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On Duality of Set-Valued Optimization*

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Abstract. An optimization problem which has set-valued objectives and inequality constraints and its dual problems are defined and discussed.

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1 Introduction and Preliminaries

Set-valued optimization is usually interpreted as vector optimization with set-valued objectives, which called set-valued set optimization, and it has been investigated for about twenty years. Against this type of set-valued optimization, set-valued set optimization has been introduced and investigated in [9, 10], recently. These two are different thought their settings are same because criteria of solutions are different. Each optimization has various applications for many fields of mathematics, economics, and so on.

In this paper, we discuss duality theory of set-valued set optimization. Now we mention our setting. Let X be a nonempty set, Y a topological vector space, Z a normed space, K, L pointed solid convex cones of Y, Z, respectively, and $F : X \to 2^Z$, $G : X \to 2^Y$ with $\mathrm{Dom}(F) = \mathrm{Dom}(G) = X$.

Our primal problem (SP) is the following:

$$\text{(SP)} \qquad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & G(x) \cap (-K) \neq \emptyset \end{array}$$

In set-valued vector optimization (see [2, 3, 4, 5, 6, 7, 11, 12, 13]), the aim is to find $x_0 \in S = \{x \in X \mid G(x) \cap (-K) \neq \emptyset\}$ and $y_0 \in F(x_0)$ satisfying

$$y_0 \in \mathrm{Min} \bigcup_{x \in S} F(x).$$

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In this paper, we investigate set-valued set optimization. To define this optimization, we introduce two set relations as follows. For two nonempty sets A and B of Z,

$$A \leq_L^l B \overset{\text{def}}{\Longleftrightarrow} A + L \supset B,$$

$$A \leq_L^u B \overset{\text{def}}{\Longleftrightarrow} A \subset B - L.$$

Using these notations $\leq_L^l$ and $\leq_L^u$, we can define two types of set-valued set optimization problems. In this paper, we use only relation $\leq_L^l$.

2 Duality of Set-Valued Set Optimization

For primal problem (SP), we define our notions of solutions based on set optimization.

Definition 2.1 (Solutions of Primal Problem) An element $x_0 \in X$ is said to be

(i) an l-type feasible solution of (SP) if $G(x) \leq_L^l \theta$;

(ii) an l-type minimal solution of (SP) if x_0 is l-type feasible and

$$x \in X, G(x) \leq_L^l \theta, F(x) \leq_L^l F(x_0) \text{ implies } F(x_0) \leq_L^l F(x).$$

(iii) an l-type weak minimal solution of (SP) if x_0 is l-type feasible and there does not exist $x \in X$ such that

$$G(x) \leq_L^l \theta \text{ and } F(x) \leq_{\text{int}L}^l F(x_0).$$

Next we define dual problems. Let $\mathcal{L}(Y,Z) = \{T : Y \to Z \mid T \text{ is linear}\}$, $\mathcal{L}_+(Y,Z) = \{T \in \mathcal{L}(Y,Z) \mid T(K) \subset L\}$, and $\Phi, \Phi_w : \mathcal{L}(Y,Z) \to 2^Z$ defined by

$$\Phi(T) = \{F(x) + T(y) \mid (x,y) \in \text{Gr}(G) \text{ is a } l\text{-type minimal solution of } (\text{SP}_T)\},$$

$$\Phi_w(T) = \{F(x) + T(y) \mid (x,y) \in \text{Gr}(G) \text{ is a } l\text{-type weak minimal solution of } (\text{SP}_T)\},$$

where

$$(\text{SP}_T) \qquad \begin{array}{ll} \text{Minimize} & F(x) + T(y) \\ \text{subject to} & (x,y) \in \text{Gr}(G) \end{array}$$

for $T \in \mathcal{L}_+(Y,Z)$. Now, we set (SD) and (wSD) as follows:

$$(\text{SD}) \qquad \begin{array}{ll} \text{Maximize} & \Phi(T) \\ \text{subject to} & T \in \mathcal{L}_+(Y,Z) \end{array}$$

$$(w\text{SD}) \qquad \begin{array}{ll} \text{Maximize} & \Phi_w(T) \\ \text{subject to} & T \in \mathcal{L}_+(Y,Z) \end{array}$$

Definition 2.2 (Solutions of Dual Problem) An element $T_0 \in \mathcal{L}(Y,Z)$ is said to be

(i) an l-type feasible solution of (SD) if

$$T_0 \in \mathcal{L}_+(Y,Z) \text{ and } \Phi(T) \neq \emptyset;$$

(ii) an l-type maximal solution of (SD) if T_0 is feasible and there exists $A_0 \in \Phi(T_0)$ such that

$$T_1 \in \mathcal{L}_+(Y,Z), A_1 \in \Phi(T_1), A_0 \leq^l_L A_1 \text{ imply } A_1 \leq^l_L A_0$$

(iii) an l-type weak maximal solution of (wSD) if T_0 is feasible and there exists $A_0 \in \Phi_w(T_0)$ such that

$$\text{there do not exist } T_1 \in \mathcal{L}_+(Y,Z),\ A_1 \in \Phi_w(T_1) \text{ such that } A_0 \leq^l_{\text{int}L} A_1.$$

Proposition 2.1 (Weak Duality)
Let x_0 be an l-type feasible solution of (SP), T_1 an l-type feasible solution of (SD), and (x_1, y_1) an element of $\text{Gr}(G)$ satisfying $F(x_1) + T_1(y_1) \in \Phi(T_1)$. Then,

$$F(x_0) \leq^l_L F(x_1) + T_1(y_1) \text{ imply } F(x_1) + T_1(y_1) \leq^l_L F(x_0).$$

Now we have one of main theorems of this paper.

Theorem 2.1 (Strong Duality)
Let the following assumptions are satisfied:

(H1): F is nonempty compact convex values;

(H2): for each $x_1,\ x_2 \in X,\ y_1 \in G(x_1),\ y_2 \in G(x_2),\ \lambda \in (0,1)$, there exists $(x,y) \in \text{Gr}(G)$ such that

$$\begin{cases} F(x) \leq^l_L (1-\lambda)F(x_1) + \lambda F(x_2) \\ y \leq_K (1-\lambda)y_1 + \lambda y_2 \end{cases}$$

(H3): Slater condition: there is $x' \in X$ such that $G(x') \cap (-\text{int}K) \neq \emptyset$.

Then for each minimal solution x_0 of (SP), there exist $y_0^* \in K^+ \setminus \{\theta\}$ and $\mu : \text{int}L \to (0, \infty)$ such that the following is satisfied:

(i) $1/\mu$ is affine on $\text{int}L$

(ii) for each $a \in \text{int}L$, there does not exist $(x,y) \in \text{Gr}(G)$ such that

$$F(x) + T_a(y) \leq^l_{\text{int}L} F(x_0)$$

where $T_a(y) = \langle y_0^*, y \rangle \mu(a) a,\ y \in Y$.

3 Lagrangian Duality of Set-Valued Set Optimization

In this paper, we define Lagrangian set-valued map $L : X \times Y \times \mathcal{L}(Y, Z) \to 2^Z$ as

$$L(x, y, T) = F(x) + T(y)$$

for $x \in X$, $y \in Y$, $T \in \mathcal{L}(Y, Z)$, and we define concepts of saddle point the following definition. Such definitions are different from ordinary definitions in set-valued vector optimization.

Definition 3.1 (Saddle Point)
A triple $(x_0, y_0, T_0) \in \mathrm{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is said to be an l-type saddle point of L if the following two conditions (i) and (ii) are satisfied:

(i) $L(x, y, T_0) \leq_L^l L(x_0, y_0, T_0), (x, y) \in \mathrm{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq_L^l L(x, y, T_0)$;
(ii) $L(x_0, y_0, T_0) \leq_L^l L(x_0, y_0, T), T \in \mathcal{L}_+(Y, Z) \Rightarrow L(x_0, y_0, T) \leq_L^l L(x_0, y_0, T_0)$.

Definition 3.2 (Weak Saddle Point)
A triple $(x_0, y_0, T_0) \in \mathrm{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is said to be an l-type weak saddle point of L if the following two conditions (i) and (ii) are satisfied:

(i) there does not $(x, y) \in \mathrm{Gr}(G)$ such that $L(x, y, T_0) \leq_{\mathrm{int}L}^l L(x_0, y_0, T_0)$;
(ii) there does not $T \in \mathcal{L}_+(Y, Z)$ such that $L(x_0, y_0, T_0) \leq_{\mathrm{int}L}^l L(x_0, y_0, T)$.

Note that a triple (x_0, y_0, T_0) satisfies (i) of Definition 3.1 if and only if (x_0, y_0) is an l-type minimal solution of (SP_T), or equivalently, $L(x_0, y_0, T_0) \in \Phi(T_0)$, and (i) of Definition 3.2 if and only if (x_0, y_0) is an l-type weak minimal solution of (SP_T), or equivalently, $L(x_0, y_0, T_0) \in \Phi_w(T_0)$.

Theorem 3.1 Assume that K is closed, L is solid, and F satisfies the following bounded condition: for each $x \in \mathrm{Dom}(F)$ there exists $y^* \in K^+$ such that

- $\langle y^*, y \rangle > 0$ for each $y \in K \setminus \{\theta\}$;
- $\inf_{y \in F(x)} \langle y^*, y \rangle > -\infty$.

If $(x_0, y_0, T_0) \in \mathrm{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is an l-type saddle point of L, then we have

(i) $y_0 \leq \theta$ and $T_0(y_0) = \theta$;
(ii) x_0 is an l-type minimal solution of (SP);
(iii) T_0 is an l-type maximal solution of (SD).

Corollary 3.1 Let the same assumption of Theorem 3.1 is fulfilled. Then, $(x_0, y_0, T_0) \in \mathrm{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is an l-type saddle point of L if and only if

(i) $L(x, y, T_0) \leq_L^l L(x_0, y_0, T_0), (x, y) \in \mathrm{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq_L^l L(x, y, T_0)$

(ii) $y_0 \leq \theta$ and $T_0(y_0) = \theta$.

Theorem 3.2 Let the assumptions of Theorem 2.1 is satisfied. Then for each minimal solution x_0 of (SP), there exist $y_0^* \in K^+ \setminus \{\theta\}$ and $\mu : \mathrm{int}L \to (0, \infty)$ such that the following is satisfied:

(i) $1/\mu$ is affine on $\mathrm{int}L$

(ii) for each $a \in \mathrm{int}L$, (x_0, y_0, T_a) is a weak saddle point of L for each $y_0 \in G(x_0) \cap (-K)$, where $T_a(y) = \langle y_0^*, y \rangle \mu(a) a$, $y \in Y$.

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