

for some positive constants C and C_0 . Then by using this estimate we can prove the part (a) of Theorem 3 in a similar way as in the proof of Theorem 2. To show the part (b), we use the semigroup property and Theorem 2 and get

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C}{t^{l+(n/4)}} \|e^{-(2/3)tL_J} f\|_{L^2(\mathbf{R}^n)}$$

for some constant C . Note that under the assumption $m_J(x) \geq m_0$, Lemma 1 yields $\inf \sigma(L_J) \geq Cm_0^2$ for some positive constant C . Here $\sigma(L_J)$ is the spectrum of the operator L_J . So, we have

$$\|e^{-(1/3)tL_J} g\|_{L^2(\mathbf{R}^n)} \leq e^{-Cm_0^2 t} \|g\|_{L^2(\mathbf{R}^n)}.$$

Using this estimate, we obtain

$$\begin{aligned} \|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} &\leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \|e^{-(1/3)tL_J} f\|_{L^2(\mathbf{R}^n)} \\ &\leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \frac{C}{t^{n/2(1/p-1/2)}} \|f\|_{L^p(\mathbf{R}^n)}. \end{aligned}$$

In the last inequality, we used $p \leq 2$ and Theorem 2.

\square

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