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Lagrange Duality of Set-Valued Optimization with Natural Criteria*

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Abstract

Set optimization problems with objective set-valued maps are considered, and some criteria of solutions are defined. Also, cone lower semicontinuities of set-valued maps are introduced, and existence theorems of solutions of such problems are established. Moreover, some duality results of these problems are investigated.

1 Introduction

We observe a set-valued optimization problem (SP) as follows:

(SP) Minimize $F(x)$
subject to $x \in S$

where X is a set, $(Y, \leq)$ an ordered vector space, F a map from X to 2^Y, and S a nonempty subset of $\mathrm{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$. This type of set-valued optimization problem has been developed as a generalization of vector-valued optimization problems for around twenty years. In many paper concerned with set-valued optimization(for example [2, 5, 4, 6, 7, 11]), we can see that a minimal solution $x_0 \in S$ is defined such as:

$$F(x_0) \cap \mathrm{Min} \bigcup_{x \in S} F(x) \neq \emptyset$$

and this problem are often called 'vector optimization with set-valued maps.' However the criterion of solutions is sometimes not suitable for set-valued optimization because it is only based on simple comparisons between vectors though our problem is set-valued optimization.

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Our aim of this paper is to introduce some natural, suitable, and proper criteria based on comparisons between values of the (set-valued) objective map for set-valued optimization, and investigate some properties concerned with the problem. In this paper, we call such criteria based on the idea above natural criteria for set-valued optimization, see [9].

The organization of this paper is as follows: in Section 2, we formulate our set-valued optimization problem and define two types of notions of solutions. In Section 3, we introduce (natural) lower semicontinuities for set-valued maps, characterize such continuities, and derive some existence theorems of solutions by using the lower semicontinuities. Finally, we show some duality theorems for our set-valued optimization in Section 4.

2 Natural Criteria of Set-Valued Optimization

First, we redefine our set-valued minimization problem (SP). Let X be a topological space, $(Y, \leq_K)$ an ordered topological space with an ordering convex cone K, F a map from X to 2^Y, and $S \subset \mathrm{Dom}(F)(= \{x \in X \mid F(x) \neq \emptyset\})$. Our set-valued minimization problem is the following:

$$\text{(SP)} \qquad \text{Minimize} \quad F(x) \qquad \text{subject to} \quad x \in S.$$

To define notions of solutions for our problem, we introduce some relations between two nonempty sets which like the order relation in topological vector spaces; though the number types of such relations is six, we treat two important relations of them, see [8].

In this paper, we define

$$A \leq^l B \overset{\text{def}}{\Longleftrightarrow} A + K \supset B,$$

$$A \leq^u B \overset{\text{def}}{\Longleftrightarrow} A \subset B - K,$$

for nonempty subsets A, B of Y. In these cases, A is said to be smaller than B with l-inequality(resp. u-inequality) if $A \leq^l B$(resp. $A \leq^u B$).

In the above notations, l means *lower* and u means *upper*. Note that $A \leq^l B$ is equivalent to $\mathrm{Min}\, A = \mathrm{Min}\, B$ and $A \leq^u B$ is equivalent to $\mathrm{Max}\, A = \mathrm{Max}\, B$.

By using the set relations above, we introduce two types criteria of minimal solutions in the following definition. In this paper, when we consider l-minimal solution, we assume that F is l-closed map, that is $F(x)$ is l-closed for each $x \in X$ for simple consideration. Also we assume similar assumption, u-closedness of F, when we consider u-minimal solution.

Definition 2.1 [9] An element $x_0 \in S$ is said to be

(i) *l-minimal solution* of (SP) if
for any $x \in S$ with $F(x) \leq^l F(x_0)$, $F(x_0) \leq^l F(x)$ is satisfied;

(ii) *u-minimal solution* of (SP) if
for any $x \in S$ with $F(x) \leq^u F(x_0)$, $F(x_0) \leq^u F(x)$ is satisfied.

3 Semicontinuities of Set-Values Maps and Existence Theorems

To consider existence of solutions of (SP) for our solutions, some cone semicontinuities were introduced in [5, 9].

Definition 3.1 [9] A set-valued map F is said to be l-type lower semicontinuous on S if for any l-closed subset A of Y, $\mathcal{L}^l(A) = \{x \in S | F(x) \leq^l A\}$ is closed.

Definition 3.2 [9] A set-valued map F is said to be l-type quasi lower semicontinuous at $x_0 \in S$ if for each net $\{x_\lambda\}$ converges to x_0 with $\{F(x_\lambda)\}$ is l-decreasing, that is, $F(x_{\lambda'}) \leq^l F(x_\lambda)$ for $\lambda < \lambda'$, $F(x_0) \leq^l \operatorname{Lim\,sup}_\lambda(F(x_\lambda) + K)$ is satisfied. A set-valued map F is said to be l-type quasi lower semicontinuous on S if it is l-type quasi lower semicontinuous at each point of S.

Definition 3.3 [5] A set-valued map F is said to be upper K-semicontinuous at $x_0 \in S$ if for any open set V with $V \leq^l F(x_0)$, there exists a neighborhood U of x_0 such that $x \in U$ implies $V \leq^l F(x)$; A set-valued map F is said to be upper K-semicontinuous on S if it is upper K-semicontinuous at each point of S.

Now we can see some characterization with respect to these lower semicontinuities.

Proposition 3.1 [9] Let F be a l-closed set-valued map. Then we have the following:

(i) upper K-semicontinuity on S implies l-type lower semicontinuity on S;
(ii) l-type lower semicontinuity on S implies l-type quasi lower semicontinuity on S.

Also, if X and Y are finite dimensional and F is locally bounded, then we have

(iv) l-type lower semicontinuity on S implies upper K-semicontinuity on S.

Moreover, Y is the real-field, and F is a singleton map, then l-type lower semicontinuity and upper K-semicontinuity are equivalent to to the ordinary lower semicontinuity of real-valued functions.

Note that quasi lower semicontinuity is more weaker than another semicontinuities.

Now, we investigate u-type lower semicontinuities of set-valued maps.

Definition 3.4 [9] A set-valued map F is said to be u-type lower semicontinuous on S if for any u-closed subset A of Y, $\mathcal{L}^u(A) = \{x \in S | F(x) \leq^u A\}$ is closed.

Definition 3.5 [9] A set-valued map F is said to be u-type quasi lower semicontinuous at $x_0 \in S$ if for each net $\{x_\lambda\}$ converges to x_0 with $\{F(x_\lambda)\}$ is u-decreasing, that is, $F(x_{\lambda'}) \leq^u F(x_\lambda)$ for $\lambda < \lambda'$, $F(x_0) \leq^u \operatorname{Lim\,sup}_\lambda(F(x_\lambda) + K)$ is satisfied. A set-valued map F is said to be u-type quasi lower semicontinuous on S if it is u-type quasi lower semicontinuous at each point of S.

Definition 3.6 [5] A set-valued map F is said to be lower K-semicontinuous at $x_0 \in S$ if for any open set V with $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 such that $x \in U$ implies $V \cap (F(x) - K) \neq \emptyset$; A set-valued map F is said to be lower K-semicontinuous on S if it is lower K-semicontinuous at each point of S.

Now we can see some characterization with respect to these lower semicontinuities.

Proposition 3.2 [9] Let F be a u-closed set-valued map. Then we have the following:

(i) u-type lower semicontinuity on S implies u-type quasi lower semicontinuity on S.

Also, if X and Y are finite dimensional and F is locally bounded, then we have

(iii) u-type lower semicontinuity on S is equivalent to lower K-semicontinuity on S.

Moreover, Y is the real-field, and F is a singleton map, then u-type lower semicontinuity and lower K-semicontinuity are equivalent to to the ordinary lower semicontinuity of real-valued functions.

Now, we consider existence theorems for l-type and u-type minimal solutions.

Theorem 3.1 [9] Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a l-type quasi lower semicontinuous and l-closed set-valued map, then there exists a l-type minimal solution of (SP).

Theorem 3.2 [9] Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a u-type quasi lower semicontinuous and u-closed set-valued map, then there exists a u-type minimal solution of (SP).

By using one of the above theorems, we can prove the following:

Corollary 3.1 Let X be a topological space, S a nonempty compact subset of X, and $f : S \to 2^Y$ is a lower semicontinuous, then there exists an element $x_0 \in S$ such that $f(x_0) = \inf_{x \in S} f(x)$.

Let Y^* be the topological dual space of Y, θ^* the null vector of Y^*, and K^+ the positive polar cone of K, that is, $K^+ = \{y^* \in Y^* | \langle y^*, k \rangle \geq 0, \forall k \in K\}$.

Theorem 3.3 [9] Let (X, d) be a complete metric space, Y an ordered locally convex space with the cone K. Also, F be a map from X to 2^Y satisfying the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that
  - $\cdot$ $\inf \langle y^*, F(x) \rangle$ is finite for each $x \in S$
  - $\cdot$ $F(x_1) \leq^l F(x_2), x_1, x_2 \in S \Rightarrow \inf \langle y^*, F(x_2) \rangle - \inf \langle y^*, F(x_1) \rangle \geq d(x_2, x_1)$

- $F : S \to 2^Y$ is l-type lower semicontinuous and l-closed.

Then, there exists a l-type minimal solution of (SP).

Theorem 3.4 [9] Let (X, d) be a complete metric space, Y an ordered locally convex space with the cone K. Also, F be a map from X to 2^Y satisfying the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that
  - $\inf \langle y^*, F(x) \rangle$ is finite for each $x \in S$
  - $F(x_1) \leq^u F(x_2), x_1, x_2 \in S \Rightarrow \sup \langle y^*, F(x_2) \rangle - \sup \langle y^*, F(x_1) \rangle \geq d(x_2, x_1)$
- $F : S \to 2^Y$ is u-type lower semicontinuous and u-closed.

Then, there exists a u-type minimal solution of (SP).

Using one of the above theorems, we can show Phelps' theorem, see [1]:

Corollary 3.2 Let $(Y, \|\cdot\|)$ be a Banach space, D a closed nonempty subset of Y, and K a convex cone of Y. If there exist $y^* \in K^+$ and $\alpha > 0$ such that

(i) $\langle y^*, \cdot \rangle$ is bounded from below on D and
(ii) $K \subset \{y \in Y \mid \langle y^*, y \rangle + \alpha \|y\| \leq 0\}$.

Then $\mathrm{Min}\, D = \mathrm{Ext}_K D \neq \emptyset$.

4 Duality of Set-Valued Optimization

In this section, we consider a l-type set-valued minimization problem with an inequality constraint (SP) and its dual problem (SD).

$$\text{(SP)} \qquad l\text{-Minimize} \quad F(x) \qquad \text{subject to} \quad G(x) \leq^l \theta$$

$$\text{(SD)} \qquad l\text{-Maximize} \quad \Phi(T) \qquad \text{subject to} \quad T \in \mathcal{L}_+(Y, Z)$$

where, X is a nonempty set, $(Y, \leq_K)$, $(Z, \leq_L)$ ordered vector spaces with ordering cones K, L, respectively, $F : X \to 2^Z$, $G : X \to 2^Y$, $\mathcal{L}(Y, Z) = \{T : Y \to Z \mid T \text{ is linear}\}$, $\mathcal{L}_+(Y, Z) = \{T \in \mathcal{L}(Y, Z) \mid T(K) \subset L\}$, $\mathrm{Gr}(G) = \{(x, y) \in X \times Y \mid y \in G(x)\}$ and $\Phi : \mathcal{L}(Y, Z) \to 2^Z$ defined by $\Phi(T) = l\text{-}\mathrm{Min}\{F(x) + T(y) \mid (x, y) \in \mathrm{Gr}(G)\}$.

Definition 4.1 (Solutions) x_0 is said to be

(i) an l-feasible solution of (SP) if $G(x) \leq^l \theta$;

(ii) an l-solution of (SP) if x_0 is l-feasible and

$$x \in X, G(x) \leq^l \theta, F(x) \leq^l F(x_0) \text{ implies } F(x_0) \leq^l F(x).$$

T_0 is said to be

(i) a feasible solution of (SD) if

$$T_0 \in \mathcal{L}_+(Y, Z) \text{ and } \Phi(T) \neq \emptyset;$$

(ii) an l-solution of (SD) if T_0 is feasible and there exists $A_0 \in \Phi(T_0)$ such that

$$T_1 \in \mathcal{L}_+(Y, Z), A_1 \in \Phi(T), A_0 \leq^l A_1 \text{ implies } A_1 \leq^l A_0$$

Proposition 4.1 (Weak Duality)
Let x_0 be an l-feasible solution of (SP), T_1 an l-feasible solution of (SD), and (x_1, y_1) an element of $\text{Gr}(G)$ satisfying $F(x_1) + T_1(y_1) \in \Phi(T_1)$. Then,

$$F(x_0) \leq^l F(x_1) + T_1(y_1) \text{ implies } F(x_1) + T_1(y_1) \leq^l F(x_0)$$

Definition 4.2 (Lagrangian Function) For $x \in X$, $y \in Y$, $T \in \mathcal{L}(Y, Z)$,

$$L(x, y, T) = F(x) + T(y).$$

In usual, y is an element of $G(x)$.

Definition 4.3 (Saddle Point)
$(x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is said to be an l-saddle point of L if

(i) $L(x, y, T_0) \leq^l L(x_0, y_0, T_0), (x, y) \in \text{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq^l L(x, y, T_0)$

(ii) $L(x_0, y_0, T_0) \leq^l L(x_0, y_0, T), T \in \mathcal{L}_+(Y, Z) \Rightarrow L(x_0, y_0, T) \leq^l L(x_0, y_0, T_0)$

Theorem 4.1 Assume that K is closed, L is solid, and F satisfies the following bounded condition: for each $x \in \text{Dom}(F)$ there exists $y^* \in K^+$ such that

- $\langle y^*, y \rangle > 0$ for each $y \in K \setminus \{\theta\}$;
- $\inf_{y \in F(x)} \langle y^*, y \rangle > -\infty$.

If $(x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is an l-saddle point of L, then we have

(i) $y_0 \leq \theta$ and $T_0(y_0) = \theta$;

(ii) x_0 is an l-solution of (SP);

(iii) T_0 is an l-solution of (SD).

Theorem 4.2 $(x_0, y_0, T_0) \in \text{Gr}(G) \times \mathcal{L}_+(Y, Z)$ is an l-saddle point of L if and only if

(i) $L(x, y, T_0) \leq^l L(x_0, y_0, T_0), (x, y) \in \text{Gr}(G) \Rightarrow L(x_0, y_0, T_0) \leq^l L(x, y, T_0)$

(ii) $y_0 \leq \theta$ and $T_0(y_0) = \theta$.

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