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Estimates of fundamental solutions for Schrödinger operators and its applications

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1 Introduction and Main results

Let $V(x)$ be a non-negative potential and consider the Schrödinger operator $-\Delta+V$ on $\mathbf{R}^{n}$, $n\geq 3$. If V is a non-negative polynomial, Zhong ([Zh]) proved that the operators $\nabla^{2}(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-1/2}$, and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators. For the potential V which belongs to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh1]) generalized Zhong's results. He proved that the operators $\nabla(-\Delta+V)^{-1/2}$, and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators and the operator $\nabla^{2}(-\Delta+V)^{-1}$ is bounded on L^{p}, $1<p<\infty$. It is well known that Calderón-Zygmund operators are bounded on L^{p}, $1<p<\infty$. He also proved that the operators $V(-\Delta+V)^{-1}$ and $V^{1/2}\nabla(-\Delta+V)^{-1}$ are bounded on L^{p}, $1\leq p\leq\infty$.

For the operators $V(-\Delta+V)^{-1}$, $V^{1/2}\nabla(-\Delta+V)^{-1}$, and $\nabla^{2}(-\Delta+V)^{-1}$, Shen's results were generalized as follows ([KS]). We replace Δ by the second order uniformly elliptic operator $L_{0}=-\sum_{i,j=1}^{n}(\partial/\partial x_{i})\{a_{ij}(x)(\partial/\partial x_{j})\}$ and suppose V satisfy the same condition as above. Then the operators $V(L_{0}+V)^{-1}$, $V^{1/2}\nabla(L_{0}+V)^{-1}$, and $\nabla^{2}(L_{0}+V)^{-1}$ are bounded on weighted L^{p} space $(1<p<\infty)$ and Morrey spaces. (We need proper conditions for a_{ij} to prove boundedness of each operator.) It is well known that Calderón-Zygmund operators are bounded on weighted L^{p} space $(1<p<\infty)$ and Morrey spaces ([CF],[St]).

We shall repeat the definitions of the reverse Hölder class (e.g.[Sh2]) and the Morrey space (e.g.[CF]).

Throughout this paper we denote the ball centered at x with radius r by $B_{r}(x)$, and the letter C stands for a constant not necessarily the same at each occurrence.

Definition 1 (Reverse Hölder class) *Let $U\geq 0$.*

(1) *For $1<p<\infty$ we say $U\in(RH)_{p}$, if $U\in L_{loc}^{p}(\mathbf{R}^{n})$ and there exists a constant C such that*

$$\left(\frac{1}{|B_{r}(x)|}\int_{B_{r}(x)}U(y)^{p}dy\right)^{1/p}\leq\frac{C}{|B_{r}(x)|}\int_{B_{r}(x)}U(y)dy \tag{1}$$

holds for every $x \in \mathbf{R}^n$ *and* $0 < r < \infty$. *If (1) holds for* $0 < r \leq 1$, *we say* $U \in (RH)_{p,loc}$.

(2) *We say* $U \in (RH)_\infty$, *if* $U \in L^p_{loc}(\mathbf{R}^n)$ *and there exists a constant C such that*

$$\|U\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} U(y)dy \tag{2}$$

holds for every $x \in \mathbf{R}^n$ *and* $0 < r < \infty$. *If (2) holds for* $0 < r \leq 1$, *we say* $U \in (RH)_{\infty,loc}$.

Remark 1 (1) *If* $P(x)$ *is a polynomial, then* $U(x) = |P(x)|^\alpha$, $\alpha > 0$, *belongs to* $(RH)_\infty$ *([Fe]).*

(2) *For* $1 < p < \infty$, *it is easy to see* $(RH)_\infty \subset (RH)_p$.

Definition 2 *For* $0 \leq \mu < n$ *and* $1 \leq p < \infty$, *the Morrey space is defined by*

$$L^{p,\mu}(\mathbf{R}^n) = \left\{ f \in L^p_{loc}(\mathbf{R}^n) : \|f\|_{p,\mu} = \sup_{\substack{r>0 \\ x\in\mathbf{R}^n}} \left(\frac{1}{r^\mu} \int_{B_r(x)} |f(y)|^p dy \right)^{1/p} < \infty \right\}.$$

Note that $L^{p,0}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$.

In this paper we consider the following magnetic Schrödinger operators. Let $\mathbf{a}(x) = (a_1(x), a_2(x), \cdots, a_n(x))$,

$$L_j = \frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x), \quad \text{for} \quad 1 \leq j \leq n, \quad n \geq 3,$$

where $\mathbf{a} \in C^2(\mathbf{R}^n)$, and let

$$H = H(\mathbf{a}, V) = \sum_{j=1}^n L_j^2 + V(x),$$

where $V \in L^\infty_{loc}(\mathbf{R}^n)$ and $V \geq 0$.

We use the following notation throughout this paper. Let $\mathbf{B}(x) = (b_{jk}(x))_{1\leq j,k\leq n}$, where

$$b_{jk}(x) = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j},$$

and for $1 \leq j \leq n$, $1 \leq k \leq n$, $1 \leq l \leq n$, let

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial^2_{jk} = \frac{\partial^2}{\partial x_j \partial x_k}, \quad |Lu(x)|^2 = \sum_j |L_j u(x)|^2, \quad |L^2 u(x)|^2 = \sum_{j,k} |L_j L_k u(x)|^2,$$

$$|L^3 u(x)|^2 = \sum_{j,k,l} |L_j L_k L_l u(x)|^2, \quad \text{and} \quad |\mathbf{B}| = |\mathbf{B}(x)| = \sum_{j,k} |b_{jk}(x)|.$$

For the operator H, Shen ([Sh2]) proved that the operators VH^{-1}, $V^{1/2}LH^{-1}$, and L^2H^{-1} are bounded on L^p, $1 < p < \infty$, if V and the magnetic field $\mathbf{B}$ satisfy certain conditions given in terms of the reverse Hölder inequality. These results are extensions of the case $\mathbf{a} \equiv 0$ which was shown by himself.

The purpose of this paper is to show the following two results. The first is that the operators VH^{-1}, $V^{1/2}LH^{-1}$, and L^2H^{-1} are bounded on Morrey spaces. The second is that the operator of the type L^2H^{-1} is a Calderón-Zygmund operator. To show this we need to assume $\mathbf{a} \in C^4(\mathbf{R}^n)$ and $V \in C^3(\mathbf{R}^n)$.

In his paper [Sh2], Shen established the estimates of the fundamental solutions of the Schrödinger operator by using the auxiliary function $m(x, U)$ which was introduced by himself. The estimate plays an important role in the proof of L^p boundedness of above operators. We need his estimates to prove our results.

We shall repeat the definition of the function $m(x, U)$.

Definition 3 ([Sh1], [Sh2]) *For $x \in \mathbf{R}^n$, the function $m(x, U)$ is defined by*

$$\frac{1}{m(x,U)} = \sup\left\{ r > 0 : \frac{r^2}{|B_r(x)|} \int_{B_r(x)} U(y)dy \le 1 \right\}.$$

Remark 2 $0 < m(x, U) < \infty$ *for* $U \in (RH)_{n/2}$, *and* $1 \le m(x, U) < \infty$ *for* $U \in (RH)_{n/2,loc}$.

We state Theorem 1 and Theorem 2 which are main results of this paper.

Theorem 1 *Suppose* $\mathbf{a} \in C^2(\mathbf{R}^n)$, $V \in L^\infty_{loc}(\mathbf{R}^n)$, $n \ge 3$, *and* $V \ge 0$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \le Cm(x, |\mathbf{B}| + V)^2, \\ |\nabla \mathbf{B}(x)| \le Cm(x, |\mathbf{B}| + V)^3. \end{cases}$$

(1) *Let* $1 < p < \infty$ *and let* $0 < \mu < n$. *Then* VH^{-1} *and* $V^{1/2}LH^{-1}$ *are bounded on* $L^{p,\mu}(\mathbf{R}^n)$.

(2) *Let* $1 < p < \infty$ *and let* $0 < \mu < n$. *In addition assume that*

$$\begin{cases} |\nabla \mathbf{a}(x)| \le Cm(x, |\mathbf{B}| + V)^2, \\ |\mathbf{a}(x)| \le Cm(x, |\mathbf{B}| + V). \end{cases}$$

Then L^2H^{-1} *is bounded on* $L^{p,\mu}(\mathbf{R}^n)$.

Remark 3 *If* $V \in (RH)_\infty$ *then there exists a constant* C *such that* $V(x) \le Cm(x, V)^2$. *In Theorem 1, if* $\mathbf{a} \equiv 0$ *then the conclusion was shown in [KS] under the assumption* $V \in (RH)_\infty$.

Theorem 2 *Suppose* $\mathbf{a} \in C^4(\mathbf{R}^n)$, $V \in C^3(\mathbf{R}^n)$, $n \geq 3$, *and* $V \geq 0$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^3 V(x)| \leq Cm(x)^5, \quad |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\ |\nabla^3 \mathbf{B}(x)| \leq Cm(x)^5, \quad |\nabla^2 \mathbf{B}(x)| \leq Cm(x)^4, \\ |\nabla^2 \mathbf{a}(x)| \leq Cm(x)^3, \quad |\nabla \mathbf{a}(x)| \leq Cm(x)^2, \quad |\mathbf{a}(x)| \leq Cm(x), \end{cases} \tag{3}$$

where $m(x) = m(x, |\mathbf{B}| + V)$. *Then* $L^2(H+1)^{-1}$ *is a Calderón-Zygmund operator.*

We denote the kernel function of the operator $(H(\mathbf{a}, V) + 1)^{-1}$ by $\Gamma(x, y)$.

We prove Theorem 2 by using Shen's estimate for $\Gamma(x, y)$ and the following inequality which holds for $\lambda = 1$. For $\lambda > 0$ and $V \geq 0$,

$$|(H(\mathbf{a}, V) + \lambda)^{-1} f(x)| \leq (-\Delta + \lambda)^{-1} |f|(x), \qquad f \in L^2(\mathbf{R}^n), \tag{4}$$

([LS, Lemma 6]).

Remark 4 *Assume the same assumption as in Theorem 2. If we use (4) which holds for all* $\lambda > 0$ *and the estimate for the kernel function of the operator* $(H + \lambda)^{-1}$, *we can prove that, for all* $\lambda > 0$, $L^2(H + \lambda)^{-1}$ *is a Calderón-Zygmund operator. This can be done in the same way as in the proof of the case* $\lambda = 1$.

Remark 5 *In Theorem 2, the condition (3) hold if the components of* $\mathbf{a}$ *are polynomials and* V *is a non-negative polynomial (see [Sh2, page 820]). If* $\mathbf{a} \equiv 0$ *then the conclusion of Remark 4 also holds for* $\lambda = 0$, *namely, it follows that the operator* $\nabla^2(-\Delta + V)^{-1}$ *with non-negative potentials* V *which satisfy the same condition as in Theorem 2 is a Calderón-Zygmund operator. This is an extension of Zhong's result on the above operator with non-negative polynomials* V *([Zh, Proposition 3.1]).*

It is known that the operator $L^2(H+1)^{-1}$ is bounded on $L^2(\mathbf{R}^n)$ ([Sh2, Theorem 0.9]). Hence, to prove Theorem 2, it suffices to show that the estimates

$$|L_j L_k \Gamma(x, y)| \leq \frac{C}{|x - y|^n}, \qquad |\partial_j L_k L_l \Gamma(x, y)| \leq \frac{C}{|x - y|^{n+1}},$$

hold. (see e.g.[Ch, page 12]). As a matter of fact, stronger estimates hold as the following two theorems state.

Theorem 3 *Let* $k > 0$ *be an integer. Suppose* $\mathbf{a} \in C^3(\mathbf{R}^n)$, $V \in C^2(\mathbf{R}^n)$, $n \geq 3$, *and* $V \geq 0$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\ |\nabla^2 \mathbf{B}(x)| \leq Cm(x)^4, \quad |\nabla \mathbf{B}(x)| \leq Cm(x)^3. \end{cases}$$

Then there exists a constant C_k such that

$$|L_j L_k \Gamma(x,y)| \le \frac{C_k}{\{1+m(x)|x-y|\}^k} \cdot \frac{1}{|x-y|^n},$$

where $m(x) = m(x, |\mathbf{B}| + V)$.

Theorem 4 *Let $k > 0$ be an integer. Assume the same assumption as in Theorem 2. Then there exists a constant C_k such that*

$$|\partial_j L_k L_l \Gamma(x,y)| \le \frac{C_k}{\{1+m(x)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n+1}}.$$

Theorem 3 and Theorem 4 can be proved by the method similar to the one used in the proof of [Sh2, Theorem 1.13].

The plan of this paper is as follows. In section 2, we prove Theorem 1. In section 3, we establish Caccioppoli type inequalities which are necessary to complete the proof of Theorem 3 and Theorem 4. In section 4, we prove Theorem 3. In section 5, we prove Theorem 4.

2 Proof of Theorem 1

Theorem 1 is easily proved by the following pointwise estimates. These estimates generalize the results in [Zh, Lemma 3.2] to magnetic Schrödinger operators.

Lemma 1 *Assume the same assumption as in Theorem 1 (1). Then there exist constants C_1,C_2 such that*

$$|m(x, |\mathbf{B}| + V)^2 f(x)| \le C_1 M(|H(\mathbf{a}, V)f + f|)(x), \qquad f \in C_0^\infty(\mathbf{R}^n), \tag{5}$$

$$|m(x, |\mathbf{B}| + V) L f(x)| \le C_2 M(|H(\mathbf{a}, V)f + f|)(x), \qquad f \in C_0^\infty(\mathbf{R}^n), \tag{6}$$

where M is the Hardy-Littlewood maximal operator.

To prove Lemma 1 we use the following estimates of the fundamental solutions.

Theorem 5 *Let $k > 0$ be an integer. Suppose $\mathbf{a} \in C^2(\mathbf{R}^n)$, $V \in L^{n/2}_{loc}(\mathbf{R}^n)$, $n \ge 3$, and $V \ge 0$. Also assume that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla \mathbf{B}(x)| \le C m(x, |\mathbf{B}| + V)^3. \end{cases}$$

Then there exists a constant C_k such that

$$|\Gamma(x,y)| \le \frac{C_k}{\{1+m(x, |\mathbf{B}| + V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}.$$

Theorem 6 *Let $k > 0$ be an integer. Suppose* $\mathbf{a} \in C^2(\mathbf{R}^n)$, $V \in L^\infty_{loc}(\mathbf{R}^n)$, $n \geq 3$, *and* $V \geq 0$. *Also assume that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \leq Cm(x, |\mathbf{B}| + V)^2, \\ |\nabla \mathbf{B}(x)| \leq Cm(x, |\mathbf{B}| + V)^3. \end{cases}$$

Then there exists a constant C_k such that

$$|L_j\Gamma(x,y)| \leq \frac{C_k}{\{1 + m(x, |\mathbf{B}| + V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-1}}.$$

Remark 6 *For $|x-y| \leq 1$, estimates for $|\Gamma(x,y)|$ and $|L_j\Gamma(x,y)|$ like above were obtained in [Sh2, Theorem 1.13, Theorem 2.8] respectively under the conditions given in terms of the inequality (1) which holds for $0 < r \leq 1$. Theorems 5 and 6 are obtained by the same way as in the proof of Shen's theorems.*

Proof of Lemma 1. Estimate (5) can be proved as follows. Let $u = H(\mathbf{a}, V)f + f$ and let $r = 1/m(x, |\mathbf{B}| + V)$. Then it follows from Theorem 5 that

$$\begin{aligned} |m(x, |\mathbf{B}| + V)^2 f(x)| &\leq \int_{\mathbf{R}^n} m(x, |\mathbf{B}| + V)^2 |\Gamma(x,y)||u(y)|dy \\ &\leq C_k \int_{\mathbf{R}^n} \frac{m(x, |\mathbf{B}| + V)^2 |u(y)|}{\{1 + m(x, |\mathbf{B}| + V)|x-y|\}^k |x-y|^{n-2}} dy \\ &\leq C_k \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < |x-y| \leq 2^j r} \frac{|u(y)|}{(1 + r^{-1}|x-y|)^k |x-y|^{n-2}} dy \\ &\leq C_k \sum_{j=-\infty}^{\infty} \int_{|x-y| \leq 2^j r} \frac{|u(y)|}{(1 + 2^{j-1})^k (2^{j-1}r)^{n-2}} dy \\ &\leq C_k \sum_{j=-\infty}^{\infty} \frac{2^{2(j-1)+n}}{(1+2^{j-1})^k} \cdot \frac{1}{(2^j r)^n} \int_{|x-y| \leq 2^j r} |u(y)|dy \\ &\leq CC_k \sum_{j=-\infty}^{\infty} \frac{2^{2j}}{(1+2^j)^k} M(|u|)(x). \end{aligned}$$

Therefore we obtain the desired estimate, if we take $k = 3$ for example.

The proof of (6) can be done in the same way as above by using Theorem 6. □

Proof of Theorem 1 (1). The boundedness of the operators $V(H+1)^{-1}$ and $V^{1/2}L(H+1)^{-1}$ immediately follows from the fact that the Hardy-Littlewood maximal operator is bounded on Morrey spaces ([CF]). Then from the argument of scale invariance (e.g.[Sh2, pp.839-840]), the desired conclusion follows. □

Proof of Theorem 1 (2). Let $f \in C_0^\infty(\mathbf{R}^n)$. Note that

$$L_j L_k = -\partial_{jk}^2 - a_j L_k - a_k L_j - \frac{1}{i}\partial_j a_k - a_j a_k,$$

$$H(\mathbf{a}, V) = -\Delta + V - 2\sum_{j=1}^{n} a_j L_j - \frac{1}{i}\mathrm{div}\mathbf{a} - |\mathbf{a}|^2.$$

Also note that for $1 < p < \infty$ an inequality

$$\int_{B_{R/2}(x_0)} |\nabla^2 f(x)|^p dx \le C \int_{B_R(x_0)} |\Delta f(x)|^p dx + \frac{C}{R^{2p}} \int_{B_R(x_0)} |f(x)|^p dx \tag{7}$$

holds ([Sh2, page 836]).

From Theorem 1 (1) it follows that

$$\begin{aligned} \|L^2 f\|_{p,\mu} &\le C\{\|H(\mathbf{a}, V)f\|_{p,\mu} + \|mLf\|_{p,\mu} + \|m^2 f\|_{p,\mu}\} \\ &\le C\{\|H(\mathbf{a}, V)f\|_{p,\mu} + \|f\|_{p,\mu}\}. \end{aligned}$$

Then from the argument of scale invariance, desired estimate follows. □

3 Caccioppoli type inequalities

In this section we prepare the following lemmas. We call these estimates Caccioppoli type inequalities.

For the rest of this paper, we let $m(x) = m(x, |\mathbf{B}| + V)$.

Lemma 2 ([Sh2, Lemma 1.2]) *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$. Then there exists a constant C such that*

$$\int_{B_{R/2}(x_0)} |Lu(x)|^2 dx \le \frac{C}{R^2} \int_{B_R(x_0)} |u(x)|^2 dx.$$

Lemma 3 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ and*

$$\begin{cases} |\nabla V(x)| \le Cm(x)^3, \\ |\nabla \mathbf{B}(x)| \le Cm(x)^3. \end{cases}$$

Then there exist constants C, k_1 such that

$$\int_{B_{R/4}(x_0)} |L^2 u(x)|^2 dx \le \frac{C\{1 + Rm(x_0)\}^{k_1}}{R^4} \int_{B_R(x_0)} |u(x)|^2 dx.$$

Remark 7 $|\nabla \mathbf{B}(x)| \le Cm(x)^3$ *implies* $|\mathbf{B}(x)| \le Cm(x)^2$ *(see [Sh2, Remark 1.8]), which is also used to prove Lemma 3.*

Lemma 4 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ and*

$$\begin{cases} |\nabla^2 V(x)| \le Cm(x)^4, & |\nabla V(x)| \le Cm(x)^3, \\ |\nabla^2 \mathbf{B}(x)| \le Cm(x)^4, & |\nabla \mathbf{B}(x)| \le Cm(x)^3. \end{cases}$$

Then there exist constants C, k_2 such that

$$\int_{B_{R/8}(x_0)} |L^3u(x)|^2dx \le \frac{C\{1+Rm(x_0)\}^{k_2}}{R^6}\int_{B_R(x_0)} |u(x)|^2dx.$$

Lemma 2 implies Lemma 3. Since we can prove Lemma 4 using the same idea as in the proof of Lemma 3, we prove only Lemma 3.

We also need following Lemma 5 to prove Lemma 3.

Lemma 5 ([Sh1, Lemma 1.4(b)]) *Suppose $U \in (RH)_{n/2}$ and $U \ge 0$. Then there exist constants C, k_0 such that*

$$m(y,U) \le C\{1+|x-y|m(x,U)\}^{k_0}m(x,U).$$

Now we give

Proof of Lemma 3. Note that, for $1 \le j \le n$, $1 \le k \le n$,

$$[L_j, L_k] = L_jL_k - L_kL_j = \frac{1}{i}(\partial_k a_j - \partial_j a_k) = \frac{1}{i}b_{jk}, \tag{8}$$

$$\begin{aligned}[L_k, L_j^2+V] &= L_j[L_k,L_j] + [L_k,L_j]L_j + [L_k,V] \\ &= \frac{2}{i}b_{kj}L_j + \frac{1}{i}\partial_k V - \partial_j b_{kj}.\end{aligned} \tag{9}$$

Hence

$$\begin{aligned}(H(\mathbf{a},V)+1)L_ku &= -[L_k, H(\mathbf{a},V)+1]u = -\sum_{j=1}^n [L_k, L_j^2+V]u \\ &= \sum_{j=1}^n \left\{-\frac{2}{i}b_{kj}L_ju - \left(\frac{1}{i}\partial_k V - \partial_j b_{kj}\right)u\right\}.\end{aligned}$$

Let $\eta \in C_0^\infty(B_{R/2}(x_0))$ such that $\eta \equiv 1$ on $B_{R/4}(x_0)$ and $|\nabla\eta| \le C/R$. Multiply the equation by η^2L_ku, integrate over $\mathbf{R}^n$ by integration by parts, we have

$$\begin{aligned}&\int_{\mathbf{R}^n} L_j(L_ku)L_j(\eta^2L_ku) \\ &\quad \le \sum_{j=1}^n \int_{\mathbf{R}^n}\left\{-\frac{2}{i}b_{kj}(L_ju)\eta^2(L_ku) - \left(\frac{1}{i}\partial_kV - \partial_jb_{kj}\right)u\eta^2(L_ku)\right\}.\end{aligned} \tag{10}$$

The left hand side of (10) is equal to

$$\int_{\mathbf{R}^n}\left\{(L_jL_ku)^2\eta^2 + \frac{2}{i}\eta(L_jL_ku)\cdot\partial_j\eta L_ku\right\}.$$

Then we have

$$\int_{\mathbf{R}^n} |L^2u(x)|^2\eta(x)^2dx \leq C\int_{\mathbf{R}^n} |\nabla\eta(x)|^2|Lu(x)|^2dx + C\int_{\mathbf{R}^n} |\mathbf{B}(x)||Lu(x)|^2\eta(x)^2dx$$
$$+C\int_{\mathbf{R}^n}(|\nabla V(x)| + |\nabla\mathbf{B}(x)|)|u(x)||Lu(x)|\eta(x)^2dx.$$

By Lemmas 2 and 5, we obtain

$$\begin{aligned}&\int_{B_{R/4}(x_0)} |L^2u(x)|^2dx \\ &\leq \frac{C}{R^2}\int_{B_{R/2}(x_0)} |Lu(x)|^2dx \\ &\quad+\frac{C\{1+Rm(x_0)\}^{2(k_0+1)}}{R^2}\int_{B_{R/2}(x_0)} |Lu(x)|^2dx \\ &\quad+\frac{C\{1+Rm(x_0)\}^{3(k_0+1)}}{R^3}\cdot R\int_{B_{R/2}(x_0)}\left(|Lu(x)|^2+\frac{1}{R^2}|u(x)|^2\right)dx \\ &\leq \frac{C\{1+Rm(x_0)\}^{k_1}}{R^4}\int_{B_R(x_0)} |u(x)|^2dx,\end{aligned}$$

where $k_1 = 3(k_0+1)$. □

4 Proof of Theorem 3

Theorem 3 follows easily from

Lemma 6 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\ |\nabla^2\mathbf{B}(x)| \leq Cm(x)^4, \quad |\nabla\mathbf{B}(x)| \leq Cm(x)^3. \end{cases}$$

Then for any positive integer k there exists a constant C_k such that

$$\sup_{y\in B_{R/2}(x_0)} |L^2u(y)| \leq \frac{C_k}{\{1+Rm(x_0)\}^k}\cdot\frac{1}{R^2}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |u(x)|^2dx\right)^{1/2}. \tag{11}$$

Assuming this lemma for the moment, we give

Proof of Theorem 3. By using (4), we have

$$|\Gamma(x,y)| \leq \frac{C}{|x-y|^{n-2}}. \tag{12}$$

Fix $x_0, y_0 \in \mathbf{R}^n$. If we put $R = |x_0 - y_0|$, then $u(x) = \Gamma(x, y_0)$ is a solution of $H(\mathbf{a}, V)u + u = 0$ on $B_{R/2}(x_0)$. Hence combining (11) and (12) we arrive at the desired estimate. □

To prove Lemma 6, we need Lemmas (3 and 5) prepared in Section 3 and the following subsolution estimates.

Lemma 7 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla \mathbf{B}(x)| \leq Cm(x)^3. \end{cases}$$

Then for any positive integer k there exists a constant C_k such that

$$\sup_{y \in B_{R/2}(x_0)} |u(y)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{13}$$

Proof. By using the same way as in the proof of [Sh2, Lemma 1.11], for all $0 < R < \infty$ we obtain the estimate for $|u(x_0)|$, i.e.

$$|u(x_0)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{14}$$

Then, (13) follows easily from (14). Indeed, for all $y \in B_{R/2}(x_0)$, $H(\mathbf{a}, V)u + u = 0$ in $B_{R/4}(y)$. Then from (14) it follows that

$$|u(y)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|B_{R/4}(y)|} \int_{B_{R/4}(y)} |u(x)|^2 dx \right)^{1/2}.$$

Then we have

$$\sup_{y \in B_{R/2}(x_0)} |u(y)| \leq \frac{CC_k}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \quad \square$$

Lemma 8 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \leq Cm(x)^2, \\ |\nabla \mathbf{B}(x)| \leq Cm(x)^3. \end{cases}$$

Then for any positive integer k there exists a constant C_k such that

$$\sup_{y \in B_{R/2}(x_0)} |Lu(y)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{15}$$

Proof. By using the same way as in the proof of [Sh2, Lemma 2.7], for all $0 < R < \infty$ we obtain the estimate for $|Lu(x_0)|$, i.e.

$$|Lu(x_0)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{16}$$

Combining (16) and the argument in the proof of Lemma 7, we arrive at (15). $\square$

To prove Lemma 6, we also need

Lemma 9 ([Sh, Lemma 1.3]) *Suppose* $H(\mathbf{a},V)u+u=f$ *in* $B_R(x_0)$. *Then there exists a constant* C *such that*

$$\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|u(x)|^q dx\right)^{1/q} \leq C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2 dx\right)^{1/2} +CR^2\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(x)|^p dx\right)^{1/p},$$

where $2\leq p\leq q\leq\infty$ *and* $1/q>1/p-2/n$.

Now we are ready to give

Proof of Lemma 6. (This lemma can be proved by the method similar to the one used in the proof of [Sh2, Lemma 2.3].) Note that, for $1\leq j\leq n$, $1\leq k\leq n$, $1\leq l\leq n$,

$$\begin{aligned}[L_kL_l, L_j^2+V] &= L_k[L_l, L_j^2+V]+[L_k, L_j^2+V]L_l \\ &= \frac{2}{i}b_{lj}L_kL_j+\frac{2}{i}b_{kj}L_jL_l-2\partial_k b_{lj}L_j+\left(\frac{1}{i}\partial_l V-\partial_j b_{lj}\right)L_k \\ &\quad+\left(\frac{1}{i}\partial_k V-\partial_j b_{kj}\right)L_l-\left(\partial_{kl}^2 V+\frac{1}{i}\partial_{kj}^2 b_{lj}\right),\end{aligned}\tag{17}$$

where we have used (9).

Hence

$$\begin{aligned}&(H(\mathbf{a},V)+1)L_kL_l u=-[L_kL_l, H(\mathbf{a},V)+1]u \\ &=-\sum_{j=1}^n[L_kL_l, L_j^2+V]u \\ &=\sum_{j=1}^n\Big\{-\frac{2}{i}b_{lj}L_kL_ju-\frac{2}{i}b_{kj}L_jL_lu+2\partial_k b_{lj}L_ju-\left(\frac{1}{i}\partial_l V-\partial_j b_{lj}\right)L_ku \\ &\quad-\left(\frac{1}{i}\partial_k V-\partial_j b_{kj}\right)L_lu+\left(\partial_{kl}^2V+\frac{1}{i}\partial_{kj}^2 b_{lj}\right)u\Big\}.\end{aligned}$$

It then follows that, if $2\leq p\leq q\leq\infty$ and $1/q>1/p-2/n$,

$$\begin{aligned}&\left(\frac{1}{|B_{R/64}(x_0)|}\int_{B_{R/64}(x_0)}|L^2u(x)|^q dx\right)^{1/q} \\ &\leq C\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|L^2u(x)|^2dx\right)^{1/2} \\ &\quad+CR^2\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}\{|\mathbf{B}(x)||L^2u(x)|\}^p dx\right)^{1/p} \\ &\quad+CR^2\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}\{(|\nabla V(x)|+|\nabla\mathbf{B}(x)|)|Lu(x)|\}^p dx\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+CR^2\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}\{(|\nabla^2 V(x)|+|\nabla^2\mathbf{B}(x)|)|u(x)|\}^p dx\right)^{1/p}\\
\leq\ &\frac{C\{1+Rm(x_0)\}^{k_1/2}}{R^2}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&+CR^2\{1+Rm(x_0)\}^{2k_0}m(x_0)^2\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|L^2u(x)|^p\right)^{1/p}\\
&+CR^2\{1+Rm(x_0)\}^{3k_0}m(x_0)^3\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|Lu(x)|^p\right)^{1/p}\\
&+CR^2\{1+Rm(x_0)\}^{4k_0}m(x_0)^4\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|u(x)|^p\right)^{1/p}\\
\leq\ &\frac{C\{1+Rm(x_0)\}^{k_3}}{R^2}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&+C\{1+Rm(x_0)\}^{2(k_0+1)}\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|L^2u(x)|^p dx\right)^{1/p},
\end{aligned}
$$

where k_3 is a constant depending only on k_0 and we have used Lemmas 5, 7, 8, and 9.

A bootstrap argument then yields that

$$
\begin{aligned}
|L^2u(x_0)|\ \leq\ &\frac{C\{1+Rm(x_0)\}^{k_4}}{R^2}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&+C\{1+Rm(x_0)\}^{k_4}\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}|L^2u(x)|^2 dx\right)^{1/2}\\
\leq\ &\frac{C\{1+Rm(x_0)\}^{k_1/2+k_4}}{R^2}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
\leq\ &\frac{C_k}{\{1+Rm(x_0)\}^k}\cdot\frac{1}{R^2}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2 dx\right)^{1/2},
\end{aligned}
$$

where k_4 is a constant depending only on n and k_0 and we have used Lemmas 3 and 7. □

5 Proof of Theorem 4

By the same argument as in the proof of Theorem 3, we obtain Theorem 4 by the following lemma.

Lemma 10 *Suppose $H(\mathbf{a}, V)u + u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^3 V(x)| \leq Cm(x)^5, \quad |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\ |\nabla^3 \mathbf{B}(x)| \leq Cm(x)^5, \quad |\nabla^2 \mathbf{B}(x)| \leq Cm(x)^4, \\ |\nabla^2 \mathbf{a}(x)| \leq Cm(x)^3, \quad |\nabla \mathbf{a}(x)| \leq Cm(x)^2, \quad |\mathbf{a}(x)| \leq Cm(x). \end{cases}$$

Then for any positive integer k there exists a constant C_k such that

$$\sup_{y \in B_{R/2}(x_0)} |\nabla L^2 u(y)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R^3} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2},$$

where $|\nabla L^2 u(x)| = \left(\sum_{j,k,l} |\partial_j L_k L_l u(x)|^2 \right)^{1/2}$.

Proof. This lemma can also be proved by the method similar to the one used in the proof of [Sh2, Lemma 2.3]. We omit the details. □

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