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NONLINEAR ERGODIC THEOREMS FOR ALMOST NONEXPANSIVE CURVES

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a subset of H. Then, a mapping T of C into itself is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for all $x,y\in C$. We denote by $F(T)$ the set of fixed points of T.

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [2]: Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If for some $x_0\in C$, $\{T^nx_0 : n\in\mathbb{N}\}$ is bounded, then for each $x\in C$, the Cesàro means

$$S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}T^kx$$

converge weakly to some $y\in F(T)$. In Baillon's theorem, putting $y=Px$ for each $x\in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT^n=T^nP=P$ for all positive integers n and $Px\in\overline{co}\{T^nx : n=1,2,\dots\}$ for each $x\in C$, where $\overline{co}A$ is the closure of the convex hull of A. Takahashi [22, 23] proved the existence of such retractions, "ergodic retractions", for noncommutative semigroups of nonexpansive mappings in a Hilbert space. Rodé [19] found a sequence of means on the semigroup, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic retraction onto the set of common fixed points. Recently Takahashi [25] proved a nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings without convexity in a Hilbert space. On the other hand, Miyadera and Kobayasi [17] introduced the notion of almost-orbits of a one-parameter nonexpansive semigroup on C and studied weak and strong convergence theorems of such almost-orbits (see also [6, 7]). Then,

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Rouhani [20, 21] introduced the notion of almost nonexpansive sequences and curves in a Hilbert space and proved weak and strong convergence theorems for such sequences and curves. Kada and Takahashi [12] introduced the notion of almost nonexpansive curves over a commutative semigroup. They studied the asymptotic behavior of such almost nonexpansive curves over a commutative semigroup.

In this article, we recall the notion of almost nonexpansive sequences and curves over a commutative semigroup and nonlinear ergodic theorems for such sequences and curves. Further, we introduce the notion of almost nonexpansive curves over a noncommutative semigroup and for any almost nonexpansive curve u, consider generalized fixed point set $F(u)$. Then, we prove nonlinear ergodic theorems for almost nonexpansive curves over a right reversible semitopological semigroup.

2. Theorems for Nonexpansive sequences and curves

Throughout this article, we assume that C is a nonempty closed convex subset of a real Hilbert space H. We also assume that D is a subspace of $B(S)$ containing constants unless other specified. We write $x_n \rightharpoonup x$ (or $w\text{-}\lim_n x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x. Similarly $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) and $x_n \overset{w^*}{\to} x$ (or $w^*\text{-}\lim_{n\to\infty} x_n = x$) will symbolize strong convergence and w^*-convergence, respectively. We denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ the set of all real numbers, nonnegative real numbers and nonnegative integer, respectively. For a subset A of H, coA and $\overline{co}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [2]:

Theorem 2.1 ([2]). *Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If for some $x_0 \in C$, $\{T^n x_0 : n \in \mathbb{N}\}$ is bounded, then for each $x \in C$, the Cesàro means*

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to a fixed point of T.

Many mathematicians obtained generalizations of Baillon's result [2] (for example, see, [17, 19, 22, 23, 25]). Among other things, by modifying the method used by B.D. Rouhani and S. Kakutani (" Ergodic theorems for nonexpansive nonlinear operators in a Hilbert space", preprint, 1984) and B.D. Rouhani ("Ergodic theorems for nonexpansive sequences in Hilbert spaces and related problems", Part I, Thesis, Yale University, and " A new proof of the weak convergence theorems for nonexpansive sequence and curves in Hilbert spaces," preprint, 1984), Rouhani [20, 21] introduced the notion of almost nonexpansive sequences and curves in a Hilbert space and studied nonlinear ergodic theorems for such sequences and curves. Let $\{x_n\}$ be a sequence in H. Then, $\{x_n\}$ is called an almost nonexpansive curve if there exists a nonnegative real-valued function $\varepsilon(\cdot,\cdot)$ on $\mathbb{N}\times\mathbb{N}$ such that

$$\|x_{i+k}-x_{j+k}\|^2 \le \|x_i-x_j\|^2+\varepsilon(i,j)$$

for every i,j and k in $\mathbb{N}$ and $\lim_{i,j}\varepsilon(i,j)=0$. In the case when $\varepsilon(s,t)=0$ for every $i,j\in\mathbb{N}$, $\{x_n\}$ is called a nonexpansive sequence (see [20]).

Remark 2.2. Let $\{x_n\}$ be a bounded sequence in H such that

$$\|x_{i+k}-x_{j+k}\| \le \|x_i-x_j\|+\varepsilon_1(s,t)$$

for every i,j and k in $\mathbb{N}$ and $\lim_{i,j}\varepsilon_1(i,j)=0$. Then, it is obvious that $\{x_n\}$ is an almost nonexpansive sequence curve with $\varepsilon(i,j)=4(\sup_{i\in\mathbf{N}}\|x_i\|)\varepsilon_1(i,j)+\varepsilon_1(i,j)^2$ (see also [20, 21]).

A sequence $\{x_n\}$ in H is called an almost-orbit of T if

$$\lim_k \sup_{n\ge 0}\|x_{n+k}-T^n x_k\|=0$$

(see [6]).

Example 2.3. Let T be a nonexpansive mapping from a closed convex subset C of H into itself. If $\{x_n\}$ is a bounded almost-orbit of T, from Remark 2.2, $\{x_n\}$ is an almost nonexpansive sequence in H. Hence, we also see that for $x\in C$, $\{T^n x\}$ is an almost nonexpansive curve from $\mathbb{R}^+$ to C if $\{T^n x\}$ is bounded (see also [20]).

Let $\{x_n\}$ be a sequence H. Then, we denote the subsets F_1 and F of H as follows: $q\in F_1$ if and only if $\|x_{i+k}-q\|\le\|x_i-q\|$ for every $i,k\in S$ and $q\in F$ if and only if $\lim_n\|x_n-q\|$ exists. We can prove that F_1 and F are closed convex subset of H and

$F_1 \subset F$ (see [20]). Rouhani [20] obtained the following nonlinear ergodic theorem for an almost nonexpansive sequence which is a generalization of Baillon's result [2]:

Theorem 2.4 ([20]). *Let $\{x_n\}$ be a bounded almost nonexpansive sequence in H. Then, $\{\frac{1}{n}\sum_{i=0}^{n-1} x_{i+k}\}$ converges weakly to $z_0 \in F$ as $n \to \infty$ uniformly in $k \in \mathbb{R}^+$. Further, z_0 is a $\overline{\lim}$-asymptotic center of $\{x_n\}$ in H, i.e., $z_0 \in \{z \in H : \overline{\lim}\text{-}\|x_n - z\| = \inf_{y \in H} \overline{\lim}\text{-}\|x_n - y\|\}$.*

We do not know whether Theorem 2.4 would hold in the case when H is a Banach space.

A family $\{T(s) : s \in \mathbb{R}^+\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(a) $s \mapsto T(s)x$ is continuous for all $x \in C$;
(b) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;
(c) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$;
(d) $T(0) = I$.

Baillon [3] proved a nonlinear ergodic theorem for a one-parameter nonexpansive semigroup in a Hilbert space:

Theorem 2.5 ([3]). *Let $\{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C. If for some $x_0 \in C$, $\{T(t)x_0 : t \in \mathbb{R}^+\}$ is bounded, then for any $x \in C$, $\{\frac{1}{t}\int_0^t T(s)x\,ds\}$ converges weakly to a fixed point of T.*

Rouhani [20, 21] also introduced the notion of almost nonexpansive curve in a Hilbert space and studied a nonlinear ergodic theorem for such a curve which is a generalization of Baillon's result [3].

Let u be a function from $\mathbb{R}^+$ into H. Then, u is called an almost nonexpansive curve if there exists a nonnegative real-valued function $\varepsilon(\cdot,\cdot)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $\|u(h+s) - u(h+t)\|^2 \leq \|u(s) - u(t)\|^2 + \varepsilon(s,t)$ for every s, t and h in $\mathbb{R}^+$ and $\lim_{s,t} \varepsilon(s,t) = 0$. In the case when $\varepsilon(s,t) = 0$ for every $s, t \in S$, u is called a nonexpansive curve (see [20]).

Remark 2.6. Let u be a bounded function from $\mathbb{R}^+$ into H such that

$$\|u(h+s) - u(h+t)\| \leq \|u(s) - u(t)\| + \varepsilon_1(s,t)$$

for every s,t and h in $\mathbb{R}^+$ and $\lim_{s,t}\varepsilon_1(s,t)=0$. Then, it is obvious that u is an almost nonexpansive curve with $\varepsilon(s,t)=4(\sup_{r\in S}\|u(r)\|)\varepsilon_1(s,t)+\varepsilon_1(s,t)^2$ (see also [20, 21]).

A continuous function u from $\mathbb{R}^+$ into C is called an almost-orbit of $\mathfrak{S}=\{T(t):t\in\mathbb{R}^+\}$ if

$$\lim_s \sup_t \|u(t+s)-T(t)u(s)\|=0$$

(see [17]).

Example 2.7. Let $\{T(s):s\in\mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C. If u is a bounded almost-orbit of $\{T(s):s\in\mathbb{R}^+\}$, from Remark 2.6, u is an almost nonexpansive curve from $\mathbb{R}^+$ to C. Hence, we also see that for $x\in C$, $\{T(t)x:t\in\mathbb{R}^+\}$ is an almost nonexpansive curve from $\mathbb{R}^+$ to C if $\{T(t)x:t\in\mathbb{R}^+\}$ is bounded (see also [20]).

Let u be a function from S into H. Then, we denote the subsets $F_1(u)$ and $F(u)$ of H as follows: $q\in F_1(u)$ if and only if $\|u(h+s)-q\|\le\|u(s)-q\|$ for every $h,s\in\mathbb{R}^+$ and $q\in F(u)$ if and only if $\lim_s\|u(s)-q\|$ exists. We can prove that $F_1(u)$ and $F(u)$ are closed convex subset of H and $F_1(u)\subset F(u)$ (see [20, 21]). Rouhani [20] proved the following nonlinear ergodic theorem for an almost nonexpansive curve which is a generalization of Baillon's result [3]:

Theorem 2.8 ([20]). *Let $\{u(s):s\in\mathbb{R}^+\}$ be a bounded continuous almost nonexpansive curve in H. Then, $\{\frac{1}{t}\int_0^t u(s+k)ds\}$ converges weakly to $z_0\in F(u)$ as $t\to\infty$ uniformly in $k\in\mathbb{R}^+$. Further, z_0 is a $\overline{\lim}$-asymptotic center of $u(\cdot)$ in H, i.e., $z_0\in\{z\in H:\overline{\lim}\text{-}\|u(t)-z\|=\inf_{y\in C}\overline{\lim}\text{-}\|u(t)-z\|\}$.*

We do not know whether Theorem 2.8 would hold in the case when H is a Banach space.

3. Theorems for commutative semigroups

In this section, we prove nonlinear ergodic theorems for almost nonexpansive curves over a commutative semigroup. At first, we state some definitions and notations.
Let S be a semitopological semigroup with identity, i.e., a semigroup with a Hausdorff

topology such that for each $t \in S$, the mappings $s \mapsto s \cdot t$ and $s \mapsto t \cdot s$ from S into itself are continuous. Then, S is called right reversible if any two closed left ideals of S have non-void intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$"on S is defined by $a \leq b$ if and only if $\overline{Sa} \supseteq \overline{Sb}, a, b \in S$. Right reversible semitopological semigroups include all commutative semigroups (see [11]).

Throughout this section, we assume that S is a right reversible semitopological semigroup with identity and D is a subspace of $B(S)$ containing constants which is r_s and l_s-invariant for each $s \in S$ unless other specified. We introduce the notion of almost nonexpansive curves over a noncommutative semigroup. Let u be a function from S into H. Then, u is called an almost nonexpansive curve if there exists a nonnegative real-valued function $\varepsilon(\cdot, \cdot)$ on $S \times S$ such that

$$\|u(hs) - u(ht)\|^2 \leq \|u(s) - u(t)\|^2 + \varepsilon(s,t)$$

for every s, t and h in S and $\lim_{s,t} \varepsilon(s,t) = 0$. In the case when $\varepsilon(s,t) = 0$ for every $s, t \in S$, u is called a nonexpansive curve (see [1, 12]).

Remark 3.1. Let u be a bounded function from S into H such that

$$\|u(hs) - u(ht)\| \leq \|u(s) - u(t)\| + \varepsilon_1(s,t)$$

for every s, t and h in S and $\lim_{s,t} \varepsilon_1(s,t) = 0$. Then, it is obvious that u is an almost nonexpansive curve with $\varepsilon(s,t) = 4(\sup_{r \in S} \|u(r)\|)\varepsilon_1(s,t) + \varepsilon_1(s,t)^2$ (see also [1, 12]).

A family $\mathfrak{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(a) $s \mapsto T(s)x$ is continuous for all $x \in C$;

(b) $T(st) = T(s)T(t)$ for all $s, t \in S$;

(c) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathfrak{S})$ the set of common fixed points of $T(t), t \in S$, that is, $F(\mathfrak{S}) = \bigcap_{t \in S} F(T(t))$. A continuous function u from S into C is called an almost-orbit of $\{T(t) : t \in S\}$ if

$$\lim_s \sup_t \|u(ts) - T(t)u(s)\| = 0$$

(see [26, 27]).

Example 3.2. Let $\mathfrak{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C. If u is a bounded almost-orbit of $\mathfrak{S}$, from Remark 3.1, u is an almost nonexpansive curve from S to C. Hence, we also see that for $x \in C$, $\{T(t)x : t \in S\}$ is an almost nonexpansive curve from S to C if $\{T(t)x : t \in S\}$ is bounded (see also [12]).

Let u be a function from S into H. Then, we denote the subsets $F_1(u)$ and $F(u)$ of H as follows: $q \in F_1(u)$ if and only if $\|u(hs) - q\| \leq \|u(s) - q\|$ for every $h, s \in S$ and $q \in F(u)$ if and only if $\lim_s \|u(s) - q\|$ exists (see [1, 12]).

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. Then, for each $s \in S$ and $f \in B(S)$, we can define elements $r_s f \in B(S)$ and $l_s f \in B(S)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for all $t \in S$, respectively. We also denote by r_s^* and l_s^* the conjugate operators of r_s and l_s, respectively. Let D be a subspace of $B(S)$ and let μ be an element of D^*, where D^* is the dual space of D. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. Sometimes, $\mu(f)$ will be denoted by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$. When D contains constants, a linear functional μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. We also know that μ is a mean on D if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in D$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on any subspace D of $B(S)$ containing constants. Further, let D be a subspace of $B(S)$ containing constants which is r_s-invariant i.e., $r_s D \subset D$ for each $s \in S$. Then, a mean μ on D is called right invariant if

$$\mu(r_s f) = \mu(f)$$

for all $s \in S$ and $f \in D$. Similarly, we can define a left invariant mean on an l_s-invariant subspace of $B(S)$ containing constants. A right and left invariant mean is called an invariant mean. We also denote by $C(S)$ the set of all bounded continuous real-valued functions on S.

The following definition which was introduced by Takahashi [22] is crucial in nonlinear ergodic theory for abstract semigroups. Let u be a bounded function from S into H such that $\langle u(\cdot), y\rangle \in D$ for every $y \in H$. Let μ be an element of D^*. Then, there exists a unique

element $u_\mu \in H$ such that $\langle u_\mu, y\rangle = \mu_s\langle u(s), y\rangle$ for all $y \in H$. If μ is a mean on D, then u_μ is contained in $\overline{co}\{u(t) : t \in S\}$ (for example, see [13, 14, 22]). Sometimes, u_μ will be denoted by $\int u(t)d\mu(t)$.

Lemma 3.3. *Suppose that D has an invariant mean μ. Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. Then,* (i), (ii) *and* (iii) *hold.*

(i) *$F(u)$ and $F_1(u)$ are closed convex subsets of H;*

(ii) $F_1(u) \subset F(u)$;

(iii) $u_\mu \in F(u)$.

Let $\{\mu_\alpha : \alpha \in I\}$ be a net of means on D. Then, $\{\mu_\alpha : \alpha \in I\}$ is said to be asymptotically invariant if

$$\mu_\alpha(f) - \mu_\alpha(r_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$$

for every $s \in S$ and $f \in D$ (see [19]). Let $\{\lambda_\alpha : \alpha \in I\}$ be a net of continuous linear functionals on D. Then, $\{\lambda_\alpha : \alpha \in I\}$ is said to be left strongly regular if the following conditions are satisfied:

(a) $\sup_\alpha \|\lambda_\alpha\| < \infty$;

(b) $\lim_\alpha \lambda_\alpha(1) = 1$;

(c) $\lim_\alpha \|\lambda_\alpha - l_s^*\lambda_\alpha\| = 0$ for every $s \in S$.

Right strong regularity is defined similarly. A strongly regular net is a left and right strongly regular net (see [10]).

Let u be a bounded function from S into C such that for any $x \in C, \|u(\cdot) - x\|^2 \in D$. Then, for a mean μ on D, the set μ-$AC(u, C)$ defined by

$$\mu\text{-}AC(u,C) = \{x \in C : \mu_s\|u(s) - x\|^2 = \inf_{y\in C} \mu_s\|u(s) - y\|^2\}$$

is called the μ-asymptotic center of u in C (see also [9, 12, 15, 18]). Similarly, the set $\overline{\lim}$-$AC(u, C)$ defined by $\overline{\lim}$-$AC(u,C) = \{x \in C : \mu_s\|u(s) - x\|^2 = \inf_{y\in C} \mu_s\|u(s) - y\|^2\}$ is called the $\overline{\lim}$-asymptotic center of u in C.

Kada and Takahashi [12] proved nonlinear ergodic theorems for almost nonexpansive curves over a commutative semigroup which are generalizations of Rouhani's results [20, 21]:

Theorem 3.4 ([12]). *Let S be a commutative semigroup with a identity and let D be a subspace of $B(S)$ containing constants which is r_s-invariant for each $s \in S$. Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. If $\{\mu_\alpha : \alpha \in I\}$ is an asymptotically invariant net of means on D, then $\{\int u(t)d\mu_\alpha(t)\}$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$. Further, $y_0 = u_\mu$ and $\overline{\lim}$-$AC(u,H) = \mu$-$AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D.*

If $\{\mu_\alpha : \alpha \in I\}$ is strongly regular net, the convergence is uniform.

Theorem 3.5 ([12]). *Let S be as in Theorem 3.4. Assume that there exists a net $\{\lambda_\beta : \beta \in J\}$ of finite means on S such that $\lim_\beta \|\lambda_\beta - l_s^*\lambda_\beta\| = \lim_\beta \|\lambda_\beta - r_s^*\lambda_\beta\| = 0$ for every $s \in S$. Let u and D be as in Theorem 3.4. Let $\{\mu_\alpha : \alpha \in I\}$ be a strongly regular net of continuous linear functionals on D. Then, $\{\int u(th)d\mu_\alpha(t)\}$ and $\{\int u(ht)d\mu_\alpha(t)\}$ converge weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$. Further, $y_0 = u_\mu$ and $\overline{\lim}$-$AC(u,H) = \mu$-$AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D.*

By using Theorem 3.5, Theorems 2.4 and 2.8 can be proved (see [12]). We do not know whether Theorems 3.4 and 3.5 would hold in the case when H is a Banach space.

4. Theorems for noncommutative semigroups

In this section, we prove nonlinear ergodic theorems for almost nonexpansive curves over a noncommutative semigroup. Throughout this section, we assume that S is a right reversible semitopological semigroup with identity and D is a subspace of $B(S)$ containing constants which is r_s and l_s-invariant for each $s \in S$ unless other specified. We denote by $\Lambda(S)$ the algebraic center of S, i.e., all $s \in S$ such that $st = ts$ for all $t \in S$.

Theorem 4.1 ([1]). *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. If $\{\mu_\alpha : \alpha \in I\}$ is an asymptotically invariant net of means on D, then $\{\int u(t)d\mu_\alpha(t)\}$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$. Further, $y_0 = u_\mu$ and $\overline{\lim}$-$AC(u,H) = \mu$-$AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D.*

We consider the case when $\{\mu_\alpha : \alpha \in I\}$ is strongly regular. Then, we obtain the following theorem:

Theorem 4.2 ([1]). *Assume that there exists a net $\{\lambda_\beta : \beta \in J\}$ of finite means on S such that $\lim_\beta \|\lambda_\beta - l_s^*\lambda_\beta\| = \lim_\beta \|\lambda_\beta - r_s^*\lambda_\beta\| = 0$ for every $s \in S$. Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. Let $\{\mu_\alpha : \alpha \in I\}$ be a strongly regular net of continuous linear functionals on D. Then, $\{\int u(th)d\mu_\alpha(t)\}$ and $\{\int u(ht)d\mu_\alpha(t)\}$ converge weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$. Further, $y_0 = u_\mu$ and $\overline{\lim}\text{-}AC(u,H) = \mu\text{-}AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D.*

To prove Theorems 4.1 and 4.2, we need the following lemmas and theorem.

The following lemma is a modification of [25] (see also [12]).

Lemma 4.3 ([1]). *Assume that D has an invariant mean μ. Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. Then,*

$$\overline{\lim}\text{-}AC(u,H) = \mu\text{-}AC(u,H) = \{u_\mu\},$$

where $\overline{\lim}\text{-}AC(u,H) = \{x \in H : \overline{\lim}_s \|u(s) - x\|^2 = \inf_{y\in H} \overline{\lim}_s \|u(s) - y\|^2\}$. Consequently, if μ and λ are invariant means on D, then $u_\mu = \mu_\lambda$.

The following theorem plays an important role in the proofs of Theorems 4.1 and 4.2 (see also [12]).

Theorem 4.4 ([1]). *Assume that D has an invariant mean μ. Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in D for all $y \in H$ and $s \in S$. Then, $F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} = \{u_\mu\}$.*

The following lemma is essential to prove Theorem 4.2.

Lemma 4.5 ([1]). *Let u be a bounded almost nonexpansive curve from S to H with $\varepsilon(\cdot,\cdot)$. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of finite means on S such that*

$$\lim_\alpha \|\mu_\alpha - l_s^*\mu_\alpha\| = \lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0 \quad \text{for every} \quad s \in S. \qquad (*)$$

Then, $\{\int u(th)d\mu_\alpha(t)\}$ *converges weakly to* $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ *uniformly in* $h \in \Lambda(S)$. *Further,* $y_0 = u_\mu$ *and* $\overline{\lim}$*-*$AC(u,H) = \mu$*-*$AC(u,H) = \{u_\mu\}$ *for every invariant mean* μ *on* D.

Sketch of the proof of Lemma 4.5. Let $\{\mu_\alpha : \alpha \in A\}$ and $\{\lambda_\beta : \beta \in B\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - l_s^*\mu_\alpha\| = \lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - l_s^*\lambda_\beta\| = \lim_\beta \|\lambda_\beta - r_s^*\lambda_\beta\| = 0$$

for every $s \in S$. Define $(\beta_1,\gamma_1) \leq (\beta_2,\gamma_2)$ if and only if $\beta_1 \leq \beta_2$ and $\gamma_1 \leq \gamma_2$. Let $\{p_{\beta,\gamma} : (\beta,\gamma) \in B \times B\}$ be a net in S.

We show that $\{\int\int u(tp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$. From Lemma 4.4, it is sufficient to show that all weak limit points of subnets of the net $\{\int\int u(tp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ are in $\bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} \cap F(u)$. Put $M = \sup_{t\in S} \|u(t)\|$. Since $\{\int\int u(tp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ is bounded, there is a subnet $\{\int\int u(tp_{\beta',\gamma'}q)d\lambda_{\beta'}(t)d\lambda_{\gamma'}(q)\}$ of $\{\int\int u(tp_{\beta,\gamma}q)d\lambda_\beta(t)\lambda_\gamma(q)\}$ such that

$$\iint u(tp_{\beta',\gamma'}q)d\,\lambda_{\beta'}(t)d\lambda_{\gamma'}(q) \rightharpoonup y_0 \in H.$$

Then, we have that for any $a \in S$,

$$\iint u(tp_{\beta',\gamma'}qa)d\lambda_{\beta'}(t)d\lambda_{\gamma'}(q) \rightharpoonup y_0 \in H. \tag{1}$$

We obtain $y_0 \in F(u)$. Indeed, let $\varepsilon > 0$. Then, there exists $t_0 \in S$ such that $\varepsilon(s,t) < \varepsilon$ for all $t \geq t_0$ and $s \geq t_0$. Let $s \geq t_0$ and $h \in S$. Then, we can show that

$$\|u(hs) - y_0\|^2 - \|u(s) - y_0\|^2 - 2\left\langle u(hs) - u(s), \iint u(tp_{\beta',\gamma'}qt_0)d\lambda_{\beta'}(t)d\lambda_{\gamma'}(q) - y_0 \right\rangle$$
$$< \varepsilon + 4M^2\|\lambda_{\beta'} - l_h^*\lambda_{\beta'}\| \cdot \|\lambda_{\gamma'}\|.$$

So, it follows from (1) that $\lim_s \|u(s) - y_0\|$ exists. This implies $y_0 \in F(u)$.

From the separation theorem, we obtain $y_0 \in \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ and hence $y_0 \in \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} \cap F(u)$. This implies that all weak limit points of subnets of the net $\{\int\int u(tp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ are in $\bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} \cap F(u)$.

Next, we prove that $\{\int u(sh)d\mu_\alpha(s)\}$ converges weakly to y_0 uniformly in $h \in \Lambda(S)$. Since $\{p_{\beta,\gamma} : (\beta,\gamma) \in B \times B\}$ is arbitrary, we see that $\{\iint u(thp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ converges weakly to y_0 uniformly in $h \in S$. Then, there exists $(\beta_0,\gamma_1) \in B \times B$ such that

$$\left|\iint \langle u(thp_{\beta,\gamma}q), x\rangle\, d\lambda_\beta(t)d\lambda_\gamma(q) - \langle y_0, x\rangle\right| < \frac{\varepsilon}{3} \tag{2}$$

for every $\beta \geq \beta_0, \gamma \geq \gamma_1$ and $h \in S$. So, since $\{\mu_\alpha\}$ satisfies $(*)$, we can show that $\{\int u(sh)d\mu_\alpha(s)\}$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$. From Theorem 4.4 and Lemma 4.3, $y_0 = u_\mu$ and $\overline{\lim}\text{-}AC(u,H) = \mu\text{-}AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D. $\square$

We can prove the following lemma as in the proof of Lemma 4.5.

Lemma 4.6 ([1]). *Let S, D, u and $\{\mu_\alpha : \alpha \in A\}$ be as in Lemma 4.5. Then, $\{\int u(ht)d\mu_\alpha(t)\}$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$. Further, $y_0 = u_\mu$ and $\overline{\lim}\text{-}AC(u,H) = \mu\text{-}AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D.*

Now, we can prove the nonlinear ergodic theorems (Theorems 4.1 and 4.2).

Sketch of the proof of Theorem 4.1. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on D. Since $\{\int u(t)d\mu_\alpha(t)\}$ is bounded, $\{\int u(t)d\mu_\alpha(t)\}$ must contain a subnet which converges weakly to a point in H. So, let $\{\int u(t)d\mu_{\alpha_\beta}(t)\}$ be a subnet of $\{\int u(t)d\mu_\alpha(t)\}$ such that

$$\int u(t)d\mu_{\alpha_\beta}(t) \rightharpoonup z_0. \tag{3}$$

Let $B_1(D^*)$ be the closed unit ball of D^*. Since $\{\mu_{\alpha_\beta}\} \subset B_1(D^*)$, there exists a subnet $\{\mu_{\alpha_{\beta_\gamma}}\}$ of $\{\mu_{\alpha_\beta}\}$ such that

$$\mu_{\alpha_{\beta_\gamma}} \overset{w^*}{\to} \mu.$$

Then, we can show that μ is an invariant mean on D. Since $\mu_{\alpha_{\beta_\gamma}} \overset{w^*}{\to} \mu$, for any $x \in H$,

$$\int \langle u(t), x\rangle d\mu_{\alpha_{\beta_\gamma}}(t) \to \int \langle u(t), x\rangle d\mu(t) = \langle u_\mu, x\rangle.$$

Then, from (3), we have that $\int u(t)d\mu_{\alpha_\beta}(t) \rightharpoonup z_0$ and $z_0 = u_\mu$. From Lemma 4.3, if λ and μ are invariant means on D, then $u_\mu = u_\lambda$. Therefore, since $\{\int u(t)d\mu_{\alpha_\beta}(t)\}$ is arbitrary,

$\{\int u(t)d\mu_\alpha(t)\}$ converges weakly to u_μ. Furthermore, $\{u_\mu\} = F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} = \mu\text{-}AC(u,H) = \overline{\lim}\text{-}AC(u,H)$. $\square$

Sketch of the proof of Theorem 4.2. Let μ be an invariant mean on D and let $\{p_{\beta,\gamma} : (\beta,\gamma) \in J \times J\}$ be a net in S. From Lemma 4.5, we have that $\{\iint u(thp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q)\}$ converges weakly to u_μ uniformly in $h \in S$. We also know $F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\} = \overline{\lim}\text{-}AC(u,H) = \mu\text{-}AC(u,H) = \{u_\mu\}$ for every invariant mean μ on D. Let $x \in H, \varepsilon > 0$. Then, there exists $(\beta_0,\gamma_1) \in J \times J$ such that

$$\left|\left\langle \iint u(thp_{\beta,\gamma}q)d\lambda_\beta(t)d\lambda_\gamma(q), x\right\rangle - \langle u_\mu, x\rangle\right| < \frac{\varepsilon}{\sup_\alpha \|\mu_\alpha\|}$$

for every $\beta \geq \beta_0, \gamma \geq \gamma_1$ and $h \in S$. Put $\lambda_0 = \lambda_{\beta_0}$, $p_0 = p_{\beta_0,\gamma_1}$ and $\lambda_1 = \lambda_{\gamma_1}$. So, since $\{\mu_\alpha\}$ is strongly regular, from

$$\begin{aligned}
&\left|\int \langle u(sh),x\rangle\, d\mu_\alpha(s) - \langle u_\mu, x\rangle\right| \\
&\leq \left|\int \langle u(sh),x\rangle\, d\mu_\alpha(s) - \iint \langle u(tsh),x\rangle\, d\lambda_0(t)d\mu_\alpha(s)\right| \\
&\quad + \left|\iint \langle u(tsh),x\rangle\, d\lambda_0(t)d\mu_\alpha(s) - \iiint \langle u(tshp_0q),x\rangle\, d\lambda_0(t)d\lambda_1(q)d\mu_\alpha(s)\right| \\
&\quad + \left|\int \left\langle \iint u(tshp_0q)d\lambda_0(t)d\lambda_1(q) - u_\mu, x\right\rangle d\mu_\alpha(s)\right| + \left|\int \langle u_\mu, x\rangle\, d\mu_\alpha(s) - \langle u_\mu, x\rangle\right|,
\end{aligned}$$

we can prove that $\{\int u(sh)d\mu_\alpha(s)\}$ converges weakly to $u_\mu \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$.

As in the above argument, we obtain that $\{\int u(hs)d\mu_\alpha(s)\}$ converges weakly to $u_\mu \in F(u) \cap \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ uniformly in $h \in \Lambda(S)$. $\square$

We do not know whether Theorems in this section would hold in the case when H is a Banach space.

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