

Title	CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS (Complex Analysis and Microlocal Analysis)
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Citation	数理解析研究所講究録 (1999), 1090: 46-49
Issue Date	1999-04
URL	<a href="http://hdl.handle.net/2433/62879">http://hdl.handle.net/2433/62879</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

## CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

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ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let  $P(x, D)$  be a microdifferential operator defined at  $x^* = (0; 0, \dots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$  of order  $m \geq 2$ , written in the form

$$(1) \quad \begin{cases} P(x, D) = D_1^m + \sum_{0 \leq j \leq m-1} P_j(x, D') D_1^j, \\ \text{ord } P_j \leq m - j. \end{cases}$$

Here we have written  $D' = (D_2, \dots, D_n)$ . We also write as  $D'' = (D_1, \dots, D_{n-1})$ ,  $D''' = (D_2, \dots, D_{n-1})$ . Let  $\sigma_m(P)(x, \xi)$  be the principal symbol of  $P(x, D)$ . We assume that

$$(2) \quad \begin{cases} \text{if } x_1 = 0, \text{ then } \sigma_m(P) = \xi_n^m; \\ \text{if } x_1 \neq 0, \text{ then the equation } \sigma_m(P) = 0 \text{ has } m \text{ distinct roots} \\ \xi_1 = \varphi_1(x, \xi'), \dots, \varphi_m(x, \xi'). \end{cases}$$

We denote by  $\mathcal{O}$  (resp.  $\mathcal{O}_{(j)}$ ) the sheaf of holomorphic functions (resp. the sheaf of functions  $f(x_1^{1/j}, x')$  such that  $f(x)$  are holomorphic). Without loss of generality, we may assume that  $\varphi_j(x, \xi') \in \mathcal{O}_{(m'), x^*}$  for some  $m' \in \mathbf{N}$ , that they are homogeneous in  $\xi'$  of degree 1, and that they vanish when  $x_1 = 0$ . From now on, we denote  $\bar{\mathcal{O}} = \mathcal{O}_{(m')}$ . It follows that

$$\begin{cases} \text{for some } q_j \in \mathbf{N}/m' \text{ and some } a_j(x, \xi') \in \bar{\mathcal{O}}_{x^*} \text{ we have} \\ \varphi_j(x, \xi') = x_1^{q_j} a_j(x, \xi'), \quad a_j(x^*) \neq 0 \quad (1 \leq j \leq m). \end{cases}$$

We also assume that

$$(3) \quad i \neq j \quad \implies \quad (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).$$

We denote by  $\mathcal{C}$  (resp.  $\mathcal{E}$ ) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

$$(4) \quad \begin{cases} Pu = 0, \\ D_1^{j-1}u(0, x') = v_j(x'), \quad 1 \leq j \leq m, \end{cases}$$

where  $u \in \mathcal{C}_{\mathbf{R}^n, x^*}$  and  $v_j \in \mathcal{C}_{\mathbf{R}^{n-1}, x^{*j}}$  ( $x^{*j} = (0; 0, \dots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1}$ ). If  $P(x, D)$  is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for  $v_1 = \dots = v_m = 0$ ), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let  $A(x', D')$  be an both-side invertible  $m \times m$  matrix whose components  $A_{(\mu, \nu)}(x', D') \in \mathcal{E}_{x^*}^{\mathbf{R}}$  are independent of  $(x_1, D_1)$ . Here we denote by  $\mathcal{E}^{\mathbf{R}}$  the sheaf of holomorphic microlocal operators (c.f. [1,6]). We choose  $r$  rows of this matrix in an arbitrary way. To be clear, let  $1 \leq j_1 < j_2 < \dots < j_r \leq m$  and choose the  $j_1, \dots, j_r$ -th rows of  $A$ . Then we obtain an  $r \times m$  matrix  $A'(x', D')$  of holomorphic microlocal operators. We say that  $v_1(x'), \dots, v_m(x') \in \mathcal{C}_{\mathbf{R}^{n-1}, x^{*j}}$  satisfy an  $r$ -relation if choosing some  $r$  rows of some  $A(x', D')$  we have  $A'(x', D')\vec{v}(x') = \vec{0}$ . Here  $\vec{v}$  denotes  ${}^t(v_1, \dots, v_m)$ . Note that even if  $v_1(x'), \dots, v_m(x')$  satisfy an  $r$ -relation and another  $s$ -relation, it does not necessarily mean an  $(r+s)$ -relation.

We next define a classification of the characteristic roots. Let  $\theta \in \{0, \pi\}$ . Let

$$(5) \quad (x, \xi') \in \mathbf{R}^n \times \mathbf{R}^{n-1}, \quad x_1 \neq 0, \quad \arg x_1 = \theta.$$

We define

$$\begin{aligned} M &= \{1, 2, \dots, m\}, \\ M_{0, \theta} &= \{\lambda \in M; \operatorname{Re}(x_1 \varphi_\lambda(x, \xi')) = 0, \text{ if } (x, \xi') \text{ satisfies (5)}\}, \\ M_{\pm, \theta} &= \{\lambda \in M; \pm \operatorname{Re}(x_1 \varphi_\lambda(x, \xi')) > 0, \text{ if } (x, \xi') \text{ satisfies (5)}\}, \\ M'_\theta &= M \setminus M_{0, \theta} \setminus M_{+, \theta} \setminus M_{-, \theta}. \end{aligned}$$

It is easy to see that  $M_{0, \theta} \cup M_{+, \theta} \cup M_{-, \theta} \cup M'_\theta = M$  is a disjoint union.

Let  $m_{0, \theta}, m_{\pm, \theta}$  be the number of the elements belonging to  $M_{0, \theta}, M_{\pm, \theta}$ , respectively. We assume that

$$(6) \quad M'_\theta = \emptyset, \quad \forall \theta \in \{0, \pi\}.$$

We also need a condition for the microfunctions. Let

$$\begin{aligned} \omega(r) &= \{(x, \xi) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n; |x| < r, |\xi''| < r \operatorname{Im} \xi_n\}, \\ \omega'(r) &= \{(x', \xi') \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1}; |x'| < r, |\xi'''| < r \operatorname{Im} \xi_n\}, \end{aligned}$$

and

$$\begin{aligned} \omega_0(r) &= \{(x, \xi) \in \omega(r); |x'| \leq r^{-1}|x_1|, |\xi''| \leq r^{-1}|x_1| \operatorname{Im} \xi_n\}, \\ \omega'_0(r) &= \{tx'^*; t > 0\}. \end{aligned}$$

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We define

$$C_0 = \varinjlim_{r>0} \Gamma_{\omega_0(r)}(\mathbb{C}\mathbb{R}^n, \omega_0(r)),$$

$$C'_0 = \varinjlim_{r>0} \Gamma_{\omega'_0(r)}(\mathbb{C}\mathbb{R}^{n-1}, \omega'_0(r)).$$

Then we have the following

**Theorem.** *We assume (1) – (3) and (6). Let  $v_1(x'), \dots, v_m(x') \in C'_0$ . Then there exists an  $m_{+,0}$ -relation and an  $m_{+,\pi}$ -relation such that the Cauchy problem (4) has a solution  $u \in C_0$  if, and only if,  $v_1(x'), \dots, v_m(x')$  satisfy these relations.*

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

*Example 0 (Lewy-Mizohata operators).* If  $P_{\pm} = D_1 \pm \sqrt{-1}x_1D_n$ , then we have  $M_{\pm,\theta} = \{1\} (= M)$ ,  $M_{\mp,\theta} = \emptyset$ . The above theorem means that  $P_-u = 0$ ,  $u(0, x') = v(x')$  is solvable for any  $v \in C'_0$  without any relations. In fact using the defining function we only need to let  $u(x) = v(x''', x_n + \sqrt{-1}x_1^2/2)$ . On the other hand,  $P_+u = 0$ ,  $u(0, x') = v(x')$  is solvable only for the case when  $v(x')$  satisfies a one-relation. This means  $v = 0$ , and  $u = 0$ . It follows that  $P_+u = 0 \Rightarrow u = 0$ , i.e.,  $P_+$  is hypo-elliptic in  $C_0$  (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to  $M_{+,\theta}$  cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case  $m = 2$ .

*Example 1 (microhyperbolic operators).* Let  $P(x, D) = D_1^2 - x_1^2D_n^2 + P'(x, D)$ ,  $\text{ord } P' \leq 1$ . Without loss of generality, we may assume that  $P'$  is a polynomial in  $D_1$  of degree 1. Since  $\varphi_1(x, \xi') = x_1\xi_n$ ,  $\varphi_2(x, \xi') = -x_1\xi_n$ , and  $\arg \xi_n = \pi/2$ , it is easy to see that  $M_{0,\theta} = \{1, 2\}$ ,  $M_{\pm,\theta} = \emptyset$  for  $\theta \in \{0, \pi\}$ . It follows that (4) is solvable for arbitrary  $v_1(x'), v_2(x') \in C'_0$  without any relations (See [3]).

*Example 2 (Tricomi operators).* Let  $P(x, D) = D_1^2 - x_1D_n^2 + P'(x, D)$ ,  $\text{ord } P' \leq 1$ . We have  $\varphi_1(x, \xi') = \sqrt{x_1}\xi_n$ ,  $\varphi_2(x, \xi') = -\sqrt{x_1}\xi_n$ . It follows that  $M_{0,0} = \{1, 2\}$ ,  $m_{+,0} = 0$ , and that  $M_{+,\pi} = \{1\}$ ,  $M_{-,\pi} = \{2\}$ ,  $m_{+,\pi} = 1$ . It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let  $\omega \subset \sqrt{-1}\mathbb{T}^*\mathbb{R}^n$  be a small neighborhood of  $x^*$ , and let  $\omega^\theta = \{(x, \xi) \in \omega; x_1 \neq 0, \arg x_1 = \theta\}$ ,  $\theta \in \{0, \pi\}$ . At first we consider an elliptic boundary value problem in  $\omega_\pi$ , giving one boundary datum on  $\{x_1 = 0\}$ . Then we can always extend this solution to the hyperbolic region  $\omega_0$ . This case was considered also by [4].

*Example 3 (hypoelliptic operators).* Let  $P(x, D) = D_1^2 + x_1^2D_n^2 + P'(x, D)$ ,  $\text{ord } P' \leq 1$ . Since  $\varphi_1(x, \xi') = \sqrt{-1}x_1\xi_n$ ,  $\varphi_2(x, \xi') = -\sqrt{-1}x_1\xi_n$ , it is easy to see that  $M_{-,\theta} = \{1\}$ ,  $M_{+,\theta} = \{2\}$ ,  $m_{+,\theta} = 1$  for  $\theta \in \{0, \pi\}$ . There exist an  $m_{+,0}$ -relation and an  $m_{+,\pi}$ -relation such that the Cauchy problem (4) uniquely has a solution  $u \in C_0$  if, and only if,  $v_1(x'), \dots, v_m(x') \in C'_0$  satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case  $v_1 = v_2 = 0$ , and  $u = 0$ . In other words,  $Pu = 0$  does not have any

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non-trivial solutions. It is well-known that this is true if the principal symbol  $\sigma_1(P')$  of the lower order term satisfies  $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1}, \sqrt{-13}, \sqrt{-15}, \dots\}$  (See [2,5]).

Of course our result applies for higher order operators, too.

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