Title	Central Limit Theorem Related to the Correlation of the Conjugacy Classes in the Infinite Symmetric Group(Recent Trends in Infinite Dimensional Non-Commutative Analysis)
Author(s)	Hora, Akihito
Citation	数理解析研究所講究録 (1998), 1035: 104-113
Issue Date	1998-04
URL	http://hdl.handle.net/2433/61924
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1035 巻 1998 年 104-113

Central Limit Theorem Related to the Correlation of the Conjugacy Classes in the Infinite Symmetric Group

Akihito Hora * 洞 彰人

Okayama University

Abstract

We consider a formal sum of elements over a conjugacy class in the group algebra of the infinite symmetric group and call it an adjacency operator. Using the idea of algebraic or combinatorial approach in central limit theorems of probability theory, we exactly compute the correlation function of these adjacency operators. The main body of this talk is based on [Ho1].

1 Introduction

Let $\mathcal{S}_\infty$ denote the infinite symmetric group:

$$\mathcal{S}_\infty := \{\text{bijection } \sigma : \mathbf{N} \longrightarrow \mathbf{N} | \sigma(k) = k \text{ except finite } k\text{'s}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n \ .$$

The adjacency operator corresponding to conjugacy class C in $\mathcal{S}_\infty$ is by definition formal sum

$$A_C := \sum_{x \in C} x \tag{1}$$

in the group algebra of $\mathcal{S}_\infty$. The conjugacy classes are parametrized by the Young diagrams through cycle representation of permutations. Let $\mathcal{D}$ denote the set of those Young diagrams which contain no rows consisting of a single box. If $\lambda \in \mathcal{D}$ contains $k^{(j)}$ rows of length j (i.e. j-cycles), we use the notation $\lambda = (2^{k^{(2)}} 3^{k^{(3)}} \cdots j^{k^{(j)}} \cdots)$ and set

$$|\lambda| := \sharp \text{ of boxes in } \lambda = \sum_{j=2}^{\infty} j k^{(j)} \ ,$$

*Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Okayama 700, Japan.

where $k^{(j)} = 0$ for sufficiently large j's. C_λ denotes the conjugacy class corresponding to $\lambda \in \mathcal{D}$. Then, $\lambda \mapsto C_\lambda$ gives a bijection between $\mathcal{D}$ and the set of the nontrivial conjugacy classes in $\mathcal{S}_\infty$. A_λ denotes the adjacency operator A_{C_λ} for $\lambda \in \mathcal{D}$.

Let $\phi := \langle \delta_e, \cdot \, \delta_e \rangle_{\ell^2(\mathcal{S}_\infty)}$ denote the (vacuum) vector state, where δ_e is the delta function on unit element e. Our aim is to discuss the correlation of the adjacency operators with respect to ϕ, namely

$$\phi(A_{\lambda_1}^{p_1} A_{\lambda_2}^{p_2} \cdots A_{\lambda_m}^{p_m}) \, , \tag{2}$$

for $\lambda_1, \cdots, \lambda_m \in \mathcal{D}$ and $p_1, \cdots, p_m \in \mathbf{N}$. Since Eq.(2) is a formal expression, we need a precise formulation. Here the idea of central limit theorem in probability theory comes to be useful. Taking partial sums in Eq.(1), appropriate normalization, and infinite volume limit, we will obtain an exact form of the correlation function. Eq.(1) is a sum of noncommutative and dependent observables, though the noncommutativity and dependence are not so strong. Thus our central limit theorem is related to what is called noncommutative or quantum probability. In §2, we briefly review those central limit theorems which lie in the background of our problem mainly from an algebraic or combinatorial viewpoint.

Now we present the main result. For given $\lambda \in \mathcal{D}$ and $n > |\lambda|$, we set

$$C_\lambda^{(n)} := C_\lambda \cap \mathcal{S}_n \, , \qquad A_\lambda^{(n)} := \sum_{x \in C_\lambda^{(n)}} x \, . \tag{3}$$

$H_r(x)$ denotes the Hermite polynomial of degree r which obeys the recurrence formula:

$$\begin{aligned} &H_{r+1}(x) = xH_r(x) - rH_{r-1}(x) \quad (r \geq 1) \\ &H_0(x) = 1 \, , \qquad H_1(x) = x \, . \end{aligned} \tag{4}$$

Theorem Let $\lambda_1, \cdots, \lambda_m \in \mathcal{D}$ and $p_1, \cdots, p_m \in \mathbf{N}$ be given. For each $i \in \{1, \cdots, m\}$, let $\lambda_i = (2^{k_i^{(2)}} 3^{k_i^{(3)}} \cdots j^{k_i^{(j)}} \cdots)$. Then we have

$$\lim_{n\to\infty} \phi\Big(\Big(\frac{A_{\lambda_1}^{(n)}}{\sqrt{\sharp C_{\lambda_1}^{(n)}}}\Big)^{p_1} \cdots \Big(\frac{A_{\lambda_m}^{(n)}}{\sqrt{\sharp C_{\lambda_m}^{(n)}}}\Big)^{p_m}\Big) = \prod_{j\geq 2} \int_{\mathbf{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \Big(\frac{H_{k_1^{(j)}}(x)}{\sqrt{k_1^{(j)}!}}\Big)^{p_1} \cdots \Big(\frac{H_{k_m^{(j)}}(x)}{\sqrt{k_m^{(j)}!}}\Big)^{p_m} dx \, . \tag{5}$$

In §3, we state an outline of the proof of Theorem as well as several remarks including bibliographical comments.

2 Algebraic Approach in Central Limit Theorem

2.1 Noncommutative probability

Noncommutative or quantum probability theory is a framework containing probabilistic interpretation of observables (through their spectral decomposition). Observables, being noncommuting operators, are regarded as random variables in an appropriate setting. While classical probability is based on measure theory, mathematical foundation of noncommutative probability is theory of operator algebras. At the same time, remembering recent progress in quantitative analysis of finite probability models, I feel that combinatorial aspects of noncommutative probability are potential research fields.

Let us recall quickly some terminology. A noncommutative probability space consists of unital (∗-) algebra $\mathcal{B}$ and unital (positive) linear functional ϕ on $\mathcal{B}$. An element $a \in \mathcal{B}$ being regarded as a noncommutative random variable, distribution μ of a is determined by $\mu(f) := \phi(f(a))$ where f is taken from $Fun(\mathbf{R})$, $Fun(\mathbf{C})$, $\mathbf{C}[x]$ etc. according to the context. In particular, if a is a self-adjoint operator on Hilbert space $\mathcal{H}$, functional calculus enables us to consider $f \in Fun(\mathbf{R}) \mapsto f(a) \in \mathcal{B}(\mathcal{H})$. Thus a is regarded as a real-valued random variable and its distribution coincides with the spectral measure of a with respect to ϕ. More generally, a (∗-) algebraic homomorphism from another (∗-) algebra $\mathcal{A}$ to $\mathcal{B}$ gives an $\mathcal{A}$-valued random variable.

2.2 What is central limit theorem

Let us recall a classical central limit theorem. Assume that $X_1, X_2, \cdots$ are independent identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with every moment to be finite. Independence means having no correlations; actually it suffices to assume

$$E(X_1^{p_1} X_2^{p_2} \cdots X_m^{p_m}) = E(X_1^{p_1}) E(X_2^{p_2}) \cdots E(X_m^{p_m}) \quad (p_1, p_2, \cdots, p_m \in \mathbf{N}) \tag{6}$$

for our purpose. If $E(X_1) = 0$, law of large numbers yields

$$(X_1 + \cdots + X_n)/n \longrightarrow 0 \quad a.s. \quad \text{as} \quad n \to \infty\,,$$

which shows macroscopic behavior in a sense. Central limit theorem yields the effect of a sum of small fluctuations in more extended scaling near 0. Assuming $E(X_1^2) = 1$ in addition, one has

$$(X_1+\cdots+X_n)/\sqrt{n} \longrightarrow \text{ standard Gaussian random variable } \quad \text{as} \quad n\to\infty$$

in distribution. This convergence is equivalent to that of all moments:

$$E\Big(\big(\frac{X_1+\cdots+X_n}{\sqrt{n}}\big)^p\Big) \longrightarrow \int_{\mathbf{R}} x^p \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \quad \text{as} \quad n\to\infty \quad \text{for} \quad \forall p\in\mathbf{N}\,.$$

(The right hand side is 0 for odd p and $(2r)!/(2^r r!)$ for even $p=2r$.)

In this argument, one uses independence of random variables in the sense of Eq.(6) and checks convergence of every moment. This procedure does not need the underlying random parameter space Ω explicitly and admits a direct extension to noncommutative situation. For observables (self-adjoint operators) $X_1, X_2, \cdots$, their distributions were the spectral measures on $\mathbf{R}$. For example, if the spectrum of X_j is $\{-1,1\}$ and X_j's have no correlations in some sense, then $X_1, X_2, \cdots$ may be regarded as a model of noncommutative coin tossing (i.e. Bernoulli sequence). Even in such a simple case, the limit behavior of $(X_1+\cdots+X_n)/\sqrt{n}$ is quite nontrivial and may obey either Gaussian or non-Gaussian limit distribution.

2.3 Noncommutative central limit theorem

One of the most famous noncommutative central limit theorems involves the free independence due to Voiculescu. See [VDN] and [Vo]. Let us recall a typical example. Let $F(n)$ be the free group generated by n elements $e_1,\cdots,e_n$ and $\phi^{(n)} := \langle\delta_e, \cdot\,\delta_e\rangle_{\ell^2(F(n))}$ denote the vacuum vector state on $F(n)$. (Alternatively, since we let n go to ∞, we may consider $F(\infty)$ and the vacuum state on it from the beginning.) Take self-adjoint element $X_j := (e_j+e_j^{-1})/\sqrt{2}$ in the group algebra of $F(n)$, where $\phi^{(n)}(X_j)=0$ and $\phi^{(n)}(X_j^2)=1$. We note that

$$\frac{X_1+\cdots+X_n}{\sqrt{n}} = \frac{1}{\sqrt{2n}}\sum_{j=1}^{n}(e_j+e_j^{-1})$$

is the adjacency operator of the Cayley graph of $F(n)$ normalized by the square root of its degree. The following central limit theorem for free groups is well-known:

$$\text{distribution of} \quad \frac{X_1+\cdots+X_n}{\sqrt{n}} \longrightarrow \frac{1}{2\pi}\sqrt{4-x^2}\, I_{[-2,2]}(x)dx \quad \text{as} \quad n\to\infty\,.$$

The right hand side is usually called the standard semi-circle distribution (of Wigner). Its odd moment is 0, while its even $p=2r$th moment is $\binom{2r}{r}/(r+1)$ $(=\sharp\{$noncrossing pair partitions of $1,\cdots,2r\}$, called a Catalan number).

This situation is generalized to the free independence case. Subalgebras $\mathcal{B}_1, \mathcal{B}_2, \cdots$ of $\mathcal{B}$ are said to be free if

$$i_1 \neq i_2 \neq \cdots \neq i_p,\ a_j \in \mathcal{B}_{i_j},\ \phi(a_j) = 0\,(j = 1, \cdots, p) \Longrightarrow \phi(a_1 \cdots a_p) = 0\ .$$

Freeness gives one meaning to "no correlations". If $\mathcal{B}_j$'s are free and $X_j \in \mathcal{B}_j$, $\phi(X_j) = 0$, $\phi(X_j^2) = 1$ are satisfied, then $(X_1 + \cdots + X_n)/\sqrt{n}$ converges in distribution to the standard semi-circle one.

In terms of Cayley graphs, the moments of $X_1 + \cdots + X_n$, a sum of elements in the group algebra, are closely related to the numbers of the closed walks in the graph, which is easily seen from

$$\phi((X_1 + \cdots + X_n)^p) = \sum_{(i_1, \cdots, i_p) \in \{1, \cdots, n\}^p} \phi(X_{i_1} X_{i_2} \cdots X_{i_p})\ .$$

Such a consideration goes to more general graphs beyond lattices — commutative group $\mathbf{Z}^n$ — and regular trees — free group $F(n)$. In order to treat central limit theorems for noncommutative sums in a systematic way, often useful is the notion of "singleton condition". For this notaion, its variants, and several combinatorial approaches in central limit theorem, see e.g. [SW], [AHO], and [Ho2]. Also in our present problem, we will perform combinatorial counting arguments in the next section.

3 Correlation Function of the Adjacency Operators on $\mathcal{S}_\infty$

3.1 Corollaries of Theorem

We mention two facts which follow from Theorem stated in Introduction. One is concerned with the limit distribution of a single adjacency operator. The other characterizes asymptotic independence of adjacency operators.

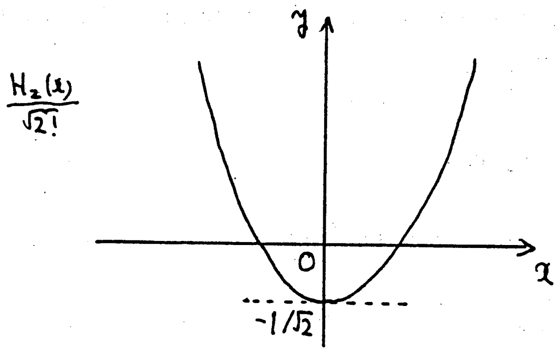
Corollary 1 Let $\lambda = (2^{k(2)} 3^{k(3)} \cdots j^{k(j)} \cdots) \in \mathcal{D}$. The distribution of $A_\lambda^{(n)}/\sqrt{\sharp C_\lambda^{(n)}}$ with respect to ϕ converges to

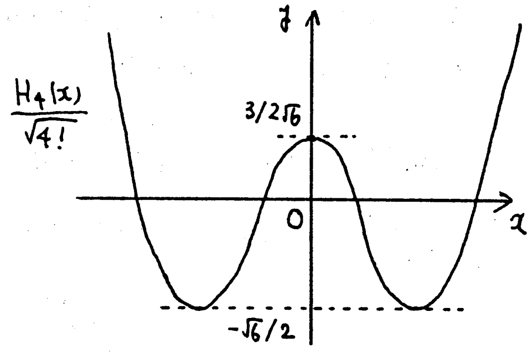
$$\Big(\prod_{j \geq 2} H_{k(j)}(x)/\sqrt{k(j)!}\Big)_* N(0,1)^{\otimes\infty}$$

as $n \to \infty$, where subscript $_*$ indicates the push-forward (i.e. image measure).

Example For two-rows diagram $(a^2) = \overbrace{\boxed{}}^{a}$ $(k^{(a)} = 2)$, the limit distribution is

$$(\sqrt{\pi}\sqrt{\sqrt{2}y+1})^{-1}e^{-(\sqrt{2}y+1)/2}I_{(-1/\sqrt{2},\infty)}(y)dy \quad \text{(a gamma distribution)}.$$





Corollary 2 If λ_i and λ_j contain no rows of equal length $(\forall i, j \in \{1, \cdots, m\},\ i \neq j)$, then $A_{\lambda_1}^{(n)}/\sqrt{\sharp C_{\lambda_1}^{(n)}}, \cdots, A_{\lambda_m}^{(n)}/\sqrt{\sharp C_{\lambda_m}^{(n)}}$ are asymptotically independent random variables.

3.2 Kerov's result

In [Ke], Kerov showed the following theorem. Let $C_k^{(n)}$ be the conjugacy class of the k-cycles in S_n (hence corresponding to $\overbrace{\boxed{}}^{k}$). For $\alpha \in \hat{S}_n$, $\chi_\alpha^{(n)}$ denotes the irreducible character and $d_\alpha^{(n)} := \dim \chi_\alpha^{(n)}$ its dimension. The Plancherel measure $M^{(n)}$ is defined by $M^{(n)}(\{\alpha\}) := d_\alpha^{(n)2}/n!$. Set

$$\varphi_k^{(n)}(\alpha) := n^{k/2}\chi_\alpha^{(n)}(C_k^{(n)})/d_\alpha^{(n)} \quad (\alpha \in \hat{S}_n)$$

where $\chi_\alpha^{(n)}(C_k^{(n)})$ indicates the value $\chi_\alpha^{(n)}(g)$ at $\forall g \in C_k^{(n)}$.

Kerov's theorem For $\forall x_2, \cdots, x_m \in \mathbf{R}$, we have

$$\lim_{n\to\infty} M^{(n)}(\{\alpha \in \hat{S}_n | \varphi_k^{(n)}(\alpha) \leq x_k,\ 2 \leq \forall k \leq m\}) = \prod_{k=2}^{m} \int_{-\infty}^{x_k} \frac{e^{-y^2/(2k)}}{\sqrt{2\pi k}} dy\ .$$

Hence $\{\varphi_k^{(n)}\}_{k=2,3,\cdots}$ is a family of asymptotically independent random variables with Gaussian limit distributions.

For an arbitrary finite group G and $\alpha \in \hat{G}$, let χ_α and $d_\alpha := \dim \chi_\alpha$ be the irreducible character and its dimension. The Plancherel measure M on $\hat{G}$ is defined by $M(\{\alpha\}) := d_\alpha^2/|G|$. For conjugacy classes $C_1, \cdots, C_p$ in G, we have

$$\phi(A_{C_1} \cdots A_{C_p}) = \int_{\hat{G}} \frac{(\sharp C_1)\chi_\alpha(C_1)}{d_\alpha} \cdots \frac{(\sharp C_p)\chi_\alpha(C_p)}{d_\alpha} M(d\alpha)\ .$$

When $G = S_n$,

$$\sharp C_k^{(n)} = n(n-1)\cdots(n-k+1)/k \sim n^k/k \quad \text{as} \quad n \to \infty .$$

Hence Corollary 1 and Corollary 2 implies that the above Kerov's theorem is equivalent to the case of one-row Young diagrams in our Theorem.

3.3 Outline of the proof of Theorem

See [Hol] for more details. Set $n^{\underline{r}} := n(n-1)\cdots(n-r+1)$. Since

$$\sharp C_\lambda^{(n)} = n^{\underline{|\lambda|}} / \prod_{j\geq 2} j^{k(j)} k(j)!$$

holds for $\lambda = (2^{k(2)}3^{k(3)}\cdots j^{k(j)}\cdots)$, we see

$$(\sharp C_{\lambda_1}^{(n)})^{p_1/2}\cdots(\sharp C_{\lambda_m}^{(n)})^{p_m/2} \asymp n^{(p_1|\lambda_1|+\cdots+p_m|\lambda_m|)/2} \quad \text{as} \quad n \to \infty . \tag{7}$$

In comparison with Eq.(7), we consider which terms survive in

$$\phi(A_{\lambda_1}^{(n)p_1}\cdots A_{\lambda_m}^{(n)p_m}) = \sum_{g_i^{(l)} \in C_{\lambda_i}^{(n)}} \phi(g_1^{(1)}\cdots g_1^{(p_1)}\cdots\cdots g_m^{(1)}\cdots g_m^{(p_m)}) \tag{8}$$

as $n \to \infty$. Let us express each $g_i^{(l)}$ in Eq.(8) as a product of cycles and set

$$\nu := \sharp \bigcup_{i=1}^{m} \bigcup_{l=1}^{p_i} \{\text{letters which move by } g_i^{(l)}\} .$$

We see that
(i) if $2\nu > p_1|\lambda_1| + \cdots + p_m|\lambda_m|$, then $g_1^{(1)}\cdots g_1^{(p_1)}\cdots\cdots g_m^{(1)}\cdots g_m^{(p_m)} \neq e$ in Eq.(8),
(ii) if $2\nu < p_1|\lambda_1| + \cdots + p_m|\lambda_m|$, then the number of such terms in Eq.(8) is of smaller order than Eq.(7),
(iii) if $2\nu = p_1|\lambda_1| + \cdots + p_m|\lambda_m|$, then such a term containing a letter which appears only once does not survive.
Thus we have only to consider the following terms.

Reduction 1 Every letter appears in $g_1^{(1)}\cdots g_1^{(p_1)}\cdots\cdots g_m^{(1)}\cdots g_m^{(p_m)}$ in Eq.(8) exactly twice or never appears.

Lemma 1 Let $g_i (\neq e) \in S_n$ be expressed as a product of cycles ($i = 1, \cdots, q$). Assume that every letter appearing in $g_1 g_2 \cdots g_q$ appears exactly twice. Then, $g_1 g_2 \cdots g_q = e$ holds if and only if $\forall$cycle S in $g_1 g_2 \cdots g_q$, $\exists S^{-1}$ in $g_1 g_2 \cdots g_q$ cleared of S.

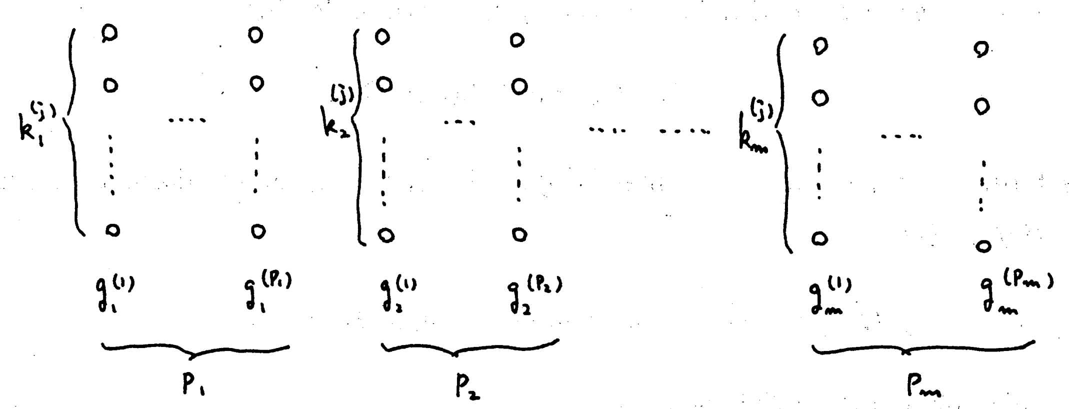
Proof is omitted. See [Hol].

Reduction 1 and Lemma 1 yield the following.

Reduction 2 The cycles in $g_1^{(1)} \cdots g_1^{(p_1)} \cdots\cdots g_m^{(1)} \cdots g_m^{(p_m)}$ form "cycle vs inverse cycle" pairs.

Under Reductions 1 and 2, we count up the numbers of terms really involved in Eq.(8).

We construct graph Γ by assigning a vertex to a cycle in $g_1^{(1)} \cdots g_1^{(p_1)} \cdots\cdots g_m^{(1)} \cdots g_m^{(p_m)}$ in Eq.(8). The j-cycles in $g_1^{(1)} \cdots g_1^{(p_1)} \cdots\cdots g_m^{(1)} \cdots g_m^{(p_m)}$ induce complete $p_1 + \cdots + p_m$-partite graph $\Gamma^{(j)}$ like



where any two vertices in the same column are not joined by an edge, while any two in different columns are joined. Set $\Gamma := \cup_{j \geq 2} \Gamma^{(j)}$. Then, a set of cycle vs inverse cycle pairs in $g_1^{(1)} \cdots g_1^{(p_1)} \cdots\cdots g_m^{(1)} \cdots g_m^{(p_m)}$ in Eq.(8) corresponds to a perfect matching in Γ. Appendix (§§3.4) explains some necessary materials in graph theory. At the moment, we freely use them to complete the proof.

Lemma 2 The limit in Theorem (i.e. LHS of Eq.(5)) coincides with

$$pm(\Gamma) / \prod_{j \geq 2} (k_1^{(j)}!)^{p_1/2} \cdots (k_m^{(j)}!)^{p_m/2} . \tag{9}$$

Proof is omitted. See [Hol].

In Eq.(9), $pm(\Gamma) = \prod_{j \geq 2} pm(\Gamma^{(j)})$ holds. Since

$$\overline{\Gamma^{(j)}} = \overbrace{K_{k_1^{(j)}} \cup \cdots K_{k_1^{(j)}}}^{p_1} \cup \cdots\cdots \cup \overbrace{K_{k_m^{(j)}} \cup \cdots \cup K_{k_m^{(j)}}}^{p_m} ,$$

using formulas in Appendix, we have

$$pm(\Gamma^{(j)}) \ = \ \int_{\mathbf{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mu(\overline{\Gamma^{(j)}}, x) dx$$

$$= \int_{\mathbf{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mu(K_{k_1^{(j)}}, x)^{p_1} \cdots \mu(K_{k_m^{(j)}}, x)^{p_m} dx$$

$$= \int_{\mathbf{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} H_{k_1^{(j)}}(x)^{p_1} \cdots H_{k_m^{(j)}}(x)^{p_m} dx .$$

Combining this with Lemma 2 completes the proof of Theorem.

3.4 Appendix

We briefly summarize the notions and the formulas in graph theory used in the previous subsection. See e.g. [Go] for details. An edge set M in graph G is called a perfect matching in G if every vertex of G lies in exactly one edge in M. Set

$$pm(G) := \sharp\{\text{perfect mathcing in } G\} .$$

An edge set $\{e_1, \cdots, e_r\}$ is called an r-matching in G if e_i and e_j do not share a common vertex for $\forall i \neq j$. Set

$$p(G,r) := \sharp\{r\text{-matching in } G\} , \quad p(G,0) := 1 .$$

The matchings polynomial of G is defined as

$$\mu(G,x) := \sum_{r \geq 0} (-1)^r p(G,r) x^{n-2r}$$

where $n := \sharp$ vertices of G. Complement $\overline{G}$ of G is the graph which has the same vertex set with G and in which two vertices are joined with an edge if and only if they are not joined in G.

Formula For any graph G, we have

$$pm(\overline{G}) = \int_{\mathbf{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mu(G,x) dx .$$

See [Go] for the proof.

K_r denotes the complete graph with r vertices. The recurrence formula Eq.(4) yields $\mu(K_r, x) = H_r(x)$. Finally, we note

$$\mu(G_1 \cup G_2, x) = \mu(G_1, x)\mu(G_2, x)$$

holds.

References

[AHO] Accardi,L., Hashimoto,Y., Obata,N., Notions of independence related to the free group, *Preprint*, 1997.

[Go] Godsil,C.D., *Algebraic combinatorics*, Chapman & Hall, New York, 1993.

[Ho1] Hora,A., Central limit theorem for the adjacency operators on the infinite symmetric group, *Commun. Math. Phys.*, *To appear.*

[Ho2] Hora,A., Central limit theorems and asymptotic spectral analysis on large graphs, *Preprint*, 1997.

[Ke] Kerov,S., Gaussian limit for the Plancherel measure of the symmetric group, *C. R. Acad. Sci. Paris* **316**, Série I, 303 – 308 (1993).

[SW] Speicher,R., von Waldenfels,W., A general central limit theorem and invariance principle, In: Accardi,L. et al (eds.), *Quantum probability and related topics* **IX**, 371 – 387, World Scientific, Singapore, 1994.

[Vo] Voiculescu,D.V., Free probability theory: random matrices and von Neumann algebras, *Proc. ICM 94*, Birkhäuser, Basel, 227 – 241, 1995.

[VDN] Voiculescu,D.V., Dykema,K.J., Nica,A., *Free random variables*, Amer. Math. Soc., 1992.