Title	Integral transforms for $ \mathcal{D}$-modules and homogeneous manifolds (Complex Analysis and Microlocal Analysis)
Author(s)	Marastoni, Corrado
Citation	数理解析研究所講究録 (1999), 1090: 1-9
Issue Date	1999-04
URL	http://hdl.handle.net/2433/62884
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1090 巻 1999 年 1-9

Integral transforms for $\mathcal{D}$-modules and homogeneous manifolds

Corrado Marastoni

1 Integral transforms, sheaves, $\mathcal{D}$-modules

Any problem of integral geometry has aspects of geometric nature (e.g. the support of the transform of a datum) and analytic nature (e.g. the differential equations describing the transform of some class of data). The idea of the approach by sheaves and $\mathcal{D}$-modules (see [8], [4], [9]) is to separate these problems in the calculations of the transform of a *constructible sheaf* (geometry) and of a *coherent $\mathcal{D}$-module* (analysis).

Complex integral transforms and real submanifolds. Since we use the theory of $\mathcal{D}$-modules, our framework will be complex, and the real transforms will be read by means of $\mathbf{R}$-constructible sheaves associated to real submanifolds (usually, locally constant sheaves of rank one). Let us explain this point a little more. Let X be a complex analytic manifold with structure sheaf $\mathcal{O}_X$ and $X^{\mathbf{R}}$ the underlying real analytic manifold: then, the functors $\cdot \otimes \mathcal{O}_X$, $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$, $Thom(\cdot, \mathcal{O}_X)$ and $R\mathcal{H}om(\cdot, \mathcal{O}_X)$ (see [8], [9]) associate a $\mathcal{D}_X$-module to any $\mathbf{R}$-constructible sheaf on $X^{\mathbf{R}}$. In particular, let M be a real analytic submanifold of $X^{\mathbf{R}}$ such that X is a complexification of M; then, denoting by $j : M \to X$ the closed embedding and by $(\cdot)^* = R\mathcal{H}om(\cdot, \mathbf{C}_X)$ the duality functor for sheaves, one has $\mathbf{C}_M \otimes \mathcal{O}_X \simeq j_!\mathcal{A}_M$ (analytic functions on M), $\mathbf{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq j_!\mathcal{C}^\infty_M$ (smooth functions), $Thom(\mathbf{C}^*_M, \mathcal{O}_X) \simeq j_!\mathcal{D}b_M$ (Schwartz's distributions) and $R\mathcal{H}om(\mathbf{C}^*_M, \mathcal{O}_X) \simeq H^{d^{\mathbf{R}}_M}_M(\mathcal{O}_X) \otimes or_{M|X} \simeq j_!\mathcal{B}_M$ (Sato's hyperfunctions).

The general integral transform. Let X and Y be complex analytic manifolds, q_j ($j = 1, 2$) the projections of $X \times Y$ on X and Y. Roughly speaking, the choice of a function (*kernel*) $k(x, y)$ on $X \times Y$ determines an integral transform from data (e.g. functions, cohomology classes) on X to data on Y by the law $(f \circ k)(y) := \int_{q_2} k(x,y)f(x)\,dx$, where dx is some volume element on X. Formally, this can be accomplished also in the categories of sheaves or $\mathcal{D}$-modules, where the pull-back of f by q_1 becomes the inverse image by q_1, the product by k the tensor product and the integration along q_2 the proper direct image by q_2.

More precisely, let $\mathbf{D}^b(\mathbf{C}_X)$ (resp. $\mathbf{D}^b(\mathcal{D}_X)$) be the derived category of sheaves of $\mathbf{C}$-vector spaces (resp. left $\mathcal{D}$-modules) on X, i.e. the complexes with bounded cohomology modulo quasi-isomorphisms. Any *kernels* $K \in \mathbf{D}^b(\mathbf{C}_{X\times Y})$ and $\mathcal{K} \in \mathbf{D}^b(\mathcal{D}_{X\times Y})$) define integral transforms by means of the following functors:

$$\begin{aligned} \cdot \circ K : \mathbf{D}^b(\mathbf{C}_X) \to \mathbf{D}^b(\mathbf{C}_Y), &\quad F \circ K = Rq_{2!}(K \otimes q_1{}^{-1}F), \\ \cdot \,\underline{\circ}\, \mathcal{K} : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_Y), &\quad \mathcal{M} \,\underline{\circ}\, \mathcal{K} = \underline{q_2}_!(\mathcal{K} \otimes_{\mathcal{O}_{X\times Y}} \underline{q_1}^{-1}\mathcal{M}), \end{aligned}$$

where $\underline{q_2}_!$ and $\underline{q_1}^{-1}$ are the direct and inverse images in the sense of $\mathcal{D}$-modules. The functor $K \circ \cdot : \mathbf{D}^b(\mathbf{C}_Y) \to \mathbf{D}^b(\mathbf{C}_X)$ is similarly defined.

A typical situation is when $\mathcal{K}$ is a regular holonomic $\mathcal{D}_{X\times Y}$-module and $K = R\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{K}, \mathcal{O}_{X\times Y})$ (i.e. the complex of holomorphic solutions of $\mathcal{K}$): by the Riemann-Hilbert correspondence in Kashiwara's formulation, K is a perverse sheaf and $\mathcal{K} \simeq \mathcal{T}hom(K, \mathcal{O}_{X\times Y})$. For example, we have the *geometric correspondences* (see [4]): let S be a smooth complex submanifold of $X\times Y$ and let $\mathcal{K} = \mathcal{B}_S$ (the holomorphic hyperfunctions along S). The Penrose transform (see [6]) is an example. In this case, one has $K \simeq \mathbf{C}_S[-cod^{\mathbf{C}}_{X\times Y} S]$. If one considers the double fibration (where f and g are the projections)

$$X \xleftarrow{\;f\;} S \xrightarrow{\;g\;} Y,$$

then it is easy to verify that $\cdot \circ \mathbf{C}_S = Rg_! f^{-1}(\cdot)$ and $\cdot \,\underline{\circ}\, \mathcal{B}_S = \underline{g}_!\underline{f}^{-1}(\cdot)$.

Adjunction formulas. The arriving point are the *adjunction formulas*, where a problem of integral geometry is divided into the problems of calculating the transforms of a *sheaf on* Y and a $\mathcal{D}$*-module on* X. For simplicity, we suppose the manifolds to be compact.

Proposition 1. ([4], [9]) *Let X and Y be compact complex analytic manifolds, $\mathcal{K}$ a regular holonomic $\mathcal{D}_{X\times Y}$-module and $K = R\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{K}, \mathcal{O}_{X\times Y})$. Assume that* $\mathrm{char}(\mathcal{K}) \cap (T^*X \times T^*_Y Y) \subset T^*_{X\times Y}(X\times Y)$. *Then, for any $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X)$ and $H \in \mathbf{D}^b(\mathbf{C}_Y)$ one has*

$$\begin{aligned} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, (K \circ H) \otimes \mathcal{O}_X) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M} \,\underline{\circ}\, \mathcal{K}, H \otimes \mathcal{O}_Y)[-d^{\mathbf{C}}_X], \\ \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om((K \circ H)^*, \mathcal{O}_X)) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M} \,\underline{\circ}\, \mathcal{K}, R\mathcal{H}om(H^*, \mathcal{O}_Y))[-d^{\mathbf{C}}_X]. \end{aligned}$$

Moreover, similar formulas hold when H has $\mathbf{R}$-constructible cohomology if one replaces $\otimes$ by $\overset{w}{\otimes}$ and $R\mathcal{H}om$ by $\mathcal{T}hom$.

In particular, we are interested in the following case (see [4]). Let $\mathcal{F}$ a holomorphic line bundle on X and $\mathcal{F}^*$. Taking $\mathcal{M} = \mathcal{D}\mathcal{F}^* = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}^*$, we get

$$\mathrm{R}\Gamma(X, (K \circ H) \otimes \mathcal{F}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}, H \otimes \mathcal{O}_Y)[-d_X^{\mathbf{C}}], \quad (1)$$

$$\mathrm{RHom}((K \circ H)^*, \mathcal{F}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}, R\mathcal{H}om(H^*, \mathcal{O}_Y))[-d_X^{\mathbf{C}}]. \quad (2)$$

Hence, (a) we shall compute the $\mathcal{D}$-module transform $\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}$, and then (b) we shall make different choices of H in order to obtain various applications.

Remark 1. Let p_j $(j = 1, 2)$ be the projections of $T^*(X \times Y)$ on T^*X and T^*Y respectively, and denote by p_j^a the composition with the antipodal map. Assuming, as above, the "non-characteristicity condition" $\mathrm{char}(\mathcal{K}) \cap (T^*X \times T_Y^*Y) \subset T_{X\times Y}^*(X \times Y)$, one has $\mathrm{char}(\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}) \subset p_2^a \mathrm{char}(\mathcal{K})$. Therefore, it is important to study the "microlocal correspondence" $T^*X \leftarrow \mathrm{char}(\mathcal{K}) \rightarrow T^*Y$ in order to get informations on the transform $\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}$.

2 Generalized flag manifolds and relations to representation theory

We specialize the preceding discussion to the case of compact homogeneous manifolds. Let G be a complex semisimple Lie group, P and Q two parabolic subgroups containing a same Borel subgroup. Let $X = G/P$ and $Y = G/Q$ be the corresponding compact homogeneous manifolds. The diagonal G-action on $X \times Y$ has a finite number of orbits, and the only closed one is $S = G/(P \cap Q)$, which is again a compact homogeneous manifold of G. Let $\mathcal{K}$ be a G-equivariant regular holonomic $\mathcal{D}_{X\times Y}$-module (e.g. the one associated to one of these orbits) and $\mathcal{F}$ be a G-equivariant holomorphic line bundle on X: then $\mathcal{D}\mathcal{F}^*$ (resp. $\mathcal{D}\mathcal{F}^* \underline{\circ} \mathcal{K}$) is a quasi G-equivariant $\mathcal{D}_X$- (resp. $\mathcal{D}_Y$-) module (we refer e.g. to [10] for all these notions).

Let G_0 be a real form of G, and let G_0 act on X and Y by restricting the G-action. Then, if H is a G_0-equivariant sheaf (e.g. we shall consider locally constant sheaves of rank one on the closed G_0-orbit in Y), so are $K \circ H$ and the duals, and the formulas (1) and (2) may be interpreted as isomorphisms in the derived category of representations of G_0.

3 The case of Grassmannians

Let $W \simeq \mathbf{C}^n$ and $G = SL_n(\mathbf{C})$. For $1 \leq p \leq n-1$, the subgroup P_p of matrices in G with the left bottom $(n-p) \times p$ block equal to zero is the "standard pth" maximal parabolic subgroup of G, and the quotient $X = G/P_p$ is naturally identified to the Grassmann manifold of p-dimensional subspaces of W. Recall that X is a compact manifold of complex dimension $p(n-p)$. The homogeneous action of G on X yields the following natural identification:

$$T^*X \simeq \{(x;\alpha) : x \in X,\ \alpha \in \mathrm{Hom}_{\mathbf{C}}(\tfrac{W}{x}, x)\}.$$

Let $1 \leq p \neq q \leq n-1$, $X = G/P_p$ and $Y = G/P_q$; assume for simplicity $p < q \leq n-p$. The diagonal G-action on $X \times Y$ has orbits

$$S_j = \{(x,y) \in X \times Y : \dim_{\mathbf{C}}(x \cap y) = j\} \qquad (j = 0, \ldots, p).$$

The closed orbit is $S_p \simeq G/(P_p \cap P_q)$ (the *flag manifold* of type (p,q) in W), S_0 is the open generic orbit and the other S_j's are smooth locally closed submanifolds. Again, for $1 \leq j \leq p$ one has the following useful identifications:

$$\begin{aligned}
T^*_{S_j}(X \times Y) &\simeq \{(x,y;\gamma) : (x,y) \in X \times Y,\ \gamma \in \mathrm{Hom}_{\mathbf{C}}(\tfrac{W}{x+y}, x \cap y)\},\\
p_1(x,y;\gamma) &= (x; \tfrac{W}{x} \xrightarrow{\pi_x} \tfrac{W}{x+y} \xrightarrow{\gamma} x \cap y \xrightarrow{i_x} x),\\
p_2^a(x,y;\gamma) &= (y; \tfrac{W}{y} \xrightarrow{\pi_y} \tfrac{W}{x+y} \xrightarrow{\gamma} x \cap y \xrightarrow{i_y} y).
\end{aligned}$$

where π and i are the natural maps.

The holomorphic line bundles on X are parametrized (up to isomorphisms) by $\lambda \in \mathbf{Z}$, and we shall denote by $\mathcal{O}_X(\lambda)$ the $-\lambda$th holomorphic tensor power of the determinant of the tautological vector bundle on X. In other words, let $F_p(W) = \{v = (v_1, \ldots, v_p) \in W^p : v_1 \wedge \cdots \wedge v_p \neq 0\}$ (the manifold of *p-frames* in W, an open dense subset of W^p) and $\pi : F_p(W) \to X$ the natural $GL_p(\mathbf{C})$-bundle assigning to any $v = (v_1, \ldots, v_p) \in F_p(W)$ the p-subspace of W spanned by the v_j's: then, for any open subset $U \subset X$ one has

$$\Gamma(U; \mathcal{O}_X(\lambda)) = \{\phi \in \Gamma(\pi^{-1}(U); \mathcal{O}_{F_p(W)}) : \phi(vA) = (\det A)^\lambda \phi(v)\ \forall A \in GL_P(\mathbf{C})\}.$$

We will write $\mathcal{D}_X(\lambda) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lambda)$ for short.

4 Applications

We announce results in two different applications.

4.1 The Grassmann duality ([11])

In the above notations, let $W \simeq \mathbf{C}^n$, $G = SL_n(\mathbf{C})$, $X = G/P_p$, $Y = G/P_{n-p}$ (we assume $p \leq n/2$), $\Omega = S_0$ and $S = (X \times Y)\setminus\Omega$. We consider the integral transform from X to Y given by $K = \mathbf{C}_\Omega$ and $\mathcal{K} = \mathcal{B}_\Omega = \mathcal{T}hom(\mathbf{C}_\Omega, \mathcal{O}_{X\times Y})$, i.e. the sheaf of meromorphic functions on $X \times Y$ with poles on S. (This choice generalizes the *projective duality* (see [5]), which is obtained for $p = 1$.) The nice geometric properties of the correspondence (e.g. for any $y \in Y$ the"slices" $\Omega_y = \{x \in X : (x,y) \in \Omega\}$ are affine charts of X) allow us to prove that :

Theorem 1a. *The functor* $\cdot \circ \mathbf{C}_\Omega : \mathbf{D}^b(\mathbf{C}_X) \to \mathbf{D}^b(\mathbf{C}_Y)$ *is an equivalence of categories preserving the objects with* $\mathbf{R}$- *or* $\mathbf{C}$-*constructible cohomologies; similarly, the functor* $\cdot \underline{\circ} \mathcal{B}_\Omega : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_Y)$ *is an equivalence of categories preserving the objects with good coherent or regular holonomic cohomologies.*

The closed singular manifold S is a non-smooth (if $p > 1$) hypersurface of $X \times Y$, Whitney-stratified by $S = \bigcup_{j=1}^p S_j$. The group G acts prehomogeneously on $X \times Y$ with singular locus S, and this action is locally isomorphic to that of $GL_p(\mathbf{C})$ on $M_p(\mathbf{C})$ whose semi-invariant is $f : M_p(\mathbf{C}) \to \mathbf{C}$, $f(a) = \det(a)$ with b-function $b(s) = (s+1)\cdots(s+p)$. This is a regular prehomogeneous vector space, and hence we get $\mathrm{char}(\mathcal{B}_\Omega) = T^*_{X\times Y}(X \times Y) \cup \bigcup_{j=1}^p T^*_{S_j}(X \times Y)$. From the above identifications, it is then easy to check that *the microlocal correspondence* $T^*X \leftarrow \mathrm{char}(\mathcal{B}_\Omega) \to T^*Y$ *induces a contact transformation between two open dense subsets* $U_X \subset \dot{T}^*X$ *and* $U_Y \subset \dot{T}^*Y$, whose graph Λ is contained in $\dot{T}^*_{S_p}(X \times Y)$, and moreover ${p_1}^{-1}(U_X) = p_2^{a-1}(U_Y) = \Lambda$. Using this fact and Theorem 1a, we obtain the following result:

Theorem 1b. *Let* $\lambda^* = -n-\lambda$*: then* $\mathcal{D}_X(-\lambda) \underline{\circ} \mathcal{B}_\Omega \simeq \mathcal{D}_Y(-\lambda^*)$ *if* $b(\lambda^*-\nu) \neq 0$ *for any* $\nu = 1, 2, \ldots$, *i.e. if* $\lambda \geq -n+p$.

Applying Theorem 1b to (1) and (2) we get the following isomorphisms for any $-n+p \leq \lambda \leq -p$ and any $H \in \mathbf{D}^b(\mathbf{C}_X)$:

$$\begin{aligned} \mathrm{R}\Gamma(X; H \otimes \mathcal{O}_X(\lambda)) &\simeq \mathrm{R}\Gamma(Y; (H \circ \mathbf{C}_\Omega) \otimes \mathcal{O}_Y(\lambda^*))[N], \\ \mathrm{R}\Gamma(X; R\mathcal{H}om(H, \mathcal{O}_X(\lambda))) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om(H \circ \mathbf{C}_\Omega, \mathcal{O}_Y(\lambda^*)))[-N], \end{aligned}$$

(where $N = p(n-p)$) and similarly for $\otimes$ and $R\mathcal{H}om$ replaced by $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ when H has $\mathbf{R}$-constructible cohomology. Hence, we are left with the choice of H and the calculation of $H \circ \mathbf{C}_\Omega$. (Using the simmetry of the transform, here we have written the formulas with H a sheaf on X rather than on Y.)

Example 1. Let Q be a hermitian form of signature $(p, n-p)$ on $W \simeq \mathbf{C}^n$, and let $G_0 = SU_{p,n-p}(Q)$ be the corresponding real form of G. The G_0-orbits in X are $U'_{i,j} = \{x \in X : Q|_x \text{ has signature } (i,j)\}$ for $0 \leq i+j \leq p$ (the only closed orbit is $U'_{0,0}$, i.e. the Q-isotropic p-subspaces, and the open orbits are $U'_{i,j}$ with $i+j = p$.). Similarly, the G_0-orbits in Y are $U''_{i,j} = \{y \in Y : Q|_y \text{ has signature } (i,j)\}$ for $0 \leq i \leq p$, $j \geq n-2p$ and $i+j \leq n-p$. Let $y_0 \in U'' = U''_{0,n-p}$, and let $E'_0 = \{x \in X : x \cap y_0 = 0\} \simeq \mathbf{C}^N$: then $U' = U'_{p,0}$ is a relatively compact open subset of E_0; similarly, fixed $x_0 \in U'$, U'' is a relatively compact open subset of the affine chart $E''_0 = \{y \in Y : x_0 \cap y = 0\} \simeq \mathbf{C}^N$. Let us consider the closure $\overline{U'} = \bigcup_{j=0}^p U'_{j,0}$, and choose $H = \mathbf{C}_{\overline{U'}}$: then it is possible to prove that $\mathbf{C}_{\overline{U'}} \circ \mathbf{C}_\Omega \simeq \mathbf{C}_{U''}$ and then from the above adjunction formulas we get

$$\mathrm{R}\Gamma(\overline{U'}; \mathcal{O}_{E'_0}) \simeq \mathrm{R}\Gamma_c(U''; \mathcal{O}_{E''_0})[N], \quad \mathrm{R}\Gamma_{\overline{U'}}(E'_0; \mathcal{O}_{E'_0}) \simeq \mathrm{R}\Gamma(U''; \mathcal{O}_{E''_0})[-N]$$

where all complexes are concentrated in degree zero.

4.2 The generalized Radon-Penrose transform ([3])

Let $W \simeq \mathbf{C}^{n+1}$, $G = SL_{n+1}(\mathbf{C})$, $X = G/P_1$, $Y = G/P_{k+1}$ (with $1 \leq k \leq n-2$) and $S = S_1$. Note that X is a n-dimensional complex projective space and S is the flag manifold of type $(1, k+1)$ in W; one has $\dim_{\mathbf{C}} X = n$, $\dim_{\mathbf{C}} Y = (k+1)(n-k)$ and $\dim_{\mathbf{C}} S = n + k(n-k)$. We consider the integral transform from X to Y given by $K = \mathbf{C}_S[-(n-k)]$ and $\mathcal{K} = \mathcal{B}_S$. (This is a natural generalization of *Penrose's twistors correspondence* (see [6]), which is obtained for $n = 3$ and $k = 1$.) We have $\mathrm{char}(\mathcal{B}_S) = \Lambda = T^*_S(X \times Y)$, and thus let us consider the microlocal correspondence $T^*X \leftarrow \Lambda \rightarrow T^*Y$: it is easy to check that $p_1|_{\dot\Lambda}$ is smooth and surjective and $p_2^a|_{\dot\Lambda}$ is a closed embedding identifying $\dot\Lambda$ to a smooth regular involutive submanifold $V \subset \dot{T}^*Y$ (in fact, it is $V \simeq \{(y;\beta) : y \in Y,\ \beta \in \mathrm{Hom}_{\mathbf{C}}(\frac{W}{y}, y),\ \mathrm{rank}(\beta) = 1\})$, which implies that *the correspondence induces microlocally a contact transformation with holomorphic parameters.* Using the theory of [4], we prove that:

Theorem 2a. *$\mathcal{D}_X(-\lambda) \underline{\circ} \mathcal{B}_S$ is concentrated in degree zero if and only if $\lambda < 0$, and $H^0(\mathcal{D}_X(-\lambda) \underline{\circ} \mathcal{B}_S)$ is a $\mathcal{D}_Y$-module with simple characteristic along V.*

For any $\lambda \in \mathbf{Z}$ we introduce a pair of G-equivariant holomorphic vector bundles $\mathcal{H}_\lambda$ and $\widetilde{\mathcal{H}}_\lambda$ on Y, and a G-invariant differential operator (the *ultra-hyperbolic system*) P_λ acting between them. The description of these objects, that will be given in detail in [3], depends upon the sign of $\lambda^* = -k - 1 - \lambda$ (*positive, null* and *negative helicity cases* in Penrose's terminology [6]): it can be partially found e.g. in [2, Ex. 9.7.1] and, in a real version, in [7].

Let $\mathcal{N}_{P_\lambda}$ be the coherent $\mathcal{D}_Y$-module associated to the differential operator P_λ, i.e. $\mathcal{N}_{P_\lambda}$ is defined by the exact sequence of $\mathcal{D}_Y$-modules (where $\mathcal{D}\mathcal{H}_\lambda^* := \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}_\lambda^*$ and P_λ^* is the transpose to P_λ):

$$\mathcal{D}\widetilde{\mathcal{H}}_\lambda{}^* \xrightarrow{P_\lambda^*} \mathcal{D}\mathcal{H}_\lambda{}^* \longrightarrow \mathcal{N}_{P_\lambda} \longrightarrow 0.$$

The $\mathcal{D}_Y$-module $\mathcal{N}_{P_\lambda}$ has simple characteristic along V, and we prove that:

Theorem 2b. *For any $\lambda < 0$, $\mathcal{D}_X(-\lambda) \underline{\circ} \mathcal{B}_S$ is isomorphic to $\mathcal{N}_{P_\lambda}$.*

Again, the application of Theorem 2b to (1) and (2) yields the following isomorphisms for any $\lambda < 0$ and any $H \in \mathbf{D}^b(\mathbf{C}_Y)$:

$$\begin{aligned} \mathrm{R}\Gamma(X, (\mathbf{C}_S \circ H) \otimes \mathcal{O}_X(\lambda)) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{N}_{P_\lambda}, H \otimes \mathcal{O}_Y)[-k], \\ \mathrm{RHom}((\mathbf{C}_S \circ H)^*, \mathcal{O}_X(\lambda)) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{N}_{P_\lambda}, R\mathcal{H}om(H^*, \mathcal{O}_Y))[-k] \end{aligned}$$

and similarly for $\otimes$ and $R\mathcal{H}om$ replaced by $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ when H has $\mathbf{R}$-constructible cohomology.

If we choose H to be a locally constant sheaf of rank one on the closed orbit of some real form G_0 of G in Y, we can recover and improve many known results of real integral geometry. We give two hints in this direction (these results will appear in [3]).

Example 2. Let $W_{\mathbf{R}}$ be a $(n+1)$-dimensional real subspace of W such that $W \simeq \mathbf{C} \otimes_{\mathbf{R}} W_{\mathbf{R}}$, and let $G_0 = SL_{n+1}(\mathbf{R})$ be the corresponding real form of G. Assuming for simplicity that $k+1 \leq (n+1)/2$, the G_0-orbits in Y are $N_j = \{y \in Y : \dim_{\mathbf{R}}(y \cap W_{\mathbf{R}}) = j\}$ $(j = 0, \ldots, k+1)$, and $N = N_{k+1}$ is naturally identified to the real Grassmann manifold of $(k+1)$-subspaces of $W_{\mathbf{R}}$. Similarly, the G_0-orbits in X are $M_i = \{x \in X : \dim_{\mathbf{R}}(x \cap W_{\mathbf{R}}) = i\}$ $(i = 0, 1)$, and $M = M_1$ is naturally identified to the real projective space of $W_{\mathbf{R}}$. It is known that N (in particular, M) is not simply connected: namely, one has $\pi_1(N) \simeq \mathbf{Z}/2\mathbf{Z}$. We denote by $\mathbf{C}_N(\epsilon)$ $(\epsilon = 0, 1)$ the two distinct locally constant sheaves on N, with the convention that $\mathbf{C}_N(0) = \mathbf{C}_N$. For example, for $\epsilon = 1$ we recover and improve the results of [7], whereas for $\epsilon = 0$ the results should be new.

Example 3. Let $1 \leq k \leq q \leq n-1$, Q a hermitian form on W of signature $(q+1, n-q)$, and let $G_0 = SU_{q+1,n-q}(Q)$ be the associated real form of G. Assuming for simplicity that $q+1 \leq (n+1)/2$, the G_0-orbits in Y are $N_{i,j} = \{y \in Y : Q|_y \text{ has signature } (i,j)\}$ for $0 \leq i+j \leq k+1$. The closed orbit is $N = N_{0,0}$, the Q-isotropic $(k+1)$-subspaces of W: one can prove that N is a generic real submanifold of Y of dimension $(k+1)(2n-3k-1)$, simply connected if $k+q+1 < n$ and affine if $k = q$. Similarly, the G_0-orbits in X are $M_{0,0}$, $M_{1,0}$ and $M_{0,1}$; the closed orbit $M = M_{0,0}$ is a simply connected real hypersurface of X, and $M_{1,0}$ and $M_{0,1}$ are the two connected components of $X \setminus M$. Here, we can extend some results known only in the case of Penrose transform (see e.g. [1]) by calculating $\mathbf{C}_S \circ \mathbf{C}_N$.

References

[1] T. Bailey, L. Ehrenpreiss and R. O. Wells jr. *Weak solutions of the massless field equations.* Proc. Roy. Soc. London, 384 (1982), p. 403–425.

[2] R. J. Baston, M. G. Eastwood. *The Penrose transform: its interaction with representation theory.* Oxford Univ. Press (1989).

[3] A. D'Agnolo, C. Marastoni. *The Radon-Penrose transform and its real forms: an approach by sheaves and $\mathcal{D}$-modules.* To appear.

[4] A. D'Agnolo, P. Schapira. *The Radon-Penrose transform for $\mathcal{D}$-modules.* Journal of Functional Analysis 139, no. 2 (1996), p. 349–382.

[5] A. D'Agnolo, P. Schapira. *Leray's quantization of projective duality.* Duke Math. Journal 84, no. 2 (1996), p. 453–496.

[6] M. G. Eastwood, R. Penrose, R. O. Wells jr. *Cohomology and massless fields.* Comm. Math. Phys. 78 (1981), p. 305–351.

[7] I. M. Gelfand, S. G. Gindikin, M. I. Graev. *Integral geometry in affine and projective spaces.* Journal of Soviet Math. 18 (1982), p. 39–167.

[8] M. Kashiwara, P. Schapira. *Sheaves on Manifolds.* Springer Grundlehren 292 (1990).

[9] M. Kashiwara, P. Schapira. *Moderate and formal cohomology associated with constructible sheaves.* Mémoire S.M.F. 64 (1996).

[10] M. Kashiwara, W. Schmid. *Quasi-equivariant $\mathcal{D}$-modules, equivariant derived category and representations of reductive Lie groups.* Preprint RIMS 980 (1994).

[11] C. Marastoni. *La dualité de Grassmann pour les $\mathcal{D}$-modules*, C. R. Acad. Sci. Paris 322 (1996); *Quantification de la dualité de Grassmann*, C. R. Acad. Sci. Paris 324 (1997); *Grassmann duality for $\mathcal{D}$-modules*, to appear in Ann. École Norm. Sup. (1998).

CORRADO MARASTONI

Dipartimento di Matematica Pura ed Applicata – Università degli studi di Padova – Via Belzoni, 7 – I-35131 Padova (Italy) – `maraston@math.unipd.it`