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Radical p-chains, Chains of radical p-subgroups and collapsing

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1 Introduction

This is an extended version of some part of my talk "p-radical chains, Dade conjecture and cohomology" given at RIMS on March 16, 1998 in the workshop on cohomology of finite groups. There I discussed two topics: the sufficient conditions for the alternating decomposition formula of the p-adic group cohomology recently found by Dwyer and Benson, and the collapsing technique (most elementary G-equivariant homotopy equivalence) which could be used to reduce the number of radical p-chains for verifining the Dade conjecture.

I choose other title for the report by the following reasons: the detail of the first part can be seen in the last section of my joint paper with S. D. Smith [SY], so I omit: it turns out that if a group satisfies $(D\mathcal{B}_p)$-property (see 2.6) then one can easily find which chains are cancelled out without collapsing them in verifying the Dade conjecture (see the last paragraph of the thrid section), so I will not discuss much about the Dade conjecture.

Instead, a foundation for the second topic, whcih I forgot to state in the talk, is explained in detail: the relation between the simplicial complex $\Delta(\mathcal{B}_p(G))$ of radical subgroups and the set $\tilde{\Phi}_p(G)$ of (reduced) radical chains is discussed, including the notion of $(D\mathcal{B}_p)$-property. It will be shown that a group of Lie type in characteristic p and the Mathieu group M_{24} satisfy this property ($p=2$ for the latter), and hence $\Delta(\mathcal{B}_p(G)) = \tilde{\Phi}_p(G)$ for these groups and primes. Explicit collapsing process is also illustrated with the latter group.

I conclude the introduction with a correction of information about radical 2-subgroups of M_{24} given in [Yo]: [1]

> two conjugacy classes of radical 2-subgroups of M_{24} are overlooked,
> and hence there are exactly 13 conjugacy classes of $\mathcal{B}_2(M_{24})$.

The arguments in [Yo, 4.2, line 17–16 from the bottom] for 2-radical subgroups containing the sextet kernel U_Σ are not enough: in fact, two radical groups arize in $3S_6 \cong G_\Sigma/U_\Sigma$ which do not correspond to radical subgroups of S_6. This yileds one new possible 2-radical subgroup $U_{\{T,\Sigma,\square\}}$, which gives another radical subgroup $U_{\{T,\square\}}$ containing the trio kernel U_T.

Consequently, in [Yo, Figure 1], we need two more boxes for $U_{\{T,\Sigma,\square\}}$ (with symbols $21a$ and $\frac{S_3}{[2^9]}$) $U_{\{T,\square\}}$ (with symbols $7a$ and $\frac{S_3\times S_3}{[2^8]}$), and five new lines joining the boxes

[1] The error was found when I checked some arguments in [AC]. I also noticed that in [AC, (5.6),p.2816], $E_4.E_{64}$ and Q should be $(E_4.E_{64})^*$ and Q^* respectively.

$U_{T,\Sigma,\square}$ and U_X for $X = \{O, T, \Sigma, \square\}$, $\{T, \Sigma\}$, $\{T, \square\}$; and joining boxes $\{T, \square\}$ and U_Y for $Y = \{O, T, \square\}$, T.

In the calculation of the Euler characteristic in [Yo, 4.3], the terms involving the classes of the overlooked radicals turns out to vanish, so that the conclusion of [Yo, 4.3] is valid. This should be the case, because we have a M_{24}-homotopy equivalence of the simplicial complex $\Delta(\mathcal{B}_2(M_{24}))$ of the poset $\mathcal{B}_2(M_{24})$ with the 2-local geometry of M_{24}, which was verified by the other method in [SY].

2 Radical p-chains and chains of radical p-subgroups

Definition 2.1 Let p be a prime divisor of the order of a finite group G. A nontrivial p-subgroup U of G is called a *radical p-subgroup* whenever U coincides with the largest normal p-subgroup $O_p(N_G(U))$ of its normalizer $N_G(U)$. (Note that $U \leq O_p(N_G(U))$ for every p-subgroup U of G.) The set of radical p-subgroups is denoted $\mathcal{B}_p(G)$:

$$\mathcal{B}_p(G) = \{U \mid 1 \neq U = O_p(N_G(U))\}.$$

For a chain of p-subgroup $C = (U_0, U_1, \ldots, U_n)$ (that is, each U_i is a p-subgroup and $U_0 < U_1 < \cdots < U_n$), the *initial i-th subchain* C_i is defined to be $(U_0, U_1, \ldots, U_i)$ $(i = 0, \ldots, n)$ and its *normalizer* $N_G(C_i)$ is defined to be $\cap_{j=0}^{i} N_G(U_j)$. The chain C is called a *radical p-chain* if $U_0 = O_p(G)$ and $U_i = O_p(N_G(C_i))$ for each $i = 1, \ldots, n$. The chain obtained from a radical p-chain by deleting the first term $U_0 = O_p(G)$ is called a *reduced radical p-chain.* The set of (resp. reduced) radical p-chains will be denoted $\Phi_p(G)$ (resp. $\tilde{\Phi}_p(G)$).

We first collect some elementary observations on radical p-chains [2].

Lemma 2.2 (0) *A Sylow p-subgroup of G is a radical p-subgroup.*

(*i*) $N_G(C_i) = N_G(C_{i-1}) \cap N_G(U_i)$ $(i = 1, \ldots, n)$.

(*ii*) *If C is a radical p-chain, then also is the initial subchain C_i $(i = 0, \ldots, n)$.*

(*iii*) *If C is a radical p-chain, then its second term U_1 is a radical p-subgroup.*

(*vi*) *A chain C of a p-subgroups is a radical p-chain if and only if $U_0 = O_p(G)$, $U_i \trianglelefteq U_j$ for $1 \leq i < j \leq n$ and U_i/U_{i-1} is a p-radical subgroup of $N_G(C_{i-1})/U_{i-1}$ for every $i = 1, \ldots, n$.*

(*v*) *If $N_G(U_1) \geq N_G(U_2) \geq \cdots \geq N_G(U_n)$, then C is a radical p-chain if and only if $U_0 = O_p(G)$ and U_i/U_{i-1} is a p-radical subgroup of $N_G(U_{i-1})/U_{i-1}$ for every $i = 1, \ldots, n$.*

[2]In this report, sometimes proofs are given to the statements which seems trivial for experts in finite group theory, because of convenience for representation theorists and algebraic topologists, who were major attendance of the workshop.

Proof. The claims (0),(i) and (ii) are immediate from the definitions. As $U_{i-1} \trianglelefteq N_G(C_i)$ $(i = 1, \ldots, n)$, it follows from Claim (i) that the condition $U_i = O_p(N_G(C_i))$ is equivalent to say that U_i is a radical p-subgroup of $N_G(C_{i-1})$. In particular, the claim (iii) follows. Furthermore, taking factor groups by U_{i-1}, it is equivalent to say that U_i/U_{i-1} is a p-radical subgroup of $N_G(C_{i-1})/U_{i-1}$. This establishes Claim (iv). Claim (v) is its corollary. □

With each radical p-subgroup U of G, we associate its normalizer $N_G(U)$. The following fundamental observation was made in [SY, Lemma 1.9].

Lemma 2.3 *For $U \neq V \in \mathcal{B}_p(G)$ with $N_G(V) \leq N_G(U)$, we have $U \trianglelefteq V$ and $V/U \in \mathcal{B}_p(N_G(U)/U)$.*

Proof. As $V \leq N_G(V) \leq N_G(U)$, the product VU is a subgroup containing V. Assume that VU properly contains V. Then it follows from a fundamental property of nilpotent groups that $N_{VU}(V)$ properly contains V. But $N_{VU}(V)$ is a p-subgroup which is normal in $N_G(V)$, as a subgroup $N_G(V)$ of $N_G(U)$ normalizes both V and U. This implies that $O_p(N_G(V)) \geq VU > V$, contradicting $V = O_p(N_G(V))$. Thus $VU = V$ or equivalently $U \leq V$. As $V \leq N_G(U)$, $U \trianglelefteq V$.

The latter claim now immediately follows, as $(N_G(U) \cap N_G(V))/U = N_G(V)/U$ and $O_p(N_G(V)/U) = O_p(N_G(V))/U = V/U$. □

Thus, to find the candidates for radical p-subgroups, we first investigate those with maximal normalizers and choose the preimages in their normalizers of p-radicals of the corresponding factor groups. This suggests that in principle we can determine $\mathcal{B}_p(G)$ reccursively. Note that a candidate V obtained from $N_G(U)/U$ may not be a radical group, as $N_G(V)$ may not be contained in $N_G(U)$. However, if $N_G(V) \leq N_G(U)$, the candidate is in fact a radical group: for, the condition $V/U \in \mathcal{B}_p(N_G(U)/U)$ is equivalent to $V/U = O_p(N_G(U) \cap N_G(V)/U)$, which is under our assumption $V/U = O_p(N_G(V)/U)$ and hence $V = O_p(N_G(V))$. These observations are summarized in the following way.

Lemma 2.4 *For a radical p-subgroup U of G, define a subset of $\mathcal{B}_p(G)$ by*

$$Red(\mathcal{B}_p)_U := \{V \in \mathcal{B}_p(G) \mid N_G(V) \leq N_G(U)\} \setminus \{U\}.$$

Then the following statements hold.

(1) *The group U is a proper normal subgroup of V for every $V \in Red(\mathcal{B}_p)_U$.*

(2) *The quotient map $\rho : V \mapsto V/U$ is an injection from $Red(\mathcal{B}_p)_U$ into $\mathcal{B}_p(N_G(U)/U)$.*

(3) *The quotient map ρ is bijective if and only if $N_G(V) \leq N_G(U)$ for every $V/U \in \mathcal{B}_p(N_G(U)/U)$.*

The following fact is also well known:

Lemma 2.5 *Let G be a finite group and p be a prime divisor of $|G|$. For every nontrivial p-subgroup U there is a radical p-subgroup W with $U \leq W$ and $N_G(U) \leq N_G(W)$.*

Proof. Starting from U, consider a chain of subgroups inductively defined as follows:

$W_0 := U$, $N_0 := N_G(W_0)$,
$W_j := O_p(N_{j-1})$, $N_j := N_G(W_j)$ $(j = 1, 2, \ldots)$

Clearly $W_{j-1} \leq W_j$ and $N_{j-1} \leq N_j$ for every $j = 1, 2, \ldots$. As G is a finite group, the increasing chain of subgroups $W_0 \leq W_1 \leq \cdots$ stops at some $W := W_m$, say. Then $W = O_p(N_G(W)$, and hence $W \in \mathcal{B}_p(G)$. By construction, $U \leq W$ and $N_G(U) \leq N_G(W)$. □

Relation between chains of radicals and radical chains. The set $\mathcal{B}_p(G)$ forms a partially ordered set with respect to inclusion. It is often convenient to consider the associated simplicial complex $\Delta(\mathcal{B}_p(G))$ (*order complex*) with the chains as its simplices, because it allows us to apply some topological method. On the other hand, though the set $\tilde{\Phi}_p(G)$ is contained in the order complex $\Delta(\mathcal{S}_p(G))$ of the poset of all nontrivial p-subgroups, it does not have the structure of a simplicial complex in general, because a subchain of a reduced radical p-chain is not a radical p-chain in general unless it is an initial subchain. This seems the most defect of the notion of radical p-chains.

If $\tilde{\Phi}_p(G)$ has the structure of a simplicial complex, then each term of a radical chain can be thought of as a radical p-chain with just one term. It is a radical p-subgroup by 2.2(iii). Thus $\tilde{\Phi}_p(G)$ is contained in the order complex $\Delta(\mathcal{B}_p(G))$.

However, in general, a simplex of $\Delta(\mathcal{B}_p(G))$ is not a radical p-chain, nor a reduced radical p-chain is not a simplex of $\Delta(\mathcal{B}_p(G))$: Take a chain $C = (U, V)$ of radical p-subgroups of length 2 for simplicity. We know $U = O_p(N_G(U))$ and $V = O_p(N_G(V))$ but this does not imply the condition $V = O_p(N_G(U) \cap N_G(V))$ required for C to be a radical p-chain. Clearly $V \cap N_G(U)$ is contained in $O_p(N_G(U) \cap N_G(V))$. Conversely, let $C = (U, V)$ be a reduced radical p-chain of length 2. By Lemma 2.2(iii), $U \in \mathcal{B}_p(G)$. But the condition $V = O_p(N_G(U) \cap N_G(V))$ does not imply $V = O_p(V)$ in general. Thus C is not a chain of radical p-subgroups. But if we have $N_G(U) \geq N_G(V)$, then $V = O_p(N_G(V))$ and C is a chain of radical p-subgroups.

These observations give us a feeling that the reduced radical p-chains $\tilde{\Phi}_p(G)$ rarely have the structure of a simplicial complex. However, we will see that even stronger result $\tilde{\Phi}_p(G) = \Delta(\mathcal{B}_p(G))$ holds for finite groups of Lie type in characteristic p and the Mathieu group M_{24} of degree 24 for $p = 2$.

We give a sufficient condition for $\Delta(\mathcal{B}_p(G)) = \tilde{\Phi}_p(G)$ (Lemma 2.7).

Definition 2.6 For a finite group G and a prime p dividing the order of G, $(D\mathcal{B}_p)$ is the following property:

> $(D\mathcal{B}_p)$: We have $N_G(U) \geq N_G(V)$ whenever radical p-subgroups U and V of G satisfy $U \leq V$.

Lemma 2.7 *If a group G satisfies the $(D\mathcal{B}_p)$ property, then $\Delta(\mathcal{B}_p(G)) = \tilde{\Phi}_p(G)$.*

Proof. Choose any chain $C = (U_1, U_2, \ldots, U_n)$ of radical p-subgroups. By assumption we have $N_G(U_1) \geq N_G(U_2) \geq \cdots \geq N_G(U_n)$. Then $N_G(C_i) = N_G(U_i)$ and $U_i = O_p(N_G(U_i)) = N_G(C_i)$ for every $i = 1, \ldots, n$. Thus C is a reduced radical p-chain.

Conversely, let $C = (U_1, U_2, \ldots, U_n)$ be any reduced radical p-chain. We will show that $U_i \in \mathcal{B}_p(G)$ for every $i = 1, \ldots, n$ by induction on the length n of C. If $n = 1$, the claim follows from Lemma 2.2(iii). Let $n > 1$. Since $C_{n-1} \in \tilde{\Phi}_p(G)$, the hypothesis of induction implies that $U_i \in \mathcal{B}_p(G)$ for all $i = 1, \ldots, n-1$. By assumption, then we have $N_G(U_1) \geq \cdots \geq N_G(U_{n-1})$ and so $N_G(C_{n-1}) = N_G(U_{n-1})$. By Lemma 2.5, there is $W \in \mathcal{B}_p(G)$ with $U_n \leq W$ and $N_G(U_n) \leq N_G(W)$. Then the radical group U_{n-1} is a subgroup of a radical group W, and hence $N_G(U_{n-1}) \geq N_G(W) \geq N_G(U_n)$ by the assumption. Thus $N_G(C) = N_G(C_{n-1}) \cap N_G(U_n) = N_G(U_{n-1}) \cap N_G(U_n) = N_G(U_n)$ and $U_n = O_p(N_G(C)) = O_p(N_G(U_n))$. Hence $U_n \in \mathcal{B}_p(G)$ as we desired. □

Lemma 2.8 *Let $\mathcal{B}_p^*(G)$ be the set of radical p-subgroups U of G for which $N_G(U)$ is maximal among the normalizers of p-radical subgroups. Assume that*

(*a*) *For every $U \in \mathcal{B}_p(G)$, there is $U_* \in \mathcal{B}_p^*(G)$ such that $N_G(V) \leq N_G(U_*)$ for every $V \in \mathcal{B}_p(G)$ containing U.*

(*b*) *$N_G(U_*)/U_*$ satisfies the $D\mathcal{B}_p$-property for every $U_* \in \mathcal{B}_p^*(G)$.*

Then G satisfies the $(D\mathcal{B}_p)$-property.

Proof. Let U and V be p-radical subgroups with $U \leq V$. Choose U_* satisfying the condition (a) for U. Then both U and V contain U_* as a normal subgroup, and U/U_* and V/U_* are radical p-subgroup by Lemma 2.3. As $U/U_* \leq V/U_*$, the condition (b) implies that the normalizer of U/U_* in $N_G(U_*)/U_*$ contains that of V/U_*. Since $N_{N_G(U_*)/U_*}(X/U_*) = (N_G(U_*) \cap N_G(X))/U_* = N_G(X)/U_*$ $(X = U, V)$, we have $N_G(U) \geq N_G(V)$. □

Lemma 2.9 *If finite groups A and B satisfy the $(D\mathcal{B}_p)$-property, then the direct product $A \times B$ satisfies the $(D\mathcal{B}_p)$-property.*

Proof. Let $U, V \in \mathcal{B}_p(A \times B)$ with $U \leq V$. By Lemma [Sa, Lemma 3.2] [3], we have $U = U_A \times U_B$ and $V = V_A \times V_B$, where $U_A = U \cap (A \times 1)$, etc. In particular, $U_A, V_A \in \mathcal{B}_p(A) \cup \{1\}$ and $U_B, V_B \in \mathcal{B}_p(B) \cup \{1\}$, identifying A with a subgroup $A \times 1$ of $A \times B$, etc. As $U \leq V$, we have $U_A = U \cap (A \times 1) \leq V \cap (A \times 1) = V_A$, and similarly $U_B \leq V_B$. As A and B satisfy $D\mathcal{B}_p$-property, $N_A(U_A) \geq N_A(V_A)$ and $N_B(U_B) \geq N_B(V_B)$. It is easy to see that $N_{A\times B}(X) = N_A(X_A) \times N_B(X_B)$ $(X = U, V)$. Thus $N_{A\times B}(V) = N_A(V_A) \times N_B(V_B) \leq N_A(U_A) \times N_B(U_B) = N_{A\times B}(U)$. □

The lemmas 2.8 and 2.9 can be slightly generalized as follows, by arguing similarly to the proofs of these lemmas. So the proofs are omitted.

[3] The result may be known before, though I don't know the proof except one given by Sawabe.

Lemma 2.10 (1) *Assume that the condition (a) in Lemma 2.8 and the following condition (b′) holds: (b′) For every $U_* \in \mathcal{B}_p^*(G)$, $\tilde{\Phi}_p(N_G(U_*)/U_*)) = \Delta(\mathcal{B}_p(N_G(U_*)/U_*))$. Then $\tilde{\Phi}_p(G) = \Delta(\mathcal{B}_p(G))$.*

(2) *If $\tilde{\Phi}_p(X) = \Delta(\mathcal{B}_p(X))$ for $X = A, B$, then $\tilde{\Phi}_p(A \times B) = \Delta(\mathcal{B}_p(A \times B))$.*

Groups of Lie type in characteristic p. Let G be a finite group of Lie type defined over a field in characteristic p and of Lie rank r. (For general reference, I recommend the reader to consult a book of Curtis and Reiner [CR], §64,65 and 69.) By parabolic theory [CR, §65], there is a complete system $\{P_F \mid F \subset I\}$ of representatives for G-conjugacy classes of parabolic subgroups which is parametrized by the power set of $I = \{1, \ldots, r\}$ and satisfies the following properties:

(i) Every proper subgroup of G containing a Borel subgroup $B := P_\emptyset$ is of the form P_F for some $F \subset I$. (This also implies that two distinct proper subgroups containing B are not conjugate under G.)

(ii) If $F, K \subset I$ then $P_{F\cap K} = P_F \cap P_K$ and $P_{F\cup K} = \langle P_F, P_K\rangle$. In particular, $P_F \leq P_{F'}$ if and only if $F \subseteq F' \subset I$.

(iii) Setting $O_p(P_F) =: U_F$ (the *unipotent radical* of P_F), $N_G(U_F) = P_F$. Thus $U_F \in \mathcal{B}_p(G)$.

Proposition 2.11 *In a finite group G of Lie type defined over a filed in characteristic p, every radical p-subgroup of G is conjugate to a unipotent radial U_F for some $F \subset I$.*

Proof. (Sketch) For a radical p-subgroup U, let P be a parabolic subgroup minimal subject to $N_G(U) \leq P$. Such a parabolic subgroup always exists by a theorem of Borel and Tits [BT], saying that a subgroup of G with non-trivial O_p is contained in a parabolic subgroup of G. Then it is not so difficult to see $U = O_p(P)$, by arguing similarly to the proof of 2.3. I left the proof as an exercise for the reader. □

Lemma 2.12 *Let G be a finite group of Lie type in characteristic p. For $U, V \in \mathcal{B}_p(G)$, the following statements are equivalent.*

(i) $U \leq V$. *(ii)* $U \trianglelefteq V$. *(iii)* $N_G(U) \leq N_G(V)$.

Proof. By Lemma 2.3, (iii) implies (ii). Obviously (ii) implies (i).

To prove the converse implications, we use [CR, (69.16)]. The readers are assumed some familiarity with notations in [CR, §69], though I follows the notation above.

(i) implies (ii): It suffices to show the claim (ii) when V is a Sylow p-subgroup of G. (For, if $U \leq V$, $U, V \in \mathcal{B}_p(G)$, take a Sylow p-subgroup S of G containing V. As $U \trianglelefteq S$, $U \trianglelefteq V$.) Note that a Sylow p-subgroup is a radical p-subgroup by definition. By Proposition 2.11, we may assume that $U = U_F$ for some $F \subset I$. Let S be a Sylow p-subgroup containing U. As $U_\emptyset = O_p(B)$ is a Sylow p-subgroup of G, $S = gU_\emptyset g = {}^g U_\emptyset$ for some $g \in G$. By

the Bruhat decomposition $G = BWB$ ([CR, (65.4)]), $g = bwb'$ for some $b, b' \in B$ and $w \in W$. As B ($\leq P_F = N_G(U_F)$) normalizes U_F and $U_\emptyset$, we have $U_F \leq {}^wU_\emptyset$. Furtheromore, U_F is normalized by W_F, the subgroup of W generated by the distinguished involutions corresponding to F, since $N_G(U_F) = P_F = BW_FB$ ([CR, (64.39)]). Writing $w = w'x$ for $w' \in W_F$ and x a distinguished double coset representative for $W_F\backslash W/W_\emptyset = W_F\backslash W$ (an element of the coset W_Fw of minimal length [CR, (64.39)]), we then have $U_F \leq {}^xU_\emptyset$.

Now we may apply [CR, (69.16)(iv)] to "I"$= F$ and "J"$= \emptyset$. Since "K"$= F \cap {}^x\emptyset = F \cap \emptyset = \emptyset$, we have

$$U_\emptyset = (P_F \cap {}^xU_\emptyset)U_F.$$

The right hand side is contatined in ${}^xU_\emptyset$) as $U_F \leq {}^xU_\emptyset$. We have $U_\emptyset = {}^xU_\emptyset$, comparing the orders. Thus $x \in W \cap N_G(U_\emptyset) = W \cap B = 1$, and therfore $g = b(w'x)b' = bw'b' \in BW_FB = P_F = N_G(U_F)$. Since $U_F \trianglelefteq U_\emptyset$ (as $U_F \leq U_\emptyset$ and $P_F \geq N_G(U_\emptyset) = B$), taking the conjugate of the both side of this equation under $g \in P_F$, we have $U = U_F = {}^gU_F \trianglelefteq {}^gU_\emptyset = S$.

(ii) implies (iii): As the arguments in the claim "$(i) \Rightarrow (ii)$" above, we may assume $U = {}^gU_K$ and $V = U_F$ for some $F, K \in I$ and $g \in G$. Furthermore g may be chosen as a distinguished double coset representative for $W_F\backslash W/W_K$, by the Bruhat decomposition $G = BWB$ and its generalizations $N_G(U_F) = P_F = BW_FB$, $N_G(U_K) = P_K = BW_KB$. By [CR, (69.16)(ii)] we have

$$U_X = (P_F \cap {}^gU_K)U_F,$$

$X = F \cap {}^gK$, identifying I with a set of distinguished generators of the Weyl group W. Note that $X \subset I$ while ${}^gK$ may not be contained in I. Since $U = {}^gU_K \leq V = U_F$, we have $U_X \leq U_F$ and hence $F \subseteq X = F \cap {}^gK$. Thus $X = F$ and $F \subseteq {}^gK$ (but this does not imply $P_F \leq {}^gP_K$, as ${}^gK$ may not be a subset of I). By the definition of Levi complement, we however have $L_X = L_F \leq {}^gL_K$. In particular, L_F normalizes ${}^gU_K = O_p({}^gP_K)$.

By our assumption $U = {}^gU_K \trianglelefteq U_F = V$, U_F also normalizes ${}^gU_K$. Thus $N_G(U_F) = P_F = L_FU_F \leq N_G({}^gU_K) = N_G(V)$. □

By Lemma 2.7 and Lemma 2.12, the following (already known) result follows.

Proposition 2.13 *For a finite group G of Lie type in characteristic p, the $(D\mathcal{B}_p)$-property holds and hence we have $\Delta(\mathcal{B}_p(G)) = \tilde{\Phi}_p(G)$.*

3 Collapsing

Definition 3.1 Let Δ be an abstract simplicial complex. Assume that there is a unique maximal simplex σ of Δ containing a simplex $\tau \in \Delta$. Then the process which deletes both σ and τ is called a *collapsing* at a pair (τ, σ). The geometric realization of the resulting complex $\Delta - \{\sigma, \tau\}$ is homotopically equivalent to that of Δ.

Collapsing for chains of subgroups. Consider a set Φ of chains of subgroups of G admitting the conjugacy action of G: for $g \in G$ and $C = (V_1, \ldots, V_n) \in \Phi$, ${}^gC := ({}^gV_1, \ldots, {}^gV_n) \in \Phi$. Typical examples are the order complex $\Delta(\mathcal{B}_p(G))$ of a poset $\mathcal{B}_p(G)$ and the set $\tilde{\Phi}_p(G)$ of reduced radical p-chains. Let $\mathcal{R}$ be a complete system of representatives of G-conjugacy classes of subgroups which appear as terms of chains of Φ.

The map f can be extended to a map on Φ by sending $C = (V_1, \ldots, V_n)$ to a sequence $f(C) := (f(U_1), \ldots, f(U_n))$ of subsets of I, where $U_i \in \mathcal{R}$ is conjugate to V_i $(i = 1, \ldots, n)$. The group G acts on Φ by conjugation, which is compatible with the map f on Φ. We call $f(C)$ the *type* of C.

Under these terminologies, when Φ has the structure of a simplicial complex, a typical example of collapsing occurs if C is a unique maximal chain of Φ containing $C^{(1)} := (V_2, \ldots, V_n)$ of type $f(C)$. The latter condition is equivalent to say that

$(*)$ if $({}^gV_1, V_2, \ldots, V_n)$ is a chain then $V_1 = {}^gV_1$.

If this condition is satisfied, then we can remove C and $C^{(1)}$ from the complex Φ without changing its homotopy type. Furthermore, since G acts on Φ, we can simultaneously remove all chains of type $f(C)$ and $f(C^{(1)})$. Thus the simplicial complex Φ is G-homotopically equivalent to $\Phi - \{D \in \Phi | f(D) = f(C), f(D) = f(C^{(1)})\}$.

This is frequently used to show that for example the order complex $\Delta(\mathcal{B}_p(G))$ of a sporadic simple group G of characteristic-p type (here we do not need the definition, see [SY]) is G-homotopically equivalent to some (much smaller) simplicial complex $\mathcal{P}(G)$, called the p-local geometry of G (see [SY]).

Even when Φ does not have the structure of a simplicial complex, we can still consider $\Phi - \{D \in \Phi | f(D) = f(C), f(D) = f(C^{(1)})\}$, if $C^{(1)} \in \Phi$. (Though the latter condition is very strong.) Take as Φ the set $\tilde{\Phi}_p(G)$ of reduced radical p-chains. Let $C = (V_1, \ldots, V_n)$ be a chain with the property $(*)$. For $x \in N_G(V_2) \cap \cdots \cap N_G(V_n)$, we have ${}^xC = ({}^xV_1, V_2, \ldots, V_n)$ and hence ${}^xV_1 = V_1$ by $(*)$. Thus $x \in N_G(V_1)$ and $N_G(C) = N_G(C^{(1)})$.

Now recall several forms of Dade conjecture (see e.g.[Ko]). Each of them claims that an alternating sum of the numbers of certain characters of $N(C)$ vanishes, when C ranges over radical p-chains. Note that as $N(C) = N(C^{(1)})$, the terms for $N(C)$ and $N(C^{(1)})$ are cancelled out *a priori* (without computing the number of certain characters!). [4] In particular, if a group G satisfies the $(D\mathcal{B}_p)$-property, then the problem is just reduced to count the number of chains of specified length which ends at a specified type. This observation is very simple, but sometimes it helps us to reduce the number of radical chains for which we should examine the Dade conjecture.

Lemma 3.2 *Assume that a finite group G satisfies the $(D\mathcal{B}_p)$-property. Then*

(1) *For each pair of radical p-subgroups U, V of G with $U \leq V$, the group U is the unique conjugate of U contained in V.*

(2) *Assume aslo that a type function is defined on the chains. Let $C = (U_1, \ldots, U_n)$ be a chain of radical p-subgroups of G and let $C^{(i)}$ be the subchain of C obtained from C by deleting U_i $(1 \leq i \leq n)$. If $i \leq n-1$, then C is the unique chain which contains $C^{(i)}$ and has the same type as C.*

[4] In [Ko] $p = 2$, this can be observed between radical 2-chains $C_{2,2}$ and $C_{1,4}$.

Proof. (1) We may assume that V is a Sylow p-subgroup of G. If U and ${}^gU$ are contained in V, then they are normal in V by the $(D\mathcal{B}_p)$-property. Then V and ${}^{g^{-1}}V$ are Sylow p-subgroups of $N_G(U)$, and hence there is $h \in N_G(U)$ with $hg^{-1} \in N_G(V)$. As $N_G(V) \leq N_G(U)$ by the $(D\mathcal{B}_p)$-property, we have $g \in N_G(U)$ and $U = {}^gU$.

Claim (2) is immediate from Claim (1). □

4 The radical 2-chains of the largest Mathieu group

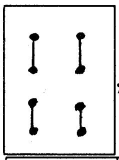
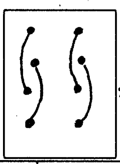
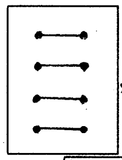
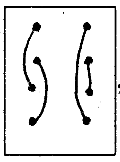
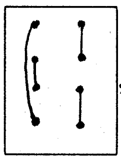
In this section, the readers are assumed to have some familiarity with the following terminologies: Steiner system $S(5,8,24)$, octads, tiros, sextets, the Mathieu group M_{24} of degree 24 as the automorphism group of $S(5,8,24)$, the structure of the stabilizers in M_{24} of an octad (trio, sextet): For a standard reference, see [CS, Chap.11]. We fix an MOG arrangement, and let O, T and Σ be the standard octad (the first brick), the standard trio (the triple of theree bricks) and the standard sextet (consisting of the six columns), respectively.

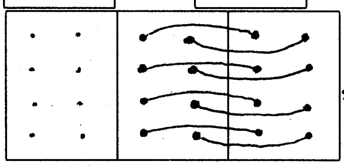
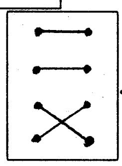
We will describe some 2-subgroups of $G := M_{24}$ which correspond to 2-radical subgroups of quotient groups $G_X/O_2(G_X)$ of stabilizers G_X of X in G for $X = O$, T and Σ.

Setting $U_X := O_2(G_X)$, we have $G_X = N_G(U_X)$. The extension G_X/U_X splits for $X = O, T, \Sigma$. We have $G_O/U_O \cong SL_4(2)$, $G_T/U_T \cong SL_2(2) \times SL_3(2)$ and $G_\Sigma/U_\Sigma \cong 3 \cdot S_6$, a nonsplit extension of S_6, in which 3 is the center of $3.A_6$. Furthermore,

$$U_O = \langle t(0,a,a), t(0,b,b), t(0,c,c), \sigma\rangle \cong 2^4,$$
$$U_T = \langle t(0,a,a), t(0,b,b), t(0,c,c), t(a,a,0), t(b,b,0), t(c,c,0)\rangle \cong 2^6 \text{ and}$$
$$U_\sigma = \langle t(0,a,a), t(0,b,b), t(a,a,0), t(b,b,0), x, y\rangle \cong 2^6,$$

where a, b, c mean the following involutive permutations on a brick, and for example, $t(a,a,0)$ means the permutation inducing a, a and the identity on the first, the second and the third bricks, respectively, x (resp. y) means the permutation inducing the following involution x' (resp. y') on each brick, and σ is the involution below. (x and y correspond to the vector $\mathbf{x} = (\omega, \bar\omega, \omega, \bar\omega, \omega, \bar\omega)$ and $\omega\mathbf{x}$ in the Hexacode: see [CS, Fig. 118(a), p. 309].)

$a =$ , $b =$ , $c =$ , $x' =$ , $y' =$ ,

$\sigma =$ , $\alpha' =$ .

We now take a dummy symbol □, and set $I := \{O, T, \Sigma, \square\}$. For $F \subseteq I \setminus \{\square\}$, we set $U_F := \langle U_X | X \in F\rangle$ and $U_{F,\square} := \langle U_F, t(a,a,0), x, \alpha\rangle$, where α is the permutation inducing the involution α' above on each brick. With this notation, we can verify that the following:

Residue at octad O The octad stabilizer G_O acts on the set of 15 trios which contain O as a member. They together with the empty symbol form a 4-dimensional vector space $V(O)$ over $\mathbf{F}_2$ under the symmetric difference. The subgroup U_O is the kernel of the action of G_O on $V(O)$, and G_O induces all linear transformations. This explains $G_O/U_O \cong SL_4(2)$. The following trios $T = T_1$, T_2, T_3, T_4 form a basis of $V(O)$, where we put the index i at the position belonging to the i-th octad of the trio:

$$T_2 = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 & 3 \\ \hline \end{array}, \quad T_3 = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 3 & 3 & 3 & 3 \\ \hline \end{array}, \quad T_4 = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 & 3 \\ \hline \end{array}.$$

With respect to the basis (T_1, T_2, T_3, T_4) we verify that $t(a,a,0)$, $t(b,b,0)$, $t(c,c,0)$, x, y and α are represented by the matrices $I + E_{41}$, $I + E_{31}$, $I + E_{21}$, $I + E_{42}$, $I + E_{32}$, and $I + E_{43}$, respectively, where E_{ij} is the matrix of degree 4 with a single non-zero entry 1 at the (i,j)-position. Thus the group $U_{\{O,T\}} = U_O\langle t(a,a,0), t(b,b,0), t(c,c,0)\rangle$ (resp. $U_{\{O,\Sigma\}} = U_O\langle t(a,a,0), t(b,b,0), x, y\rangle$ and $U_{\{O,\square\}} = U_O\langle t(a,a,0), t(b,b,0), x, \alpha\rangle$) corresponds to the unipotent radical for the stabilizer of a projective point (resp. a line and a plane), as you see below. Similarly, U_F with $F \ni O$ corresponds to the standard unipotent radicals for $SL_4(2)$. (Though the suffix here is complementary to that in the preceeding section.)

$$U_{O,T} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & & 1 & \\ * & & & 1 \end{pmatrix}, \; U_{O,\Sigma} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{pmatrix}, \; U_{O,\square} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{pmatrix},$$

$$U_{O,T,\Sigma} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{pmatrix}, \; U_{O,\Sigma,\square} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{pmatrix}, \; U_{O,T,\square} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & & 1 & \\ * & * & * & 1 \end{pmatrix},$$

$$U_{O,T,\Sigma,\square} = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{pmatrix}.$$

Residue at trio T There are 3 octads contained in T and 7 sextets refining T. The latter form a 3-dimensional space $V(T)$ over $GF(2)$ with the empty symbol under symmetric difference. The trio stabilizer G_T induces $SL_2(2) \cong S_3$ on the former and $SL_3(2)$ on the latter, with kernel U_T on the whole objects. This explains $G_T/U_T \cong SL_2(2) \times SL_3(2)$. We may choose the following sextets A, B and Σ as the basis of $V(T)$, and with respect to them x, y and α are represented as $I_3 + E_{31}$, $I_3 + E_{21}$ and $I + E_{32}$ respectively.

$$A = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ \hline 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}, \quad B = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ \hline 2 & 2 & 4 & 4 & 6 & 6 \\ \hline 1 & 1 & 3 & 3 & 5 & 5 \\ \hline 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}.$$

Thus $U_{\{T,\Sigma\}} = U_T\langle x, y\rangle$ (resp. $U_{\{T,\Box\}}$ and $U_{\{T,\Sigma,\Box\}}$) is the unipotent radical corresponding to the projective point $p = (1,0,0)$ (resp. line $l = \langle(1,0,0),(0,1,0)\rangle$ and the flag (p,l)). The subgroup $U_{\{T,O\}}$ corresponds to a subgroup of order 2 in the factor $S_3 \cong SL_2(2)$ of $G_T/U_T \cong SL_2(2) \times SL_3(2)$.

Residue at sextet Σ and $\mathcal{B}_2(3S_6)$. Though the residue at Σ is a generalized quadrangle of order $(2,2)$ on which the group $S_6 \cong Sp_4(2)$ of Lie type of rank 2 acts faithfully, we have $G_\Sigma/U_\Sigma \cong 3.S_6$, not S_6 itself. This makes the situation a bit complicated, because $U_{\{\Sigma,X\}}$ does not correspond to a unipotent radical of $S_6 \cong Sp_4(2)$, where $X = O$, T or $\{O,T\}$: For example, for $X = O$, the elements $t(0,c,c)$, σ, $t(c,c,0)$ and α induce the permutations $(34)(56)$, $(35)(46)$, $(12)(34)$ and $(12)(34)(56)$ on the six columns of Σ, respectively. Thus $U_{\{\Sigma,O\}}$ and $U_{\{\Sigma,O,\Box\}}$ correspond to subgroups $E_1 := \langle(34)(56),(35)(46)\rangle$ and $F_1 := \langle(34)(56),(35)(46),(12)\rangle$ of S_6 respectively. The former is not a radical 2-subgroup of S_6, as $N_{S_6}(E_1) = F_1\langle(345),(12)(34)\rangle$ and its O_2 is F_1, not E_1. However, the inverse image of (12) in $3S_6$ (written by the same symbol) inverts the center Z of $3S_6$, and $N_{3S_6}(E_1) = (Z\langle(12)\rangle \times E_1)\langle(345),(12)(34)\rangle$, and hence its O_2 is in fact E_1. Thus E_1 is a radical 2-subgroup of $3S_6$. We may also see that F_1 is a radical 2-subgroup of $3S_6$.

Moreover, $U_{\{\Sigma,T\}}$, $U_{\{\Sigma,T,\Box\}}$, $U_{\{\Sigma,O,T\}}$ and $U_{\{\Sigma,O,T,\Box\}}$ induce the subgroups $\langle(34),(56)\rangle$, $\langle(12),(34),(56)\rangle$, $\langle(34)(56),(35)(46),(12)(34)\rangle$ and $\langle(34)(56),(35)(46),(12)(34),(12)\rangle$ of S_6 respectively. Similar argument as above shows that their inverse images in $3S_6$ are radical 2-subgroups. It is also straightforward to verify that every radical 2-subgroup of $3S_6$ is conjugate to exactly one of the six subgroups $U_{\{\Sigma,F\}}$, $\emptyset \neq F \subseteq \{O,T,\Box\}$ with $F \neq \Box$.

Let U be a radical 2-subgroup of G. By [Yo, Lemma 4.5], $N_G(U)$ is conjugate to a subgroup of the stabilizer G_X of $X = O$, T or Σ. Thus by Lemma 2.3 and the above description of the 2-radical subgroups of $N_G(U_X)/U_X$, the subgroups U_F for a nonempty subset F of $I = \{O,T,\Sigma,\Box\}$ except $F = \{\Box\}$ and $\{\Sigma,\Box\}$ exhaust all candidates for the radical 2-subgroups of M_{24} up to conjugacy.

In fact, we can verify the following by observing the normalizer of each U_F.

Lemma 4.1 *A radical 2-subgroup of M_{24} is conjugate to one of the 13 subgroups U_F, where F ranges over all non-empty subsets of I execpt $\{\Box\}$ and $\{\Sigma,\Box\}$.*

At the same time, we can also check the following: (Note that the minimal radicals are those conjugate to U_O, U_T or U_Σ.)

Lemma 4.2 *If $F \subseteq K \subseteq I$, then we have $U_F \leq U_K$. Furthermore, for $|F| = 1$, $U_F \leq U_K$ and ${}^gU_F \leq U_K$ implies that $g \in N_G(U_F)$. In particular, the assumption (a) in Lemma 2.10 is satisfied.*

As $N_G(U_O)/U_O \cong SL_4(2)$ is a group of Lie type in characteristic 2, it satisfies the $(D\mathcal{B}_2)$-property. Information on $\mathcal{B}_2(3S_6)$ given in the above paragraph is enough to see that the same conclusion holds for $N_G(U_\Sigma)/U_\Sigma$. Finally $N_G(U_T)/U_T$ is a direct product of two groups $SL_2(2)$ and $SL_3(2)$ of Lie type in characteristic 2. Thus it also satisfies the $(D\mathcal{B}_2)$-property by Lemma 2.9. Hence Lemma 2.8 yields:

Proposition 4.3 *The Mathieu group M_{24} of degree 24 satisfies the $(D\mathcal{B}_2)$-property. That is, for $U, V \in \mathcal{B}_2(M_{24})$, the following conditions are equivalent.*

$$(i)\ U \leq V \qquad (ii)\ U \trianglelefteq V \qquad (iii)\ N_G(U) \geq N_G(V)$$

In particular, $\tilde{\Phi}_2(M_{24}) = \Delta(\mathcal{B}_2(M_{24}))$.

Finally we will show that $\tilde{\Phi}_2(M_{24}) = \Delta(\mathcal{B}_2(M_{24}))$ is M_{24}-homotopically equivalent to the subcomplex $\mathcal{P}_2(M_{24})$ consisting of chains of subgroups conjugate to U_F for $\square \notin F$. (The simplicial complex $\mathcal{P}_2(M_{24})$ is referred to as the *2-local geometry* for M_{24}.)

Extending the type map $U_F \mapsto F$, we may naturally associate the type with each chain of radical 2-subgroups. Types are increasing chains of subsets of $I = \{O, T, \Sigma, \square\}$. In particular, each maximal chain is of length 4 (i.e., has four terms).

If C is a chain of length 3 with the initial term of type $X\square$ for $X = O$ or T (we write for example $\{O, T, \square\}$ by $OT\square$ etc. for short), there is a unique chain $\tilde{C}$ including C with the initial term of type X, because there is no radical groups of type $\square$ and by Lemma 3.2(2). As $\tilde{C}$ is maximal, we may remove both C and $\tilde{C}$. In the complex of the remaining chains, each chain of type $(X, X\square, OT\square)$ is maximal, and it is a unique chain containing its last two terms. Thus they can be removed. In the remaining simplices, $(X, X\square)$ and $(X\square)$ are the only possible types containing $X\square$ for $X = O, T$. They can be removed as there is a unique chain of type $(X, X\square)$ (which is maximal now) containing its last term.

In the complex Δ' of the remaining chains, each simplex does not contain any term of type $X\square$ for $X = O$ or T. Thus if the type of a term of a chain $C \in \Delta'$ contains $\square$, then it is $OT\square$, $T\Sigma\square$ or $O\Sigma\square$. (Note that there is no radical group of type $\Sigma\square$.) Chains of length 4 in Δ' can be removed as follows, where for example the symbol

$$(T, T\Sigma, OT\Sigma\square) - (T, T\Sigma, T\Sigma\square, OT\Sigma\square)$$

means that by Lemma 3.2(2) a chain of type $(T, T\Sigma, OT\Sigma\square)$ is contained in a unique chain of type $(T, T\Sigma, T\Sigma\square, OT\Sigma\square)$, which is maximal in Δ', and therefore we can collapse chains of types $(T, T\Sigma, OT\Sigma\square)$ and $(T, T\Sigma, T\Sigma\square, OT\Sigma\square)$. Note that ther are no overlaps among the types appearing in the list, so we can remove these chains simultaneously.

$$\begin{array}{ll}
(T, T\Sigma, OT\Sigma\square) - (T, T\Sigma, T\Sigma\square, OT\Sigma\square), & (\Sigma, T\Sigma, OT\Sigma\square) - (\Sigma, T\Sigma, T\Sigma\square, OT\Sigma\square), \\
(O, O\Sigma, OT\Sigma\square) - (O, O\Sigma, O\Sigma\square, OT\Sigma\square), & (\Sigma, O\Sigma, OT\Sigma\square) - (\Sigma, O\Sigma, O\Sigma\square, OT\Sigma\square), \\
(O, OT, OT\Sigma\square) - (O, OT, OT\square, OT\Sigma\square), & (O, OT, OT\Sigma\square) - (T, OT, OT\square, OT\Sigma\square), \\
(T, OT\Sigma, OT\Sigma\square) - (T, T\Sigma, OT\Sigma, OT\Sigma\square), & (T\Sigma, OT\Sigma, OT\Sigma\square) - (\Sigma, T\Sigma, OT\Sigma, OT\Sigma\square), \\
(\Sigma, OT\Sigma, OT\Sigma\square) - (\Sigma, O\Sigma, OT\Sigma, OT\Sigma\square), & (O\Sigma, OT\Sigma, OT\Sigma\square) - (\Sigma, O\Sigma, OT\Sigma, OT\Sigma\square), \\
(O, OT\Sigma, OT\Sigma\square) - (O, OT, OT\Sigma, OT\Sigma\square), & (OT, OT\Sigma, OT\Sigma\square) - (T, OT, OT\Sigma, OT\Sigma\square).
\end{array}$$

The complex Δ'' of remaining chains does not contain chains of length 4. In Δ'', we then collapse as follows:

$$
\begin{array}{cc}
(T\Sigma, OT\Sigma\Box) - (T\Sigma, T\Sigma\Box, OT\Sigma\Box), & (O\Sigma, OT\Sigma\Box) - (O\Sigma, O\Sigma\Box, OT\Sigma\Box), \\
(OT, OT\Sigma\Box) - (OT, OT\Box, OT\Sigma\Box), & (T, OT\Sigma\Box) - (T, T\Sigma\Box, OT\Sigma, \Box), \\
(\Sigma, OT\Sigma\Box) - (T, O\Sigma\Box, OT\Sigma, \Box), & (O, OT\Sigma\Box) - (T, OT\Box, OT\Sigma, \Box), \\
(OT\Sigma, OT\Sigma\Box) - (T, OT\Sigma, OT\Sigma, \Box), & (O\Sigma\Box, OT\Sigma\Box) - (O, O\Sigma\Box, OT\Sigma, \Box), \\
(T\Sigma\Box, OT\Sigma\Box) - (\Sigma, T\Sigma\Box, OT\Sigma, \Box), & (O, OT\Box) - (O, OT, OT\Box), \\
(OT, OT\Box) - (T, OT, OT\Box), & (O, O\Sigma\Box) - (O, O\Sigma, O\Sigma\Box), \\
(T\Sigma, T\Sigma\Box) - (T, T\Sigma, T\Sigma\Box), & (T\Sigma, T\Sigma\Box) - (\Sigma, T\Sigma, T\Sigma\Box).
\end{array}
$$

In the remaining complex, we finally remove the chains of the following types:

$$
\begin{array}{l}
(OT\Sigma\Box) - (OT\Sigma, OT\Sigma\Box),\ (OT\Box) - (T, OT\Box), \\
(O\Sigma\Box) - (\Sigma, O\Sigma\Box),\ (T\Sigma\Box) - (\Sigma, T\Sigma\Box).
\end{array}
$$

We removed all the chains with terms of type containing $\Box$. Hence

Proposition 4.4 *The simplicial complex $\Delta(\mathcal{B}_2(M_{24}))$ is M_{24}-homotopically equivalent to the subcomplex $\mathcal{P}_2(M_{24})$ (the 2-local geometry for M_{24}) consisting of chains of subgroups conjugate to U_F for $\Box \notin F$.*

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