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Multiple Solutions of Boundary Value Problems for Semilinear Wave Equations

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Introduction

We consider nonlinear wave equation:

$$u_{tt} - u_{xx} + g(u) = f(x,t) \text{ in } \Omega \tag{1}$$

$$u(0,t) = u(\pi,t) = 0, \qquad 0 \leq t \leq \pi, \tag{2}$$

$$u(x,0) = \varphi_0(x), u(x,\pi) = \varphi_1(x), \qquad 0 \leq x \leq \pi, \tag{3}$$

where $\Omega = (0,\pi) \times (0,\pi)$, g is an odd function, $\varphi_0, \varphi_1 \in C_0^2([0,\pi]) = \{\phi \in C^2([0,\pi])\ ;\ \phi(0) = \phi(\pi) = 0\}$, $f \in L^\infty(\Omega)$.

The prime motive that we consider the problem (1)-(3) is the boundary problems for ordinary differential equations of second-order in $\mathbf{R}^N$ which can be regarded as a finite-dimensional case of (1)-(3). Ekeland, Ghoussoub, Tehrani [6] considered the following Bolza problem

$$\begin{cases} \frac{d^2x}{dt^2} + V'(x) = 0, \quad 0 < t < T \\ x(0) = x_0 \text{ and } x(T) = x_1 \end{cases}$$

where $V \in C^1(\mathbf{R}^N : \mathbf{R})$ is even and satisfies $V(x) \sim |x|^p$, $|x|$ large for some $p \in (2,4)$. They showed that the above problem has infinitely many solutions. The proof relies on the variational principle of Rabinowitz ('86) that is, perturbation results. We apply this method to the problem (1)-(3).

It is also known that there are infinitely many solutions for

$$u_{tt} - u_{xx} + u|u|^{p-2} = f(x,t) \text{ in } (0,\pi) \times \mathbf{R} \tag{4}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbf{R}, \tag{5}$$

$$u(x,t) = u(x,t+2\pi), \qquad 0 \le x \le \pi, t \in \mathbf{R} \tag{6}$$

where $p > 2$ is a constant and $f \in L^\infty$ is 2π-periodic in t. ([14], [15], [3]). The duality method is used there.

We define weak solution of (1)-(3) as follows.

Definition 1 *A function $u \in L^1(\Omega)$ is a weak solution of (1)-(3) if $g(u) \in L^1(\Omega)$ and*

$$\int_0^\pi \int_0^\pi \Big\{(u-z)(\zeta_{tt} - \zeta_{xx}) + (g(u) - f - z_{xx})\zeta\Big\} dxdt = 0$$

holds for all $\zeta \in C_0^2 \equiv \{w \in C^2(\overline{\Omega})\,;\, w = 0 \text{ on } \partial\Omega\}$, where $z = z(x,t) = \frac{t}{\pi}\varphi_1(x) + (1 - \frac{t}{\pi})\varphi_0(x)$.

Our main results are as follows:

Theorem 1 *Suppose that $g \in C(\mathbf{R};\mathbf{R})$ has the following properties.*

(1°) *g is an odd function.*

(2°) *g is a strictly increasing function.*

(3°) *$^\exists p \in (2, 1+\sqrt{3}), R_0 \ge 0\ ;\ 0 < pG(u) := p\int_0^u g(v)dv \le ug(u)$ for all u, $|u| \ge R_0$*

(4°) *$^\exists C > 0\ ;\ |g(u)| \le C(|u|^{p-1} + 1)$ for all u*

Then for any $\varphi_0, \varphi_1 \in C_0^2([0,\pi]), f \in L^\infty(\Omega)$, the problem (1)-(3) has an unbounded sequence of weak solutions $(u_k)_{k=1,2,3\cdots}$.

Furthermore,

Theorem 2 *Suppose that $g \in C(\mathbf{R};\mathbf{R})$ has the following properties.*

(1°) *g is an odd function.*

(2°) *g is a strictly increasing function.*

(3°) *$^\exists p \in (2, 2+\sqrt{2}), R_0 \ge 0\ ;\ 0 < pG(u) := p\int_0^u g(v)dv \le ug(u)$ for all u*

(4°) *$^\exists C > 0\ ;\ |g(u)| \le C(|u|^{p-1} + 1)$ for all u*

Then for any $f \in L^\infty(\Omega)$, *the problem (1)-(3) with* $\varphi_0, \varphi_1 = 0$ *has an unbounded sequence of weak solutions* $(u_k)_{k=1,2,3\cdots}$.

Remark 1 *If* $\Omega = (0, L) \times (0, T)$ *and* $\frac{L}{T} \in \mathbf{Q}$, *then the same results hold.*

Remark 2 *If* φ_0, φ_1 *satisfies that there exists a function* $z \in W^{2,\infty}(\Omega) = C^{1,1}(\overline{\Omega})$ *such that* $z(0,t) = z(\pi,t) = 0$, $(0 \le t \le \pi)$, $z(x,0) = \varphi_0(x)$, $z(x,\pi) = \varphi_1(x)$, $(0 \le x \le \pi)$, *then the same results hold with* φ_0, φ_1

Remark 3 *If a weak solution of (1)-(3)* u *belongs to* $C^2(\overline{\Omega})$, *then* u *is a classical solution of (1)-(3).*

In what follows, we reformulate the problem in a way such that duality methods can be applied similar to known results. Moreover, We construct an operator $K = A^{-1}$ from the suitable function space to $C(\overline{\Omega})$ which is the key lemma.

Proof of Theorem 1 and Theorem 2

Let p' denote the conjugate number of p , that is

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

then $p' \in (1, 2)$. Moreover let

$$N = \Big\{p(t+x) - p(t-x) \ ; \ p \in L^1_{\text{loc}}(\mathbf{R}), p \text{ is } 2\pi\text{-periodic and even function} ,$$

$$\int_0^{2\pi} p(\tau)d\tau = 0\Big\},$$

$$\tilde{N} = \Big\{p(t+x) - p(t-x) \ ; \ p \in L^1_{\text{loc}}(\mathbf{R}), p \text{ is } 2\pi\text{-periodic} , \int_0^{2\pi} p(\tau)d\tau = 0\Big\}.$$

The operator $A = \partial_t^2 - \partial_x^2$ has infinitely many positive and negative eigenvalues and also posesses an infinite-dimensional kernel. The element of N belongs to the kernel of A. Hereafter we regard $w \in C_0^2$ as an extension of w to $[0, \pi] \times \mathbf{R}$ which satisfies

$$w(x, -t) = -w(x,t), w(x, t + 2\pi) = w(x,t).$$

Let $D = (0, \pi) \times (-\pi, \pi)$. Here is a key lemma to the proof.

Lemma 3 *There exists a linear operator* K: $V_1 = \Big\{v \in L^1(\Omega)\ ;\ \int_\Omega v\phi = 0$ *for all* $\phi \in N \cap L^\infty\Big\} \to V_2 = \Big\{v \in C([0,\pi] \times \mathbf{R})\ ;\ v(x,t+2\pi) = v(x,t), v(x,-t) = -v(x,t),$ *for all* $t, \int_D v\phi = 0$ *for all* $\phi \in \widetilde{N} \cap L^\infty\Big\}$ *which has the following properties.*

$$\int_\Omega (Kv)(A\zeta) = \int_\Omega v\zeta, \text{ for all } v \in V_1, \zeta \in C_0^2 \tag{7}$$

$$\|Kv\|_\infty \le C\|v\|_1, \text{ for all } v \in V_1 \tag{8}$$

$$\int_\Omega (Kv_1)v_2 = \int_\Omega v_1(Kv_2), \text{ for all } v_1, v_2 \in V_1 \tag{9}$$

$$\|Kv\|_{0,\alpha} \le C\|v\|_q, \ \alpha = 1 - \frac{1}{q}, \text{ for all } v \in V_1 \cap L^q \tag{10}$$

Proof. Introduce function spaces $W_1 = \Big\{v \in L^1(D)\ ;\ \int_D v\phi = 0,$ for all $\phi \in \widetilde{N} \cap L^\infty\Big\}$, $W_2 = \Big\{v \in C([0,\pi] \times \mathbf{R})\ ;\ v(x,t+2\pi) = v(x,t),$ for all $t, \int_D v\phi = 0,$ for all $\phi \in \widetilde{N} \cap L^\infty\Big\}$ and also define a linear operator $\widetilde{K} : W_1 \to W_2$ satisfying $A\widetilde{K} = id$ as follows. For given $v \in W_1$,

$$\widetilde{K}v(x,t) = \psi(x,t) - (p(t+x) - p(t-x)) \tag{11}$$

where ψ is constructed from a 2π-periodic extension of v to $[0,\pi] \times \mathbf{R}$ by using the fundamental solution of the wave operator ; that is

$$\psi(x,t) = -\frac{1}{2}\int_x^\pi d\xi \int_{t-(\xi-x)}^{t+(\xi-x)} v(\xi,\tau)d\tau + c\frac{\pi - x}{\pi} \tag{12}$$

with

$$c = \frac{1}{2}\int_0^\pi d\xi \int_{t-\xi}^{t+\xi} v(\xi,\tau)d\tau \tag{13}$$

Note that c is a constant; here the fact that $v \in W_1$ is used. Moreover periodicity of v implies periodicity of $\widetilde{K}v$. Finally choosing

$$p(s) = \frac{1}{2\pi}\int_0^\pi \{\psi(\xi, s-\xi) - \psi(\xi, s+\xi)\}\, d\xi \tag{14}$$

ensures that $\int_D (\widetilde{K}v)\phi = 0$ for all $\phi \in \widetilde{N} \cap L^\infty$. Hence (11)-(14) determine the operator $\widetilde{K}$ from W_1 to W_2 as desired. Noting that $A\widetilde{K}v = v$ for a smooth function v, there holds

$$\int_D (\widetilde{K}v)(A\zeta) = \int_D v\zeta, \qquad \text{for all } v \in W_1, \zeta \in C_0^2 \tag{15}$$

and also

$$\int_D (\widetilde{K}v_1)v_2 = \int_D v_1(\widetilde{K}v_2) \tag{16}$$

for all $v_1, v_2 \in W_1$. Moreover (11)-(14) and Hölder's inequality imply that

$$\|\widetilde{K}v\|_{L^\infty(D)} \le C\|v\|_{L^1(D)} \tag{17}$$

for all $v \in W_1$. Also for $v \in W_1 \cap L^q$, $q > 1$, we have $\widetilde{K}v \in C^{0,\alpha}(\overline{D})$, with $\alpha = 1 - \frac{1}{q} > 0$ and

$$\|\widetilde{K}v\|_{C^{0,\alpha}(\overline{D})} \le C\|v\|_{L^q(D)}. \tag{18}$$

For each $v \in V_1$, let ιv denote an odd extension of v to D. We can see $\iota v \in W_1$ as follows. Choose $\phi(x,t) = p(t+x) - p(t-x) \in \widetilde{N} \cap L^\infty$, $p \in L^1_{\text{loc}}$, p is 2π-periodic, $\int p(\tau)d\tau = 0$. We may write $p = p_e + p_o$ a.e., p_e is even, p_o is odd, p_e and p_o are 2π-periodic. Letting $\phi_o(x,t) = p_e(t+x) - p_e(t-x)$, $\phi_e(x,t) = p_o(t+x) - p_o(t-x)$, we have $\phi(x,t) = \phi_e(x,t) + \phi_o(x,t)$, $\phi_o(x,-t) = -\phi_o(x,t)$, $\phi_e(x,-t) = \phi_e(x,t)$. Therefore,

$$\begin{aligned} \int_D (\iota v)\phi &= \int_D (\iota v)\phi_e + \int_D (\iota v)\phi_o \\ &= 2\int_\Omega (\iota v)\phi_o \\ &= 0, \end{aligned} \tag{19}$$

which yields the desired result. By the definition of $\widetilde{K}$, we have

$$\widetilde{K}\iota v(x,-t) = -\widetilde{K}\iota v(x,t) \text{ for } v \in V_1 \tag{20}$$

Since $\widetilde{K}\iota v \in W_2$, this implies that $\widetilde{K}\iota v \in V_2$. Hence $\widetilde{K}\iota$ defines the desired linear operator $:V_1 \to V_2$. Noting that the product of two odd functions is an even function, the properties (7)-(10) easily follow. □

Next we define the functional by using the operator K. Let h be the inverse function of g. By assumption, h is strictly increasing function continuous odd function. Also let $H(u) = \int_0^u h(v)dv$ be the primitive of h (H is the conjugate convex function of G). By assumption, there exists $a_1, a_2, a_3, a_4 > 0$ such that

$$0 \le a_1|u|^{p'-1} \le |h(u)| + a_2 \le a_3|u|^{p'-1} + a_4 \tag{21}$$

for all $u \in \mathbf{R}$. Define

$$E = \left\{u \in L^{p'}(\Omega) \ ; \ \int_\Omega u\phi = 0 \text{ for all } \phi \in N \cap L^p\right\},$$

$$\tilde{f} = f + z_{xx},$$

then there is a constant $C > 0$ such that

$$\int_\Omega |H(u+\tilde{f}) - H(u)| \le C\left[\left\{\int_\Omega H(u+\tilde{f})\right\}^{(p'-1)/p'} + 1\right] \tag{22}$$

for all $u \in E$. For $u \in E$ let

$$I(u) = \frac{1}{2}\int_\Omega (Ku)u + \int_\Omega H(u+\tilde{f}) - \int_\Omega zu.$$

If $v \in E$ is the critical point of I,

$$0 = \int_\Omega \left\{(Kv)w + h(v+\tilde{f})w - zw\right\}$$

for $w \in E$. Here for $\zeta \in C_0^2$ substitute $w = A\zeta \in E$,

$$0 = \int_\Omega \left\{v\zeta + \left(h(v+\tilde{f}) - z\right)(A\zeta)\right\}.$$

Then $u = h(v+\tilde{f})$ satisfies

$$0 = \int_\Omega \left\{\left(g(u) - \tilde{f}\right)\zeta + (u-z)(A\zeta)\right\}$$

which yields that u is the desired weak solution. Hence the assertion of the theorem is equivalent to the claim that the functional I possesses an unbounded sequence of critical points in E.

Introduce a modified functional

$$J(u) = \frac{1}{2}\int_\Omega (Ku)u + \int_\Omega H(u) + \psi(u)\int_\Omega \left(H(u+\tilde{f}) - H(u) - zu\right)$$

where $\psi(u) = \chi\left(\Psi(u)^{-1}\int_\Omega(-Ku)u\right)$, $\Psi(u) = a\left(I^2(u)+1\right)^{1/2}$ with $a = \frac{6p'+4}{2-p'} > 1$ is a constant and χ is a function in $C^\infty(\mathbf{R};[0,1])$ which is equal to 1 on $(-\infty,1]$, to 0 on $[2,\infty)$ and such that $\chi'(t) \in (-2,0)$ for $t \in (1,2)$.

Lemma 4 *(i) There is a constant $\beta > 0$ such that*

$$|J(u) - J(-u)| \le \beta(|J(u)|^{1/p'} + 1)$$

for any $u \in E$

(ii) If $z = 0$ ($\varphi_0 = \varphi_1 = 0$), then there is a constant $\beta > 0$ such that

$$|J(u) - J(-u)| \le \beta(|J(u)|^{(p'-1)/p'} + 1)$$

for any $u \in E$

Proof. (i) We can estimate

$$|J(u) - J(-u)| \leq \psi(u)A + \psi(-u)B$$

where

$$A := \int_\Omega |H(u+\tilde{f}) - H(u) - zu|, \tag{23}$$

$$B := \int_\Omega |H(-u+\tilde{f}) - H(-u) + zu|. \tag{24}$$

If $u \in \text{supp}\,\psi$, then we obtain that

$$\begin{aligned} \int_\Omega H(u+\tilde{f}) &= I(u) + \frac{1}{2}\int_\Omega (-Ku)u + \int_\Omega zu \\ &\leq 2\Psi(u) + \int_\Omega zu \\ &\leq 2\Psi(u) + \frac{1}{2}\int_\Omega H(u+\tilde{f}) + C \\ &\leq C\Psi(u) + \frac{1}{2}\int_\Omega H(u+\tilde{f}) \end{aligned}$$

since $\int_\Omega (-Ku)u \leq 2\Psi(u)$. It follows that

$$\int_\Omega H(u+\tilde{f}) \leq C\Psi(u) \tag{25}$$

for $u \in \text{supp}\,\psi$. Hence by (22),(25)

$$\begin{aligned} A &\leq C\int_\Omega H(u+\tilde{f})^{(p'-1)/p'} + C\left\{\int_\Omega |u|^{p'}\right\}^{1/p'} + C \\ &\leq C\int_\Omega H(u+\tilde{f})^{1/p'} \\ &\leq C\int_\Omega \Psi(u)^{1/p'} \end{aligned} \tag{26}$$

for $u \in \text{supp}\,\psi$. Therefore (23),(26) implies that there is a universal constant $C > 0$ such that

$$\begin{aligned} \psi(u)A &\leq C\psi(u)\left(|I(u)|^{1/p'} + 1\right) \\ &\leq C\psi(u)\left[|J(u)|^{1/p'} + A^{1/p'} + \psi(u)^{1/p'}A^{1/p'} + 1\right] \end{aligned} \tag{27}$$

for any $u \in E$. (In the case $u \notin \text{supp}\,\psi$, this inequality obviously holds). Similarly

$$\begin{aligned}\psi(-u)B &\le C\psi(-u)\left(|I(-u)|^{1/p'}+1\right)\\ &\le C\psi(-u)\left[|J(u)|^{1/p'}+B^{1/p'}+\psi(u)^{1/p'}A^{1/p'}+1\right]. \quad (28)\end{aligned}$$

Therefore by (27),(28) and Young's inequality

$$\psi(u)A+\psi(-u)B \le C(|J(u)|^{1/p'}+1).$$

(ii) If $z=0$, we can replace (26) by

$$A \le C\int_\Omega \Psi(u)^{(p'-1)/p'}.$$

Other estimates proceeds similarly. □

Lemma 5 *(i) There is a constant $M>0$ such that $J(u)>M$ and $J'(u)=0$ implies that $I(u)=J(u)$ and $I'(u)=0$.*
(ii) J satisfies (P.-S.) on $\{u \in E; J(u) \ge M\}$.

Proof. First we shall show that there is a constant $M>0$ such that $J(u)>M$ and $\|J'(u)\|<1$ implies that $\psi(u)=1$. By the definition of ψ, this will be the case if

$$\int_\Omega(-Ku)u \le \Psi(u). \quad (29)$$

Since (29) is obvious if $\int_\Omega(-Ku)u \le 0$, we may asuume that $\int_\Omega(-Ku)u>0$. Note that

$$\begin{aligned}\langle u, J'(u)\rangle &= \int_\Omega (Ku)u+\int_\Omega h(u)u+\psi(u)\int_\Omega\left\{h(u+\tilde f)-z-h(u)\right\}u\\ &\quad +\langle u,\psi'(u)\rangle\int_\Omega\left\{H(u+\tilde f)-zu-H(u)\right\}\end{aligned}$$

where

$$\langle u,\psi'(u)\rangle=\chi'(\theta(u))\left[2\Psi^{-1}(u)-a^2I(u)\Psi^{-3}(u)\langle u,I'(u)\rangle\right]\int_\Omega(-Ku)u,$$

$$\theta(u)=\Psi(u)^{-1}\int_\Omega(-Ku)u,$$

$$\langle u,I'(u)\rangle=\int_\Omega(Ku)u+\int_\Omega\left\{h(u+\tilde f)-z\right\}u.$$

Regrouping terms shows that

$$\begin{aligned}\langle u, J'(u)\rangle &= (1-\psi(u))\int_\Omega h(u)u + (\psi(u) - T_1(u))\int_\Omega h(u+\tilde{f})u \\ &\quad - (1 - T_2(u))\int_\Omega (-Ku)u\end{aligned}$$

where

$$\begin{aligned}T_1(u) &= a^2\chi'(\theta(u))\Psi^{-3}(u)I(u)\int_\Omega (-Ku)u \times \\ &\quad \times \int_\Omega \left\{H(u+\tilde{f}) - zu - H(u)\right\},\end{aligned}$$

$$\begin{aligned}T_2(u) &= T_1(u) + \chi'(\theta(u))\Psi^{-3}(u)\left\{2\Psi^2(u) + a^2 I(u)\int_\Omega zu\right\} \times \\ &\quad \times \int_\Omega \left\{H(u+\tilde{f}) - zu - H(u)\right\},\end{aligned}$$

We will show $T_1(u), T_2(u) \to 0$ as $M \to \infty$. By the definition of T_1,

$$\begin{aligned}|T_1(u)| &\le C\Psi(u)^{-2}\int_\Omega (-Ku)u\left[\left\{\int_\Omega H(u+\tilde{f})\right\}^{(p'-1)/p'} + \int_\Omega |u| + 1\right] \\ &\le C\Psi(u)^{-2}\int_\Omega (-Ku)u \times \\ &\quad \times \left[\left\{\int_\Omega H(u+\tilde{f})\right\}^{(p'-1)/p'} + \left\{\int_\Omega H(u+\tilde{f})\right\}^{1/p'} + 1\right].\end{aligned}$$

If $u \notin \operatorname{supp}\psi$, then $T_i(u) = 0\,(i = 1, 2)$. Otherwise, since

$$\begin{aligned}I(u) &\ge J(u) + (1-\psi(u))\int_\Omega \left\{H(u+\tilde{f}) - zu - H(u)\right\} \\ &\ge M - C\left\{\int_\Omega H(u+\tilde{f})\right\}^{(p'-1)/p'} - C\left\{\int_\Omega H(u+\tilde{f})\right\}^{1/p'} - C\end{aligned}$$

and (25),

$$\begin{aligned}\Psi(u) &\ge I(u) \\ &\ge M - C\Psi(u)^{(p'-1)/p'} - C\Psi(u)^{1/p'} - C.\end{aligned}$$

This implies that

$$\Psi(u) + C\Psi(u)^{(p'-1)/p'} + C\Psi(u)^{1/p'} \ge M - C,$$

$$\frac{3}{2}\Psi(u) \geq M - C.$$

Here, letting $M > 2C$,

$$\Psi(u) \geq \frac{1}{3}M. \tag{30}$$

By $\int_\Omega (-Ku)u \leq 2\Psi(u)$, (25) and (30)

$$\begin{aligned} |T_1(u)| &\leq C(\Psi(u)^{-(p'-1)/p'} + \Psi(u)^{-1/p'} + \Psi(u)^{-1}) \\ &\leq CM^{-(p'-1)/p'} \end{aligned}$$

which goes to 0 as $M \to \infty$. Similarly we have

$$|T_2(u)| \leq |T_1(u)| + CM^{-(p'-1)/p'}.$$

Therefore we may assume that for M sufficiently large, $|T_i(u)| < \frac{1}{2}$ for $(i = 1, 2)$ and

$$\frac{1 - T_2(u)}{p'(1 - T_1(u))} - \frac{1}{2} \geq \frac{1}{2}\left(\frac{1}{p'} - \frac{1}{2}\right) \equiv b > 0.$$

Noting that $\inf_{u\in\mathbf{R}}(p'H(u) - uh(u)) > -\infty$ by assumption (3°), (22), (21) and the fact that $|T_i(u)|$ are sufficiently small, simple estimates show

$$\begin{aligned} & I(u) - \frac{1}{p'(1 - T_1(u))}\langle u, J'(u)\rangle \\ =\ & \left(\frac{1 - T_2(u)}{p'(1 - T_1(u))} - \frac{1}{2}\right)\int_\Omega (-Ku)u \\ & + \frac{1 - \psi(u)}{p'(1 - T_1(u))}\int_\Omega \left\{p'H(u + \tilde{f}) - p'H(u)\right\} \\ & + \frac{1 - \psi(u)}{p'(1 - T_1(u))}\int_\Omega \{p'H(u) - uh(u)\} \\ & + \frac{\psi(u)}{p'(1 - T_1(u))}\int_\Omega \left\{p'H(u + \tilde{f}) - (u + \tilde{f})h(u + \tilde{f})\right\} \\ & + \frac{\psi(u)}{p'(1 - T_1(u))}\int_\Omega h(u + \tilde{f})\tilde{f} \\ & - \frac{T_1(u)}{p'(1 - T_1(u))}\int_\Omega p'H(u + \tilde{f}) \\ & + \frac{T_1(u)}{p'(1 - T_1(u))}\int_\Omega uh(u + \tilde{f}) - \int_\Omega zu \\ \geq\ & b\int_\Omega (-Ku)u - \frac{b}{6}\int_\Omega H(u + \tilde{f}) - C. \end{aligned} \tag{31}$$

On the other hand, by the assumption $\|J'(u)\|_{E^*} < 1$,

$$\left|\frac{\langle u, J'(u)\rangle}{p'(1-T_1(u))}\right| \le \frac{2}{p'}\|J'(u)\|\|u\|_{p'} \le 2\|u\|_{p'} \le \frac{b}{6}\int_\Omega H(u+\tilde{f}) + C. \quad (32)$$

Hence adding $bI(u) = \frac{b}{2}\int_\Omega (Ku)u + b\int_\Omega H(u+\tilde{f}) - b\int_\Omega zu$ to (31) and using (32),

$$\begin{aligned}(1+b)I(u) \ &\geq\ \frac{b}{2}\int_\Omega (-Ku)u \\ &\quad + \left[\frac{b}{2}\int_\Omega H(u+\tilde{f}) - C\right]. \end{aligned} \quad (33)$$

Since by the assumption $\int_\Omega (-Ku)u > 0$,

$$\begin{aligned} M \ &<\ J(u) \\ &<\ \int_\Omega H(u) + \psi(u)\int_\Omega \left\{H(u+\tilde{f}) - H(u)\right\} \\ &\leq\ 2\int_\Omega H(u+\tilde{f}) + C, \end{aligned} \quad (34)$$

we have

$$\int_\Omega H(u+\tilde{f}) \to \infty \text{ as } M \to \infty.$$

Hence (33) implies

$$(1+b)I(u) \geq \frac{b}{2}\int_\Omega (-Ku)u, \qquad \text{for } M \text{ large}.$$

Thus $\int_\Omega (-Ku)u \le aI(u) \le \Psi(u)$. This proves (29) and hence lemma 5(i).

For the proof of lemma 5(ii), let (u_n) be (P.-S.) sequence for J such that $M < J(u_n)$. Since for large n, $J(u_n) = I(u_n)$ and $J'(u_n) = I'(u_n)$, (u_n) is also (P.-S.) sequence for I. Hence, it suffices to show that I satisfies (P.-S.). Let (u_n) be (P.-S.) sequence for I. We may write

$$\langle u_n, I'(u_n)\rangle = \int_\Omega w_n u_n \quad (35)$$

where $w_n \in L^p$, $\|w_n\|_p \to 0$. Hence

$$\begin{aligned} C + o(1)\|u_n\|_{p'} \ &\geq\ I(u_n) - \frac{1}{2}\langle u_n, I'(u_n)\rangle \\ &=\ \int_\Omega \left\{H(u_n+\tilde{f}) - \frac{1}{2}h(u_n+\tilde{f})u_n\right\} - \frac{1}{2}\int_\Omega zu_n. \end{aligned}$$

Since this implies that

$$\begin{aligned} c\|u_n\|_{p'}^{p'} - C &\le \left(\frac{1}{p'} - \frac{1}{2}\right)\int_\Omega (u_n+\tilde{f})h(u_n+\tilde{f}) \\ &\le \int_\Omega \left\{H(u_n+\tilde{f}) - \frac{1}{2}h(u_n+\tilde{f})(u_n+\tilde{f})\right\} \\ &\le C + C\|u_n\|_{p'}, \end{aligned}$$

(u_n) is bounded in $L^{p'}$. Extracting a subsequence if necessary, we may assume that $u_n \rightharpoonup u_0 \in E$ (weak in E). Noting that the oerator $K : E \to E^*$ is compact,

$$\begin{aligned} &\int_\Omega \left\{h(u_n+\tilde{f}) - h(u_0+\tilde{f})\right\}(u_n - u_0) \\ &\qquad = \int_\Omega \left\{w_n - Ku_n + z - h(u_0+\tilde{f})\right\}(u_n-u_0) \\ &\qquad \to 0. \end{aligned}$$

Hence a subsequence of (u_n) satisfies

$$(h(u_n+\tilde{f}) - h(u_0+\tilde{f}))(u_n-u_0) \to 0 \quad \text{a.e. in } \Omega,$$

$$(h(u_n+\tilde{f}) - h(u_0+\tilde{f}))(u_n-u_0) \le l_1(x,t) \quad \text{a.e. in } \Omega$$

where $l_1 \in L^1$. By monotonicity of h and (21),

$$u_n(x,t) \to u_0(x,t) \quad \text{a.e. in } \Omega,$$

$$|u_n(x,t)| \le l_2(x,t) \quad \text{a.e. in } \Omega$$

where $l_2 \in L^{p'}$. Thus by the Lebesgue convergence theorem, $u_n \to u_0$ strong in E. The proof is completed. □

Now we can show J has an unbounded sequence of critical values. Note that K defines a compact self-adjoint operator in $\{v \in L^2(\Omega)\,;\, \int_\Omega v\phi = 0 \text{ for all } \phi \in N \cap L^2\} = \overline{\text{span}}\{\sin ix \cdot \sin jt; i = 1,2,\ldots, j = 1,2,\ldots, i \ne j\}$. Its eigenvalues are $\sigma(K) = \{\frac{1}{i^2-j^2}\,;\, i,j = 1,2,\ldots, i \ne j\} = \{\pm\mu_k\,;\, k = 1,2,\ldots\}$ where μ_k are positive eigenvalues such that

$$\mu_1 \ge \mu_2 \ge \mu_3 \ge \ldots > 0$$

Let $e_k = \frac{2}{\pi}\sin ix \cdot \sin jt$ be the eigenfunction corresponding to the negative eigenvalue $-\mu_k = \frac{1}{i^2-j^2}$ and $f_k = \frac{2}{\pi}\sin ix \cdot \sin jt$ be the eigenfunction corresponding to the positive eigenvalue $\mu_k = \frac{1}{i^2-j^2}$. Let

$$E_k = \text{span}\{e_1, \ldots e_k\}$$

$$(E_k)^*_+ = \{u + te_{k+1}; u \in E_k, t \geq 0\}$$

Since L^2-norm and $L^{p'}$-norm are equivalent in E_k, There is a constant $C > 0$ depending on E_k such that

$$\begin{aligned} J(u) &\leq -\frac{1}{2}\mu_k\|u\|_2^2 + C\|u\|_{p'}^{p'} + C\|u\|_1 + C \\ &\leq -\frac{1}{2}\mu_k\|u\|_2^2 + C(\|u\|_{p'}^{p'} + 1) \\ &\leq -\frac{1}{2}\mu_k\|u\|_{p'}^2 + C(\|u\|_{p'}^{p'} + 1) \end{aligned} \tag{36}$$

for all $u \in E_k$. Hence choose $R_k > 0$ such that

$$u \in E_k, \|u\|_{p'} \geq R_k \Rightarrow J(u) \leq 0.$$

Since $E_k \subset E_{k+1}$, we may assume that $R_k \leq R_{k+1}$ for all k. Let

$$\begin{aligned} \Gamma_k = \{h \in C(E; E); h \text{ is odd, } {}^\forall j \leq k \\ u \in E_j, \|u\|_{p'} \geq R_j \Rightarrow h(u) = u\} \end{aligned}$$

$$\Gamma = \{h \in C(E; E); h \text{ is odd, } \max\{J(u), J(-u)\} \leq 0 \Rightarrow h(u) = u\}$$

Note that $\Gamma \subset \Gamma_{k+1} \subset \Gamma_k$. Define

$$b_k = \inf_{h \in \Gamma_k} \sup_{u \in E_k} J(h(u))$$

$$\widetilde{b_k} = \inf_{h \in \Gamma} \sup_{u \in E_k} J(h(u))$$

$$\widetilde{b_k}^* = \inf_{h \in \Gamma} \sup_{u \in (E_k)^*_+} J(h(u))$$

Obviously $\widetilde{b_k}^* \geq \widetilde{b_k} \geq b_k$ holds. Recall the following variational principle of Rabinowitz [11] which is the key to the perturbation method.

Proposition 1 *Suppose $J \in C^1(E)$ satisfies (P.-S.) condition on $\{u \in E\,;\, J(u) \geq M\}$ for some $M \in [0, +\infty)$. Let $W \subset E$ be a finite dimensional subspace of E, $w^* \in E \backslash W$ and let $W^* = W \oplus span\{w^*\}$; also let*

$$W^*_+ = \{w + tw^*\,;\, w \in W, t \geq 0\}$$

denote the "upper half-space" in W^*. Suppose

(1) $^{\exists}R > 0$; $^{\forall}u \in W$: $\|u\| \geq R \Rightarrow J(u) \leq 0$,

(2) $^{\exists}R^* \geq R$; $^{\forall}u \in W^*$: $\|u\| \geq R^* \Rightarrow J(u) \leq 0$,

and let

$$\Gamma = \{h \in C(V,V) \, ; \, h \text{ is odd }, \max\{J(u), J(-u)\} \leq 0 \Rightarrow h(u) = u\}$$

Then, if

$$\beta^* := \inf_{h\in\Gamma} \sup_{u\in W_+^*} J(h(u)) > \beta := \inf_{h\in\Gamma} \sup_{u\in W} J(h(u)) \geq M,$$

the functional J possesses a critical value $\geq \beta^*$.

Lemma 6 *For any $\delta > 0$, there are constants $\alpha > 0$ and $k_1 \in \mathbf{N}$ such that*

$$\tilde{b}_k \geq b_k \geq \alpha k^{\frac{2(p'-1)}{2-p'} - \delta}$$

for all $k \geq k_1$.

Proof. Letting $W_k = \text{span}\,\{e_j, f_i \, ; \, j \geq k, i \geq 1\}$, $S_r = \{u \in E \, ; \, \|u\|_{p'} = r\}$, we have $h \in \Gamma_k$, $r > 0$ for any $h \in \Gamma_k$, $r > 0$. (See [12] intersection lemma II.6.4). On the other hand, considering lattice points of (i,j) plane, there is a constant $\gamma > 0$ such that

$$\sharp\{(i,j) \in \mathbf{N} \times \mathbf{N} \, ; \, 0 < j^2 - i^2 < M\} \leq \gamma M \log M$$

for all $M > 1$. Hence by the definition of μ_k, for any $\delta > 0$ there is a constant $C = C(\delta)$ depending on δ such that

$$\mu_k \leq C k^{-1+\delta} \tag{37}$$

for $k \in \mathbf{N}$. if $u \in L^2 \cap W_k$, then since e_k, f_k are orthonormal basis in L^2, we may write

$$u = \sum_{i=k}^{\infty} c_i e_i + \sum_{i=1}^{\infty} d_i f_i$$

By Hölder's inequality ($\frac{1}{r} + \frac{2}{p} = 1$) and Housdorff-Young's inequality

$$\begin{aligned} \int_\Omega (-Ku)u &\leq \sum_{i=k}^{\infty} \mu_i |c_i|^2 \\ &\leq \left(\sum_{i=k}^{\infty} \mu_i^r\right)^{1/r} \left(\sum_{i=k}^{\infty} |c_i|^p\right)^{2/p} \\ &\leq C \left(\sum_{i=k}^{\infty} \mu_i^r\right)^{1/r} \|u\|_{p'}^2 \end{aligned}$$

By the density argument,

$$\int_\Omega (-Ku)u \le a_k \|u\|_{p'}^2$$

for all $u \in W_k$, where $a_k = C\left(\sum_{i=k}^\infty \mu_i^r\right)^{1/r}$. By (37), for any $\delta > 0$ there is a constant $C = C(\delta) > 0$ depending on δ such that

$$\begin{aligned} a_k &\le C\left(\sum_{i=k}^\infty i^{(-1+\delta)r}\right)^{1/r} \\ &\le Ck^{1/r-1+\delta} \\ &= Ck^{-\frac{2(p'-1)}{p'}+\delta} \end{aligned}$$

for all $k \in \mathbf{N}$. This implies that for any $\delta > 0$ there is a constant $k_0 \in \mathbf{N}$ such that

$$a_k \le k^{-\frac{2(p'-1)}{p'}+\delta}$$

for all $k \ge k_0$. Since for $u \in W_k \cap S_r$,

$$\begin{aligned} J(u) &\ge -\frac{1}{2}\int_\Omega (-Ku)u + \int_\Omega H(u) - C\left\{\int_\Omega H(u+\tilde{f})\right\}^{(p'-1)/p'} \\ &\quad - C\|u\|_1 - C \\ &\ge -\frac{1}{2}\int_\Omega (-Ku)u + \frac{1}{2}\int_\Omega H(u) - C \\ &\ge -\frac{1}{2}\int_\Omega (-Ku)u + C\|u\|_{p'}^{p'} - C \\ &\ge -\frac{1}{2}a_k r^2 + Cr^{p'} - C \end{aligned}$$

we obtain

$$\begin{aligned} b_k &\ge \sup_{r>0} \inf_{u \in W_k \cap S_r} J(u) \\ &\ge \sup_{r>0}\left(-\frac{1}{2}a_k r^2 + C_1 r^{p'} - C_2\right) \\ &= \left(1 - \frac{p'}{2}\right) C_1^{2/(2-p')} \left(\frac{a_k}{p'}\right)^{-p'/(2-p')} - C_2 \\ &\ge \alpha k^{\frac{2(p'-1)}{2-p'}-\delta} - C_2 \end{aligned}$$

for $k \ge k_0$. The proof is complete. □

Conclusion. (i)Let $z \neq 0$. Suppose that there is a constant $k_2 \in \mathbf{N}$ such that $\widetilde{b_k}^* = \widetilde{b_k}$ for all $k \geq k_2$. By lemma 4(i), $\widetilde{b_{k+1}} \leq \widetilde{b_k} + \beta(|\widetilde{b_k}|^{1/p'} + 1)$ for $k \geq k_2$. Hense for $k \geq k_3 = \max\{k_1, k_2\}$, there holds

$$\begin{aligned} \widetilde{b_{k+1}} &\leq \widetilde{b_k} + C\widetilde{b_k}^{1/p'} \\ &\leq \widetilde{b_k}(1 + C\widetilde{b_k}^{(1-p')/p'}) \end{aligned}$$

with an uniform constant C. By iteration technique,

$$\begin{aligned} \widetilde{b_{k_3+l}} &\leq \widetilde{b_{k_3}} \prod_{k=k_3}^{k_3+l-1} \left(1 + C\widetilde{b_k}^{(1-p')/p'}\right) \\ &\leq \widetilde{b_{k_3}} \exp\left(\sum_{k=k_3}^{k_3+l-1} \log\left(1 + C\widetilde{b_k}^{(1-p')/p'}\right)\right) \\ &\leq \widetilde{b_{k_3}} \exp\left(C \sum_{k=k_3}^{k_3+l-1} \widetilde{b_k}^{(1-p')/p'}\right) \end{aligned}$$

Since $p' \in (1 + \frac{\sqrt{3}}{3}, 2)$ by assumption $p \in (2, 1 + \sqrt{3})$, there is a constant $\delta > 0$ such that

$$\mu \equiv \frac{1-p'}{p'}\left(\frac{2(p'-1)}{2-p'} - \delta\right) < -1$$

Therefore by lemma 6, we can uniformly estimate

$$\begin{aligned} \widetilde{b_{k_3+l}} &\leq \widetilde{b_{k_3}} \exp\left(C \sum_{k=k_3}^{\infty} k^{\mu}\right) \\ &\leq C' < \infty \end{aligned}$$

for all $l \in \mathbf{N}$, which contradicts lemma 6. Hence there are infinitely many k such that $\widetilde{b_k}^* > \widetilde{b_k}$. By proposition 1, J has a sequence of critical values which diverges to $+\infty$. (note that lemma 5(ii)). By lemma 5(i), so does I.

(ii) Let $z = 0$. Suppose that there is a constant $k_2 \in \mathbf{N}$ such that $\widetilde{b_k}^* = \widetilde{b_k}$ for all $k \geq k_2$. By lemma 4(ii), $\widetilde{b_{k+1}} \leq \widetilde{b_k} + \beta(|\widetilde{b_k}|^{(p'-1)/p'} + 1)$. Hense since for $k \geq k_3 = \max\{k_1, k_2\}$

$$\widetilde{b_{k+1}} \leq \widetilde{b_k}(1 + C\widetilde{b_k}^{-1/p'}),$$

there holds

$$\widetilde{b_{k_3+l}} \leq \widetilde{b_{k_3}} \exp\left(C \sum_{k=k_3}^{k_3+l-1} \widetilde{b_k}^{-1/p'}\right)$$

for $l \in \mathbf{N}$. But since $p' \in (\sqrt{2}, 2)$ by $p \in (2, 2+\sqrt{2})$, there is a constant $\delta > 0$ such that

$$\mu \equiv -\frac{1}{p'}\left(\frac{2(p'-1)}{2-p'} - \delta\right) < -1$$

which yields the desired contradiction similarly. □

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