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FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $Sp(2,\mathbf{R})$; THE CASE OF P_J-PRINCIPAL SERIES AND DISCRETE SERIES

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1.Introduction

In this note, we study a kind of generalized Whittaker models, or equally, of generalized spherical functions associated with automorphic forms on the real symplectic group of degree two. We call these spherical functions 'Fourier-Jacobi type', since these are closely connected with the coefficients of the 'Fourier-Jacobi expansions' of (holomorphic or non-holomorphic) automorphic forms. Also these can be considered as a non-holomorphic analogue of the local Whittaker-Shintani functions on $Sp(2,\mathbf{R})$ of Fourier-Jacobi type in the paper of Murase and Sugano [6].

2.Preliminaries

2.1.Groups and algebras. We denote by $\mathbf{Z}_{\geq m}$ the set of integers n such that $n \geq m$. Moreover, we use the convention that unwritten components of a matrix are zero.

Let G be the real symplectic group $Sp(2,\mathbf{R})$ of degree two given by

$$Sp(2,\mathbf{R}) = \left\{ g \in M_4(\mathbf{R}) \;\middle|\; {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \det g = 1 \right\}.$$

Let $\theta(g) = {}^t\bar{g}^{-1}$ $(g \in G)$ be a Cartan involution of G and K be the set of fixed points of θ. Then K becomes a maximal compact subgroup of G which is isomorphic to the unitary group $U(2)$.

Let $\mathfrak{g} = \{X \in M_4(\mathbf{R}) \mid J_2 X + {}^t X J_2 = 0\}$ be the Lie algebra of G. If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^t\bar{X}$ $(X \in \mathfrak{g})$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of θ in $\mathfrak{g}$, respectively, and hence

$$\mathfrak{k} = \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A, B \in M_2(\mathbf{R}), {}^t A = -A, {}^t B = B \right\},$$
$$\mathfrak{p} = \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \;\middle|\; A, B \in M_2(\mathbf{R}), {}^t A = A, {}^t B = B \right\}.$$

Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Of course, $\mathfrak{k}$ is the Lie algebra of K which is isomorphic to the unitary algebra $\mathfrak{u}(2)$.

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$. Let $\mathfrak{h}$ be a compact Cartan subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{h} = \left\{ H(\theta_1, \theta_2) = \left(\begin{array}{cc|cc} & & \theta_1 & \\ & & & \theta_2 \\ \hline -\theta_1 & & & \\ & -\theta_2 & & \end{array} \right) \,\middle|\, \theta_i \in \mathbf{R} \right\}.$$

Now we identify a linear form $\beta : \mathfrak{h}_{\mathbf{C}} \to \mathbf{C}$ with $(\beta_1, \beta_2) \in \mathbf{C}^2$ via $\beta = \beta_1 e_1 + \beta_2 e_2$, where $e_i(H(\theta_1, \theta_2)) = \sqrt{-1}\theta_i$. Then the set of roots $\Delta = \Delta(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ of $(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ is given by

$$\Delta = \{\pm(2,0),\ \pm(0,2),\ \pm(1,1),\ \pm(1,-1)\}.$$

Fix a positive root system $\Delta^+ = \{(2,0),\ (0,2),\ (1,1),\ (1,-1)\}$, and put Δ_c^+ and Δ_n^+ the set of compact and non-compact positive roots, respectively. Then

$$\Delta_c^+ = \{(1,-1)\}, \qquad \Delta_n^+ = \{(2,0),\ (0,2),\ (1,1)\}.$$

If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_\beta$, then we have a decomposition $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_+ = \sum_{\beta \in \Delta_n^+} \mathfrak{g}_\beta$ and $\mathfrak{p}_- = \sum_{\beta \in \Delta_n^+} \mathfrak{g}_{-\beta}$.

Put P_J the Jacobi maximal parabolic subgroup of G with the Langlands decomposition $P_J = M_J A_J N_J$, where

$$M_J = \left\{ \left(\begin{array}{cc|cc} \varepsilon & & & \\ & a & & b \\ \hline & & \varepsilon & \\ & c & & d \end{array} \right) \,\middle|\, \varepsilon \in \{\pm 1\},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}) \right\} \simeq \{\pm I\} \times SL(2,\mathbf{R}),$$

$$N_J = \left\{ n(x,y;z) = \left(\begin{array}{cc|cc} 1 & y & & \\ & 1 & & \\ \hline & & 1 & \\ & & -y & 1 \end{array} \right) \cdot \left(\begin{array}{cc|cc} 1 & & z & x \\ & 1 & x & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \,\middle|\, x, y, z \in \mathbf{R} \right\},$$

and $A_J = \{\mathrm{diag}(a, 1, a^{-1}, 1) \,|\, a > 0\}$. Remark that the unipotent radical N_J of P_J is isomorphic to the 3-dimensional Heisenberg group $\mathcal{H}_1$. The Levi part $M_J A_J$ of P_J acts on N_J via the conjugate action, and M_J gives the centralizer of the center $Z(N_J) = \{n(0,0;z) \,|\, z \in \mathbf{R}\} \simeq \mathbf{R}$ of N_J in $M_J A_J$. Now we define the Jacobi group R_J by the semidirect product $M_J^\circ \ltimes N_J \simeq SL(2,\mathbf{R}) \ltimes \mathcal{H}_1$, where $M_J^\circ \simeq SL(2,\mathbf{R})$ is the identity component of M_J.

2.2.Representations. First we investigate the irreducible unitary representations of the Jacobi group R_J. Since $Z(R_J) = Z(N_J) \simeq \mathbf{R}$, the central characters of elements in $\hat{R}_J$ and $\hat{N}_J$ are parametrized by the real numbers. Then we call an irreducible unitary representation of R_J and N_J *of type m* if its central character is of the form $z \mapsto e^{2\pi\sqrt{-1}mz}$ with $m \in \mathbf{R}$. Let $\nu \in \hat{N}_J$ of type m. According to the theorem of Stone-von Neumann (*cf.* Corwin-Greenleaf [1; pp.46-47, 51-52]), ν is a character if $m = 0$ and ν is infinite dimensional if $m \neq 0$. Moreover ν of type $m \neq 0$ is uniquely determined by m up to unitary equivalence. Now we fix an irreducible unitary representation $(\nu_m, \mathcal{U}_m)$ of N_J of type $m \neq 0$. From the theory of the Weil representation, $(\nu_m, \mathcal{U}_m)$ can be extended to a continuous true projective unitary representation $(\tilde{\nu}_m, \mathcal{U}_m)$ of R_J by $\tilde{\nu}_m(\tilde{n}) = W_m(g)\nu_m(n)$ for $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$ with the Weil representation W_m on M_J°. Here $\tilde{\nu}_m$ has a factor set α which is a proper 2-cocycle.

Lemma 2.1. (Satake [7; Appendix I, Proposition 2]) *Let $\tilde{\nu}_m$ $(m \neq 0)$ as above. For every irreducible projective unitary representation π of M_J° with factor set α^{-1}, put $\rho(\tilde{n}) = \pi(g) \otimes \tilde{\nu}_m(\tilde{n})$ for $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$. Then ρ is an irreducible unitary representation of R_J. Conversely, all irreducible unitary representations of R_J of type $m \neq 0$ are obtained in this manner. Moreover ρ is square-integrable iff π is so.*

Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J of type $m \neq 0$. From the above lemma, we can regard $(\rho, \mathcal{F}_\rho) \in \hat{R}_J$ as a tensor product representation $(\pi_1 \otimes \tilde{\nu}_m, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$. Here, if we write $\widetilde{M_J^\circ}$ for the double cover of $M_J^\circ \simeq SL(2, \mathbf{R})$, $(\tilde{\nu}_m, \mathcal{U}_m)$ is a unitary representation of $\widetilde{M_J^\circ} \ltimes N_J$ which is extended from $(\nu_m, \mathcal{U}_m) \in \hat{N}_J$ as above and $(\pi_1, \mathcal{W}_{\pi_1})$ is a unitary representation of $\widetilde{M_J^\circ}$ which does not factor through M_J°. On the other hand, the unitary dual of $\widetilde{M_J^\circ}$ is given as follows.

Proposition 2.2. (*cf.* Gelbert[2; Lemma 4.1, 4.2]) *The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of $\widetilde{SL}(2, \mathbf{R})$.*

(1) *(unitary principal series) $\mathcal{P}_s^\tau$, $s \in \sqrt{-1}\mathbf{R}$, $\tau = 0, 1, \pm\frac{1}{2}$ except for the case $(s, \tau) = (0, 1)$.*
(2) *(complementary series) $\mathcal{C}_s^\tau$, $0 < s < 1$ for $\tau = 0, 1$ and $0 < s < \frac{1}{2}$ for $\tau = \pm\frac{1}{2}$.*
(3) *((limit of) discrete series) $\mathcal{D}_k^\pm$, $k \in \frac{1}{2}\mathbf{Z}_{\geq 2}$.*
(4) *(quotient representation) $\mathcal{D}_{\frac{1}{2}}^-$, $\mathcal{D}_{\frac{1}{2}}^+$.*
(5) *The trivial representation of $\widetilde{SL}(2, \mathbf{R})$.*

In the above, the representations $\mathcal{P}_s^\tau$, $\mathcal{C}_s^\tau$ for $\tau = 0,\ 1$, $\mathcal{D}_k^\pm$ for $k \in \mathbf{Z}_{\geq 1}$ and (5) factor through $SL(2, \mathbf{R})$, and the otherwise not.

Hence we take as $(\pi_1, \mathcal{W}_{\pi_1})$ one of the irreducible unitary representations $\mathcal{P}_s^\tau$, $\mathcal{C}_s^\tau$ with $\tau = \pm\frac{1}{2}$ and $\mathcal{D}_k^\pm$ with $k \in \frac{1}{2}\mathbf{Z}\backslash\mathbf{Z}$, $k \geq \frac{1}{2}$.

Remark 2.3. The Weil representation W_m considered as the representation of $\widetilde{M_J^\circ}$ has the following irreducible decomposition;

$$W_m = \begin{cases} \mathcal{D}_{\frac{1}{2}}^+ \oplus \mathcal{D}_{\frac{3}{2}}^+, & \text{if } m > 0, \\ \mathcal{D}_{\frac{1}{2}}^- \oplus \mathcal{D}_{\frac{3}{2}}^-, & \text{if } m < 0. \end{cases}$$

Next, we treat the irreducible unitary representations of K. Since Δ_c^+ is also a positive system of $\Delta(\mathfrak{k}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$, then the set of the Δ_c^+-dominant weights, and thus

$\hat{K}$, is parametrized by the set

$$\Lambda = \{\lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2\}$$

(*cf.* Knapp[4; Theorem 4.28]). We denote by $(\tau_\lambda, V_\lambda)$ the element of $\hat{K}$ corresponding to $\lambda = (\lambda_1, \lambda_2) \in \Lambda$. Here $\dim V_\lambda = d_\lambda + 1$ with $d_\lambda = \lambda_1 - \lambda_2$.

Both of $\mathfrak{p}_\pm$ become K-modules via the adjoint representation of K, and we have isomorphisms $\mathfrak{p}_+ \simeq V_{(2,0)}$ and $\mathfrak{p}_- \simeq V_{(0,-2)}$. For a given irreducible K-module V_λ with the parameter $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, the tensor product K-modules $V_\lambda \otimes \mathfrak{p}_+$ and $V_\lambda \otimes \mathfrak{p}_-$ have the irreducible decompositions

$$V_\lambda \otimes \mathfrak{p}_+ \simeq \bigoplus_{\beta \in \Delta_n^+} V_{\lambda+\beta}, \qquad V_\lambda \otimes \mathfrak{p}_- \simeq \bigoplus_{\beta \in \Delta_n^+} V_{\lambda-\beta}.$$

For each $\beta \in \Delta_n^+$, let $P^\beta : V_\lambda \otimes \mathfrak{p}_+ \to V_{\lambda+\beta}$ and $P^{-\beta} : V_\lambda \otimes \mathfrak{p}_- \to V_{\lambda-\beta}$ be the projectors into the irreducible factors of $V_\lambda \otimes \mathfrak{p}_\pm$.

In this note, we consider the following two series of representations of G; one is the principal series induced from P_J, and the other is the discrete series. We explain these representations in the remaining of this section.

Let $\sigma = (\varepsilon, D)$ be a representation of $M_J \simeq \{\pm I\} \times SL(2, \mathbf{R})$ with a character $\varepsilon : \{\pm I\} \to \mathbf{C}^\times$ and a discrete series representation $D = \mathcal{D}_n^\pm$ $(n \in \mathbf{Z}_{\geq 2})$ of $SL(2, \mathbf{R})$, and take a quasi-character ν_z $(z \in \mathbf{C})$ of A_J such that $\nu_z(\mathrm{diag}(a, 1, a^{-1}, 1)) = a^z$. Then we can construct a induced representation $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ of G from the Jacobi maximal parabolic subgroup $P_J = M_J A_J N_J$ by the usual manner (*cf.* Knapp[4; Chapter VII]), and call $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ *the P_J-principal series representation of G*. The following lemma is derived from the Frobenius reciprocity for induced representations.

Lemma 2.4. *$\tau_\lambda \in \hat{K}$ with the parameter $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ such that $\lambda_1 < n$ (resp. $\lambda_2 > -n$) does not occur in the K-type of $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ for $D = \mathcal{D}_n^+$ (resp. $\mathcal{D}_n^-$). The 'corner' K-types $\tau_\lambda \in \hat{K}$ of $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ with the parameter $\lambda \in \Lambda$ given below occur with multiplicity one.*

(1) *$\lambda = (n, n)$ for $\varepsilon(\gamma) = (-1)^n$ and $D = \mathcal{D}_n^+$,*
(2) *$\lambda = (n, n-1)$ for $\varepsilon(\gamma) = -(-1)^n$ and $D = \mathcal{D}_n^+$,*
(3) *$\lambda = (-n, -n)$ for $\varepsilon(\gamma) = (-1)^n$ and $D = \mathcal{D}_n^-$,*
(4) *$\lambda = (-n+1, -n)$ for $\varepsilon(\gamma) = -(-1)^n$ and $D = \mathcal{D}_n^-$.*

Here $\gamma = \mathrm{diag}(-1, 1, -1, 1)$.

In order to parametrize the discrete series representations of G, we enumerate all the positive root systems compatible to Δ_c^+:

(I) $\Delta_{\mathrm{I}}^+ = \{(1,-1), (2,0), (1,1), (0,2)\}$,
(II) $\Delta_{\mathrm{II}}^+ = \{(1,-1), (2,0), (1,1), (0,-2)\}$,
(III) $\Delta_{\mathrm{III}}^+ = \{(1,-1), (2,0), (0,-2), (-1,-1)\}$,
(IV) $\Delta_{\mathrm{IV}}^+ = \{(1,-1), (0,-2), (-1,-1), (-2,0)\}$.

Let J be a variable running over the set of indices I, II, III, IV, and let us denote the set of non-compact positive roots for the index J by $\Delta_{J,n}^+ = \Delta_J^+ - \Delta_c^+$. Define a subset Ξ_J of Δ_c^+-dominant weights by

$$\Xi_J = \{\Lambda = (\Lambda_1, \Lambda_2),\ \Delta_c^+ - \text{dominant weight} \mid \langle \Lambda, \beta \rangle > 0, \forall \beta \in \Delta_{J,n}^+\}.$$

The set $\bigcup_{J=\mathrm{I}}^{\mathrm{IV}} \Xi_J$ gives the Harish-Chandra parametrization of the discrete series representation of G. Let us write by π_Λ the discrete series representation of G with the Harish-Chandra parameter $\Lambda \in \bigcup_{J=\mathrm{I}}^{\mathrm{IV}} \Xi_J$. Then π_Λ is called *the holomorphic* discrete series representation if $\Lambda \in \Xi_{\mathrm{I}}$ and *the anti-holomorphic* one if $\Lambda \in \Xi_{\mathrm{IV}}$. Moreover if $\Lambda \in \Xi_{\mathrm{II}} \cup \Xi_{\mathrm{III}}$, a discrete series representation π_Λ is called *large* (in the sense of Vogan[8]).

The Blattner formula gives the description of the K-types of π_Λ. In particular, the minimal K-type $(\tau_\lambda, V_\lambda)$ of π_Λ is given by the formula $\lambda = \Lambda - \rho_c + \rho_n$, where ρ_c (resp. ρ_n) is the half sum of compact (resp. non-compact) positive roots in Δ_J^+. We call such λ *the Blattner parameter of* π_Λ.

3.Fourier-Jacobi type spherical functions

3.1. Radial parts. Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J and let (τ, V_τ) be a finite dimensional K-module. We denote by $C_{\rho,\tau}^\infty(R_J\backslash G/K)$ the space of smooth functions $F : G \to \mathcal{F}_\rho \otimes V_\tau$ with the property

$$F(rgk) = \big(\rho(r) \otimes \tau(k)^{-1}\big)F(g), \qquad (r, g, k) \in R_J \times G \times K.$$

On the other hand, let $C^\infty(A_J; \rho, \tau)$ be the space of smooth functions $\varphi : A_J \to \mathcal{F}_\rho \otimes V_\tau$ satisfying

$$\big(\rho(m) \otimes \tau(m)\big)\varphi(a) = \varphi(a), \qquad m \in R_J \cap K = M_J^\circ \cap K,\ \ a \in A_J.$$

Because of an Iwasawa decomposition of G, we have $G = R_J A_J K$. Also we remark that all elements in $M_J^\circ \cap K$ are commutative with $a \in A_J$. Then the restriction to A_J gives a linear map from $C_{\rho,\tau}^\infty(R_J\backslash G/K)$ to $C^\infty(A_J; \rho, \tau)$, which is injective. For each $f \in C_{\rho,\tau}^\infty(R_J\backslash G/K)$, we call $f|_{A_J} \in C^\infty(A_J; \rho, \tau)$ *the radial part of* f, where $|_{A_J}$ means the restriction to A_J.

Let $(\tau', V_{\tau'})$ be also a finite dimensional K-module. For each $\mathbf{C}$-linear map $u : C_{\rho,\tau}^\infty(R_J\backslash G/K) \to C_{\rho,\tau'}^\infty(R_J\backslash G/K)$, we have a unique $\mathbf{C}$-linear map $\mathcal{R}(u) : C^\infty(A_J; \rho, \tau) \to C^\infty(A_J; \rho, \tau')$ with the property $(uf)|_{A_J} = \mathcal{R}(u)(f|_{A_J})$ for $f \in C_{\rho,\tau}^\infty(R_J\backslash G/K)$. We call $\mathcal{R}(u)$ *the radial part of* u.

3.2. Fourier-Jacobi type spherical functions. Let $(\rho, \mathcal{F}_\rho)$ be as above and consider a C^∞-induced representation $C^\infty\mathrm{Ind}_{R_J}^G(\rho)$ with the representation space

$$C_\rho^\infty(R_J\backslash G) = \{F : G \to \mathcal{F}_\rho,\ C^\infty \mid F(rg) = \rho(r)F(g), \quad (r, g) \in R_J \times G\}$$

on which G acts by the right translation. Then $C_\rho^\infty(R_J\backslash G)$ becomes a smooth G-module and a $(\mathfrak{g}_{\mathbf{C}}, K)$-module naturally. Moreover let $(\tau, V_\tau) \in \hat{K}$ and take an irreducible Harish-Chandra module π of G with the K-type τ^*, where τ^* is the contragredient representation of τ. Now we consider the intertwining space

$$\mathcal{I}_{\rho,\pi} := \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi, C^\infty\mathrm{Ind}_{R_J}^G(\rho))$$

between $(\mathfrak{g}_{\mathbb{C}}, K)$-modules and its restriction to the K-type τ^* of π.

Let $i : \tau^* \to \pi|_K$ be a K-equivariant map and let i^* be the pullback via i. Then the map

$$\mathcal{I}_{\rho,\pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, C_\rho^\infty(R_J\backslash G)) \simeq C_{\rho,\tau}^\infty(R_J\backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\rho,\pi}$ to the K-type τ^* and we denote the image of T in $C_{\rho,\tau}^\infty(R_J\backslash G/K)$ by T_i. Now the space $\mathcal{J}_{\rho,\pi}(\tau)$ of *the algebraic Fourier-Jacobi type spherical functions of type* $(\rho, \pi; \tau)$ *on* G is defined by

$$\mathcal{J}_{\rho,\pi}(\tau) := \bigcup_{i \in \mathrm{Hom}_K(\tau^*, \pi|_K)} \{T_i \mid T \in \mathcal{I}_{\rho,\pi}\}.$$

Moreover put

$$\mathcal{J}^\circ_{\rho,\pi}(\tau) = \{f \in \mathcal{J}_{\rho,\pi}(\tau) \big| f|_{A_J}(\mathrm{diag}(a,1,a^{-1},1)) \text{ is of moderate growth as } a \to \infty\}.$$

We call $f \in \mathcal{J}^\circ_{\rho,\pi}(\tau)$ *a Fourier-Jacobi type spherical functions of type* $(\rho, \pi; \tau)$.

In this note, we investigate the space $\mathcal{J}^\circ_{\rho,\pi}(\tau)$ for the following triplet $(\rho, \pi; \tau)$: As $\pi \in \hat{G}$ and $\tau^* \in \hat{K}$, we take either the P_J-principal series representation and the corner K-type or the discrete series representation and the minimal K-type, and also as $\rho \in \hat{R}_J$ the one with the non-trivial central character, i.e. of type $m \neq 0$.

4.Differential equations

4.1.Differential operators. In this subsection, we introduce some differential operators acting on $C_{\rho,\tau}^\infty(R_J\backslash G/K)$.

Take an orthonormal basis $\{X_i\}$ of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$. Now we define a first order gradient type differential operator

$$\nabla_{\rho,\tau} : C_{\rho,\tau}^\infty(R_J\backslash G/K) \to C_{\rho,\tau\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}}^\infty(R_J\backslash G/K)$$

by

$$\nabla_{\rho,\tau} f = \sum_i R_{X_i} f \otimes X_i, \qquad f \in C_{\rho,\tau}^\infty(R_J\backslash G/K),$$

where

$$R_X f(g) = \frac{d}{dt} f(g \cdot \exp(tX))\Big|_{t=0}, \qquad X \in \mathfrak{g}_{\mathbb{C}},\ g \in G.$$

This differential operator $\nabla_{\rho,\tau}$ is called *the Schmid operator*. Then $\nabla_{\rho,\tau}$ can be decomposed as $\nabla^+_{\rho,\tau} \oplus \nabla^-_{\rho,\tau}$ with $\nabla^\pm_{\rho,\tau} : C_{\rho,\tau}^\infty(R_J\backslash G/K) \to C_{\rho,\tau\otimes\mathrm{Ad}_{\mathfrak{p}_\pm}}^\infty(R_J\backslash G/K)$ corresponding to the decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. For each $\beta \in \Delta_n^+$, *the shift operator* $\nabla^{\pm\beta}_{\rho,\tau_\lambda} : C_{\rho,\tau_\lambda}^\infty(R_J\backslash G/K) \to C_{\rho,\tau_{\lambda\pm\beta}}^\infty(R_J\backslash G/K)$ is defined as the composition of

$\nabla^{\pm}_{\rho,\tau_\lambda}$ with the projector $P^{\pm\beta}$ from $V_{\tau_\lambda}\otimes\mathfrak{p}_\pm$ into the irreducible component $V_{\tau_{\lambda\pm\beta}}$; $\nabla^{\pm\beta}_{\rho,\tau_\lambda}=(1_{\mathcal{F}_\rho}\otimes P^{\pm\beta})\nabla^{\pm}_{\rho,\tau_\lambda}$.

On the other hand, the Casimir element Ω is defined by $\Omega=\sum X_i-\sum Y_j$, where $\{Y_j\}$ is an orthonormal basis of $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. It is well known that Ω is in the center $Z(\mathfrak{g}_{\mathbb{C}})$ of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

4.2.Differential equations. In this subsection, we consider the system of differential equations satisfied by the Fourier-Jacobi type spherical functions.

First we discuss the case of the P_J-principal series representation $\pi\in\hat{G}$ and the corner K-type τ^*. It is well known that the Casimir element $\Omega\in Z(\mathfrak{g}_{\mathbb{C}})$ acts on π, hence on $\mathcal{J}_{\rho,\pi}(\tau)$, as the scalar operator χ_Ω (*cf.* Knapp[4; Corollary 8.14]). Let $\pi=\mathrm{Ind}_{P_J}^{G}(\sigma\otimes\nu_z\otimes 1_{N_J})$ with data $\sigma=(\varepsilon,\mathcal{D}_n^+)$, $\varepsilon(\gamma)=(-1)^n$, and $\tau^*=\tau^*_\lambda$ be the corner K-type of π, i.e. $\lambda=(-n,-n)$. Since $\tau^*_{\lambda+(2,2)}=\tau_{(n-2,n-2)}\in\hat{K}$ does not occur in the K-types of π from Lemma 2.4, an element in $\mathcal{J}_{\rho,\pi}(\tau)$ is annihilated by the action of the composition of the shift operators

$$\nabla^{(0,2)}_{\rho,\tau_{\lambda+(2,0)}}\circ\nabla^{(2,0)}_{\rho,\tau_\lambda}: C^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K)\to C^\infty_{\rho,\tau_{\lambda+(2,2)}}(R_J\backslash G/K).$$

Hence we have a system of differential equations satisfied by f in $\mathcal{J}_{\rho,\pi}(\tau)$;

$$(4.1)\qquad \begin{cases}\Omega f=\chi_\Omega f,\\ \nabla^{(0,2)}_{\rho,\tau_{\lambda+(2,0)}}\circ\nabla^{(2,0)}_{\rho,\tau_\lambda}f=0.\end{cases}$$

Let $\pi=\mathrm{Ind}_{P_J}^{G}(\sigma\otimes\nu_z\otimes 1_{N_J})$ with data $\sigma=(\varepsilon,\mathcal{D}_n^+)$, $\varepsilon(\gamma)=-(-1)^n$, and $\tau^*=\tau^*_\lambda$ be the corner K-type of π, i.e. $\lambda=(-n+1,-n)$. Since $\tau^*_{\lambda+(1,1)}=\tau_{(n-2,n-1)}\in\hat{K}$ does not occur in the K-types of π from Lemma 2.4, therefore an element in $\mathcal{J}_{\rho,\pi}(\tau)$ vanishes by the action of the shift operator

$$\nabla^{(1,1)}_{\rho,\tau_{\lambda+(1,1)}}: C^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K)\to C^\infty_{\rho,\tau_{\lambda+(1,1)}}(R_J\backslash G/K).$$

Hence we have a system of differential equations satisfied by f in $\mathcal{J}_{\rho,\pi}(\tau)$;

$$(4.2)\qquad \begin{cases}\Omega f=\chi_\Omega f,\\ \nabla^{(1,1)}_{\rho,\tau_{\lambda+(1,1)}}f=0.\end{cases}$$

For the case with the data $\sigma=(\varepsilon,\mathcal{D}_n^-)$, we have similar systems of equations from the Casimir operator and the shift operators.

Let $\pi=\pi_\Lambda$ be a discrete series representaiton of G with the Harish-Chandra parameter $\Lambda\in\Xi_J$ and $\tau^*=\tau^*_\lambda\in\hat{K}$ be the minimal K-type of π. Now we refer the following proposition which enables us to identify the intertwining space $\mathcal{I}_{\rho,\pi}$ with a solution space of differential equations for any $\rho\in\hat{R}_J$.

Proposition 4.1. (Yamashita [9; Theorem 2.4]) *Let $\pi=\pi_\Lambda\in\hat{G}$ and $\tau^*=\tau^*_\lambda\in\hat{K}$ be as above. Then we have a linear isomorphism*

$$\mathcal{I}_{\rho,\pi}\simeq\bigcap_{\beta\in\Delta^+_{J^*,n}}\ker(\nabla^{-\beta}_{\rho,\tau})\subset C^\infty_{\rho,\tau}(R_J\backslash G/K)$$

for any $\rho \in \hat{R}_J$. *In particular,*

$$\mathcal{J}_{\rho,\pi}(\tau) = \{F \in C^{\infty}_{\rho,\tau}(R_J\backslash G/K) \mid \nabla^{-\beta}_{\rho,\tau} F = 0, \quad \forall \beta \in \Delta^{+}_{J^*,n}\}.$$

Here the index J^* *means* IV, III, II *and* I *for* $J =$ I, II, III *and* IV, *respectively.*

5.Result

Solving the systems of the differential equations given by (4.1), (4.2) and Proposition 4.1, we obtain the following theorem.

Theorem 5.1. *Let* π *be a* P_J*-principal series representation (resp. a discrete series representation) of* $G = Sp(2,\mathbf{R})$ *and* τ^* *be the 'corner'* K*-type (resp. the minimal* K*-type) of* π. *For each irreducible unitary representation* ρ *of* R_J *of type* $m \neq 0$, *we have*

$$\dim \mathcal{J}^{\circ}_{\rho,\pi}(\tau) \leq 1.$$

Moreover the radial parts of the functions in $\mathcal{J}^{\circ}_{\rho,\pi}(\tau)$ *are expressed by the Meijer's* G*-function* $G^{3,0}_{2,3}\left(x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix}\right)$ *or more degenerate similar functions.*

Here the Meijer's G-function $G^{3,0}_{2,3}(x) = G^{3,0}_{2,3}\left(x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix}\right)$ with the complex parameters a_i, b_j $(1 \leq i \leq 2,\ 1 \leq j \leq 3)$ is the many-valued function defined by the integral

$$G^{3,0}_{2,3}(x) = G^{3,0}_{2,3}\left(x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix}\right) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{\prod_{j=1}^{3} \Gamma(b_j - t)}{\prod_{i=1}^{2} \Gamma(a_i - t)} x^t dt$$

of Mellin-Barnes type, where the contour L is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - t)$ $(1 \leq j \leq 3)$ once in the negative direction. It is known that, up to constant multiple, $G^{3,0}_{2,3}(x)$ is the unique solution of the linear differential equation of 3-rd order

$$\left\{x^3 \frac{d^3}{dx^3} + \alpha_2(x) x^2 \frac{d^2}{dx^2} + \alpha_1(x) x \frac{d}{dx} + \alpha_0(x)\right\} y = 0$$

with

$$\begin{aligned} \alpha_2(x) &= 3 - b_1 - b_2 - b_3 + x, \\ \alpha_1(x) &= (1 - b_1)(1 - b_2)(1 - b_3) + b_1 b_2 b_3 + (3 - a_1 - a_2)x, \\ \alpha_0(x) &= -b_1 b_2 b_3 + (1 - a_1)(1 - a_2)x, \end{aligned}$$

which decays exponentially as $|x| \to \infty$ in $-\frac{3}{2}\pi < \arg x < \frac{1}{2}\pi$ (See the Meijer's original paper [5] for details).

Remark 5.2. Let π be a holomorphic discrete series representation of G and τ^* be the minimal K-type of π. Moreover, put $\rho = \pi_1 \otimes \tilde{\nu}_m \in \hat{R}_J$ as in §2. For each $m \neq 0$, there is at most finitely many ρ such that $\dim \mathcal{J}^{\circ}_{\rho,\pi}(\tau) = 1$, and then the π_1-factors of such ρ's are the holomorphic discrete series representations of $\widetilde{SL}(2,\mathbf{R})$. Moreover, the radial parts of the functions in $\mathcal{J}^{\circ}_{\rho,\pi}(\tau)$ are expressed by the function of the form $x^p e^{qx}$ for some constant p, q.

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