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Existence, uniqueness and continuous dependence of weak solutions of damped sine-Gordon equations

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1 Introduction

In this paper we establish the existence, uniqueness and continuous dependence of weak global solutions of the damped Sine-Gordon equations.

In physical situation the Sine-Gordon equation represents the dynamics of a Josephson junction driven by a current sourse. If we consider the continuous case of a coupled Josephson junction by taking into account of damping effects the sine-Gordon equation leads the partial differential equation of second order in time

$$\frac{\partial^2 y}{\partial t^2}+\alpha\frac{\partial y}{\partial t}-\beta\Delta y+\gamma\sin y=f, \tag{1.1}$$

where $\alpha,\beta,\gamma>0$ are physical constants and f is a forcing function. In their study of complex system described by (1.1), Bishop, Fesser and Lomdall [1] have observed chaotic behaviours of solutions of (1.1) by a great deal of numerical experiments. Their numerical results are very interesting, but their mathematical analysis has not been given in [1]. In this paper we study the basic problems such as existence, uniqueness and continuous dependence of solutions of (1.1).

The existence and uniqueness of the strong solutions of the Cauchy problem for (1.1) with Dirichlet and Neumann boundary conditions has been studied by J. L. Lions [5] and R.Temam[9] in the evolution equation setting. In this paper we give the variational formulation of the problem due to Dautray and Lions [2] and prove the existence, uniqueness and continuous dependence of weak solutions of the problem. We note that the proof by Temam is a sketch for more general equations and the detailed proof is not given in [9].

2 Existence of weak solutions

Let Ω be an open bounded set of R^n with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. We consider the damped sine-Gordon equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial y}{\partial t} - \beta\Delta y + \gamma \sin y = f \quad \text{in} \quad Q, \tag{2.1}$$

where $\alpha, \beta, \gamma > 0$, Δ is a Laplacian and f is a given function. In physical situation, $\alpha, \beta, \gamma > 0$ are constants representing the gratitude of damping, diffusion and non-linearity effects and f is proportional to the current intensity applied to the function. The boundary condition considered in this paper is the Dirichlet condition

$$y = 0 \quad \text{on} \quad \Sigma, \tag{2.2}$$

and the initial values are given by

$$y(0,x) = y_0(x) \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega. \tag{2.3}$$

We define two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. We endow these spaces with the usual inner products and norms

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi,\psi)^{1/2}, \quad \text{for all} \quad \phi, \psi \in L^2(\Omega), \tag{2.4}$$

$$((\psi, \phi)) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)dx, \quad \|\psi\| = ((\psi,\psi))^{1/2}, \quad \text{for all} \quad \phi, \psi \in H_0^1(\Omega). \tag{2.5}$$

Then the pair (V, H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. To use a variational formulation let us introduce the bilinear form

$$a(\phi, \varphi) = \int_\Omega \beta\nabla\phi \cdot \nabla\varphi dx = \beta((\phi, \varphi)), \quad \forall \phi, \varphi \in H_0^1(\Omega). \tag{2.6}$$

The form (2.6) is symmetric, bounded on $H_0^1(\Omega) \times H_0^1(\Omega)$ and coercive

$$a(\phi, \phi) \geq \beta\|\phi\|^2, \quad \forall \phi \in H_0^1(\Omega). \tag{2.7}$$

Then we can define the bounded operator $A \in \mathcal{L}(V, V')$ and the problem (2.1)-(2.3) is reduced to the following Cauchy problem in H:

$$\begin{cases} \dfrac{d^2y}{dt^2} + \alpha\dfrac{dy}{dt} + Ay + \gamma \sin y = f(t) \ \text{ in } \ (0,T), \\ y(0) = y_0 \in V, \ \ \dfrac{dy}{dt}(0) = y_1 \in H. \end{cases} \tag{2.8}$$

For general treatments of the damped second order equations of this type including control theoretical applications, we refer to Ha [4] and Lions [5].

The operator A in (2.8) is an isomorphism from V onto V' and it is also considered as a self-adjoint operator in H with dense domain $\mathcal{D}(A)$ in V and in H,

$$\mathcal{D}(A) = \{\phi \in V : \ A\phi \in H\}.$$

In this case A in (2.8) is an unbounded selfadjoint operator in H (cf. Tanabe [7]).

We shall write $g' = \frac{dg}{dt}, g'' = \frac{d^2g}{dt^2}$ and define a (solution) space by

$$W(0,T) = \{g : \ g \in L^2(0,T;V), \ g' \in L^2(0,T;H), \ g'' \in L^2(0,T;V')\}.$$

$\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

Now we give two definitions of solutions of the problem (2.8) (see Dautray and Lions [2] and Temam [9]).

DEFINITION 1 A function y is said to be a weak solution of (2.8) if $y \in W(0,T)$ and y satisfies

$$\langle y''(\cdot), \phi\rangle_{V',V} + \alpha(y'(\cdot), \phi) + \beta((y(\cdot), \phi)) + \gamma(\sin y, \phi) = (f(\cdot), \phi)$$
$$\text{for all } \ \phi \in V \ \text{ in the sense of } \ \mathcal{D}'(0,T), \tag{2.9}$$

$$y(0) = y_0, \ \ \frac{dy}{dt}(0) = y_1. \tag{2.10}$$

Here in Definition 1 the symbol $\langle \cdot, \cdot \rangle_{V',V}$ denotes a dual pairing between V and V'.

DEFINITION 2 A function y is said to be a strong solution of (2.8) if $y \in C([0,T]; \mathcal{D}(A))$, $y' \in C([0,T]; V)$ $y'' \in C([0,T]; H)$ and y satisfies the equations in (2.8).

For the strong solution of the sine-Gordon equation, Lions [5] and Temam [9] proved the following theorem under more general form of nonlinear terms including sine function.

THEOREM 1 Let $\alpha, \beta, \gamma > 0$ and $f,\ y_0,\ y_1$ be given satisfying

$$f \in C^1([0,\ T]; H), \quad y_0 \in \mathcal{D}(A), \quad y_1 \in V. \tag{2.11}$$

Then the problem (2.8) has a unique strong solution y.

For the weak solutions of (2.8), we can state the following theorem.

THEOREM 2 Let $\alpha, \beta > 0, \gamma \in \mathbf{R}$ and $f,\ y_0,\ y_1$ be given satisfying

$$f \in L^2(0,\ T; H), \quad y_0 \in V, \quad y_1 \in H. \tag{2.12}$$

Then the problem (2.8) has a unique weak solution y in $W(0,T)$.

The existence and uniqueness of weak solutions of (2.8) is also proved in Temam [9] under the stronger assumption that $f \in C([0,T]; H)$, but the proof is a sketch and the detailed proof is not given there. In this paper we give a complete proof of Theorem 2.

Since the embedding of V into H is compact, there exists an orthonormal basis of H, $\{w_j\}_{j=1}^{\infty}$ consisting of eigenfunctions of A such that

$$\begin{cases} Aw_j = \lambda_j w_j, \quad \forall j, \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty \quad \text{as} \quad j \to \infty. \end{cases} \tag{2.13}$$

We denote by P_m the orthogonal projection in H(or V) onto the space spanned by $\{w_1, \cdots, w_m\}$.

We divide the proof of Theorem 2 into the existence part and the uniqueness part and the uniqueness part is proved in next section.

Existence proof of Theorem 2.

Step 1. Approximate solutions

We implement a Faedo-Galerkin method as used in [2]. As a basis $\{w_m\}_{m=1}^{\infty}$ we use the set of eigenfunctions w_j of the operator A which is orthonormal in H.. For each $m \in N$ we define an approximate solution of the problem (2.8) by

$$y_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j, \tag{2.14}$$

where $y_m(t)$ satisfies

$$\begin{cases} \frac{d^2}{dt^2}(y_m(t), w_j) + \alpha\frac{d}{dt}(y_m(t), w_j) + \beta((y_m(t), w_j)) + \gamma(\sin y_m(t), w_j) \\ = (f(t), w_j),\ t \in [0,T],\ 1 \le j \le m, \\ y_m(0) = P_m y_0, \\ \frac{d}{dt}y_m(0) = P_m y_1. \end{cases} \tag{2.15}$$

We set $y_{0m} = P_m y_0$ and $y_{1m} = P_m y_1$. Then

$$y_{0m} \to y_0 \ \text{ in } \ V, \quad y_{1m} \to y_1 \ \text{ in } H \quad \text{as} \ m \to \infty. \tag{2.16}$$

Then the equation (2.15) can be written as m vector differential equation

$$\frac{d^2}{dt^2}\vec{g}_m + \alpha\frac{d}{dt}\vec{g}_m + \beta\Lambda\vec{g}_m = \vec{k}(t, \vec{g}_m)$$

with initial values $\vec{g}_m(0) = [(y_{0m}, w_1), \cdots, (y_{0m}, w_m)]^t$ and
$\frac{d}{dt}\vec{g}_m(0) = [(y_{1m}, w_1), \cdots, (y_{1m}, w_m)]^t$. Here $\vec{g}_m = [g_{1m}, \cdots, g_{mm}]^t$,
$\Lambda = diag\ (\lambda_i : i = 1, \cdots, m)$, and

$$\vec{k}(t, \vec{g}_m) = [(f(t), w_1) - \gamma(\sin(\sum_{j=1}^{m} g_{jm}w_j), w_1), \cdots, (f(t), w_m) - \gamma(\sin(\sum_{j=1}^{m} g_{jm}w_j), w_m)]^t,$$

where $[\cdots]^t$ denotes the transpose of $[\cdots]$. The nonlinear forcing function vector $\vec{k}$ is Lipschitz continuous. Indeed, for $\vec{g}_m = \sum_{j=1}^{m} g_{jm}w_j, \vec{h}_m = \sum_{j=1}^{m} h_{jm}w_j$, it follows by

$$\int_\Omega |\sin\psi(x) - \sin\phi(x)|^2 dx \le \int_\Omega |\psi(x) - \phi(x)|^2 dx, \quad \forall \psi, \phi \in H \tag{2.17}$$

and Schwartz inequality that

$$|\vec{k}(t, \vec{g}_m) - \vec{k}(t, \vec{h}_m)|^2 = \gamma^2 \sum_{i=1}^{m} |(\sin(\sum_{j=1}^{m} g_{jm}w_j) - \sin(\sum_{j=1}^{m} h_{jm}w_j), w_i)|^2$$

$$\le \ \gamma^2 m |\sin(\sum_{j=1}^{m} g_{jm}w_j) - \sin(\sum_{j=1}^{m} h_{jm}w_j)|^2$$

$$\le \ \gamma^2 m^2 \sum_{j=1}^{m} |g_{jm} - h_{jm}|^2 = \gamma^2 m^2 |\vec{g}_m - \vec{h}_m|^2.$$

Therefore this second order vector differential equation admits a unique solution $\vec{g}_m$ on $[0,T]$, by reducing this to a first order system and applying Carathéodory type existence theorem. Hence we can construct the approximate solutions $y_m(t)$ of (2.15).

Step 2. A priori estimates

In this step we shall derive a priori estimates of $y_m(t)$. We multiply both sides of the equation (2.15) by $g'_{jm}(t)$ and sum over j to have

$$(y''_m(t), y'_m(t)) + \alpha(y'_m(t), y'_m(t)) + \beta((y_m(t), y'_m(t))) = (f(t), y'_m(t)) - \gamma(\sin y_m(t), y'_m(t)). \tag{2.18}$$

It is easily verified that

$$((y_m(t), y'_m(t))) = \frac{1}{2}\frac{d}{dt}\|y_m(t)\|^2, \quad (y''_m(t), y'_m(t)) = \frac{1}{2}\frac{d}{dt}|y'_m(t)|^2. \tag{2.19}$$

Then by substituting (2.19) to (2.18), we have

$$\frac{1}{2}\frac{d}{dt}[\beta\|y_m(t)\|^2 + |y'_m(t)|^2] + \alpha|y'_m(t)|^2 = (f(t), y'_m(t)) - \gamma(\sin y_m(t), y'_m(t)). \tag{2.20}$$

Let $\epsilon > 0$ be an arbitrary real number and c_1 be the imbedding constant such that $|\phi| \le c_1\|\phi\|_V$ for all $\phi \in V$. From (2.12) and (2.17) and we obtain

$$\begin{aligned}
&2\left|\int_0^t (f(\sigma), y'_m(\sigma))d\sigma\right| + 2\left|\int_0^t \gamma(\sin(y_m(\sigma)), y'_m(\sigma))d\sigma\right| \\
\le\ & \frac{1}{\epsilon}\int_0^t |f(\sigma)|^2 d\sigma + \epsilon\int_0^t |y'_m(t)|^2 d\sigma + 2|\gamma|\int_0^t |\sin(y_m(\sigma))| \cdot |y'_m(\sigma)| d\sigma \\
\le\ & \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;H)} + \epsilon\int_0^t |y'_m(\sigma)|^2 d\sigma + |\gamma|\int_0^t \left(\frac{1}{\epsilon}|y_m(\sigma))|^2 + \epsilon|y'_m(\sigma)|^2\right) d\sigma \\
\le\ & \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;H)} + (|\gamma|+1)\epsilon\int_0^t |y'_m(t)|^2 d\sigma + \frac{|\gamma|c_1^2}{\epsilon}\int_0^t \|y_m(\sigma)\|^2 d\sigma.
\end{aligned} \tag{2.21}$$

Integrating (2.20) on $[0, t]$ and using (2.21), we obtain the following inequality

$$\begin{aligned}
&\beta\|y_m(t)\|^2 + |y'_m(t)|^2 + 2\alpha\int_0^t |y'_m(\sigma)|^2 d\sigma \\
\le\ & \beta\|y_{0m}\|^2 + |y_{1m}|^2_H \\
&+ \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;H)} + \frac{|\gamma|c_1^2}{\epsilon}\int_0^t \|y_m(t)\|^2 d\sigma + (|\gamma|+1)\epsilon\int_0^t |y'_m(\sigma)|^2 d\sigma.
\end{aligned} \tag{2.22}$$

Since $\|y_{0m}\| \le \|y_0\|$ and $|y_{1m}| \le |y_1|$ (see (2.16)), it follows from (2.22) that

$$\begin{aligned}
&\beta\|y_m(t)\|^2 + |y'_m(t)|^2 + (2\alpha - (|\gamma|+1)\epsilon)\int_0^t |y'_m(\sigma)|^2 d\sigma \\
\le\ & \beta\|y_0\|^2 + |y_1|^2 + \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;H)} + \frac{|\gamma|c_1^2}{\epsilon}\int_0^t \|y_m(\sigma)\|^2 d\sigma.
\end{aligned} \tag{2.23}$$

Let us divide (2.23) by $\beta_1 = \min\{\beta, 1\} > 0$. We choose ϵ such that $2\alpha = (|\gamma|+1)\epsilon$ and set

$$C_1 = \frac{1}{\beta_1}[\beta\|y_0\|^2 + |y_1|^2 + \epsilon^{-1}\|f\|_{L^2(0,T;H)}], \quad C_2 = \frac{\gamma c_1^2}{\beta_1\epsilon}.$$

Then (2.23) implies

$$\|y_m(t)\|^2 + |y_m'(t)|^2 \le C_1 + C_2 \int_0^t (\|y_m(\sigma)\|^2 + |y_m'(\sigma)|^2) d\sigma. \tag{2.24}$$

Thus it follows by Bellman-Gronwall's inequality that

$$\|y_m(t)\|^2 + |y_m'(t)|^2 \le C_1 \exp(C_2 t) \le C_1 \exp(C_2 T). \tag{2.25}$$

Step 3. Passage to the limit

The estimate (2.25) implies that $\{y_m\}$ is bounded in $L^\infty(0,T;V)$ and $\{y_m'\}$ is bounded in $L^\infty(0,T;H)$. Therefore, by the extraction theorem of Rellich's, we can find a subsequence $\{y_{m_l}\}$ of $\{y_m\}$ and find $z \in L^\infty(0,T;V) \subset L^2(0,T;V)$, $\overline{z} \in L^\infty(0,T;H) \subset L^2(0,T;H)$ such that

$$y_{m_l} \to z \text{ weakly star in } L^\infty(0,T;V) \text{ and weakly in } L^2(0,T;V), \tag{2.26}$$

$$y_{m_l}' \to \overline{z} \text{ weakly star in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;H). \tag{2.27}$$

By the classical compactness theorem (cf. Temam [8; Thm. 2.3, Chap.III]) the conditions (2.26) and (2.27) imply

$$y_{m_l} \to z \text{ strongly in } L^2(0,T;H). \tag{2.28}$$

Hence by (2.17),

$$\sin y_{m_l} \to \sin z \text{ strongly in } L^2(0,T;H). \tag{2.29}$$

We shall show that $\overline{z} = z'$ and $z(0) = y_0$. For $t \in [0,T)$

$$y_{m_l}(t) = y_{m_l}(0) + \int_0^t y_{m_l}'(\sigma) d\sigma \tag{2.30}$$

in the V(and hence H) sense. Moreover, $y_{m_l}(0) = y_{0m_l} \to y_0$ in the V and hence H sense, whereas for each t, $\int_0^t y_{m_l}'(\sigma)d\sigma \to \int_0^t \overline{z}(\sigma)d\sigma$ in H by (2.27). Hence, taking the limit in the weak H sense in (2.30) we obtain

$$z(t) = y_0 + \int_0^t \overline{z}(\sigma) d\sigma \text{ for } t \in [0,T). \tag{2.31}$$

This shows that $z'(t)$ exists a.e. in the H sense and $\overline{z} = z' \in L^2(0,T;H)$, $z(0) = y_0$(cf. [4, p.564]).

Let j be fixed. Multiply both sides of (2.15) by the scalar function $\zeta(t)$ with

$$\zeta \in C^1([0,T]), \quad \zeta(T) = 0, \tag{2.32}$$

and put $\phi_j = \zeta(t)w_j$. Integrating these over $[0,T]$ for $m_l > j$ and using integration by parts, we have

$$\int_0^T [-(y'_{m_l}(t), \phi'_j(t)) + \alpha(y'_{m_l}(t), \phi_j(t)) + \beta((y_{m_l}(t), \phi_j(t))) + \gamma(\sin y_{m_l}(t), \phi_j(t))]dt$$
$$= \int_0^T (f(t), \phi_j(t))dt - (y_{1m_l}, \phi_j(0))_H. \tag{2.33}$$

If we take $l \to \infty$ in (2.33) and use (2.26)-(2.29), then we have

$$\int_0^T [-(z'(t), \phi'_j(t)) + \alpha(z'(t), \phi_j(t)) + \beta((z(t), \phi_j(t))) + \gamma(\sin z(t), \phi_j(t))]dt$$
$$= \int_0^T (f(t), \phi_j(t))dt - (y_1, \phi_j(0))_H, \tag{2.34}$$

so that

$$\int_0^T \zeta'(t)(-z'(t), w_j)dt$$
$$+ \int_0^T \zeta(t)\{\alpha(z'(t), w_j) + \beta((z(t), w_j)) + \gamma(\sin z(t), w_j) - (f(t), w_j)\}dt$$
$$= -\zeta(0)(y_1, w_j). \tag{2.35}$$

It we take $\zeta \in \mathcal{D}(0,T)$ in (2.35), then

$$\frac{d}{dt}(z'(\cdot), w_j) + \alpha(z'(\cdot), w_j) + \beta((z(\cdot), w_j)) + \gamma(\sin z(\cdot), w_j) = (f(\cdot), w_j) \tag{2.36}$$

in the sense of distribution $\mathcal{D}'(0,T)$. Since $\{\sum_{j=1}^m \xi_j w_j | \xi_j \in \mathbf{R}, m \in N\}$ is dense in V, we conclude by (2.36) that $z'' = -Az - \alpha z' - \gamma \sin z - f \in L^2(0,T;V')$, so that $z \in W(0,T)$, and for all $\phi \in V$

$$\langle z''(\cdot), \phi\rangle_{V',V} + \alpha(z'(\cdot), \phi) + \beta((z(\cdot), \phi)) + \gamma(\sin z(\cdot), \phi) = (f(\cdot), \phi) \tag{2.37}$$

in the sense of $\mathcal{D}'(0,T)$. Multiplying both sides of (2.36) by ζ in (2.32) and using integration by parts, we have from (2.34)

$$(z'(0), w_j)\zeta(0) = (y_1, w_j)\zeta(0),$$

and that $(z'(0), w_j) = (y_1, w_j)$. Since $\{w_j\}_{j=1}^{\infty}$ is dense in H, we obtain $z'(0) = y_1$. This proves that z is a weak solution of the problem (2.8). This completes the existence proof of Theorem 2.

3 Uniqueness and continuous dependence

In this section we study the uniqueness and continuous dependence of weak solutions of (2.8). For this we need the following result on energy equality.

THEOREM 3 Assume that the assumption in Theorem 2 holds. Let y be a weak solution of (2.8). Then, for each $t \in [0,T]$ we have the following equality

$$\begin{aligned}
&\beta\|y(t)\|^2 + |y'(t)|^2 + 2\alpha\int_0^t |y'(\sigma)|^2 d\sigma + 2\gamma\int_0^t (\sin y(\sigma), y'(\sigma))d\sigma \\
= \;&\beta\|y_0\|^2 + |y_1|^2 + 2\int_0^t (f(\sigma), y'(\sigma))d\sigma. \qquad (3.1)
\end{aligned}$$

Proof. Since $\sin y(t) \in L^2(0,T;H)$, by considering this nonlinear term as a forcing function term, the equality (3.1) can be proved by the regularization method for linear equations as proved in Lions and Magenus [6, page 276-279].

The uniqueness proof of Theorem 2 follows immediately from the following continuous dependence result.

THEOREM 4 Assume that the assumption in Theorem 2 holds. Let y_i, $(i=1,2)$ be the weak solution of (2.8) with initial values $(y_0^i, y_1^i) \in V\times H$ and $f^i \in L^2(0,T;H)$. Then there exists a constant $C>0$ depending only on α, β, γ and T such that

$$\begin{aligned}
&\|y_1(t) - y_2(t)\|^2 + |y_1'(t) - y_2'(t)|^2 \\
\le \;& C\left(\|y_0^1 - y_0^2\|^2 + |y_1^1 - y_1^2|^2 + \int_0^t |f^1(\sigma) - f^2(\sigma)|^2 d\sigma\right),\ t \in [0,T].
\end{aligned}$$

Proof. Let $z = y_1 - y_2$. Since z is a weak solution of (2.8) with $\gamma = 0$ and $f(t) = f^1(t) - f^2(t) - \gamma(\sin y_1(t) - \sin y_2(t))$, and with initial values $y_0 = y_0^1 - y_0^2$, $y_1 = y_1^1 - y_1^2$, by Theorem 3 we have

$$\begin{aligned}
&\beta\|z(t)\|^2 + |z'(t)|^2 + 2\alpha\int_0^t |z'(\sigma)|^2 d\sigma \\
+ \;& 2\gamma\int_0^t (\sin y_1(\sigma) - \sin y_2(\sigma), z'(\sigma))d\sigma \\
= \;& \beta\|z(0)\|^2 + |z'(0)|^2 + 2\int_0^t (f_1(\sigma) - f_2(\sigma), z'(\sigma))d\sigma. \qquad (3.2)
\end{aligned}$$

We can easily verify that from (2.17)

$$2\int_0^t |(\sin y_1(\sigma) - \sin y_2(\sigma), z'(\sigma)|d\sigma \le 2\int_0^t |z(\sigma)|\cdot|z'(\sigma)|d\sigma \le \int_0^t \{c_1^2\|z(\sigma)\|_V^2 + |z'(\sigma)|^2\}d\sigma.$$

Since

$$\begin{aligned} 2\int_0^t |(f_1(\sigma)-f_2(\sigma),z'(\sigma)|d\sigma &\le 2\int_0^t |f_1(\sigma)-f_2(\sigma)|\cdot|z'(\sigma)|d\sigma \\ &\le \int_0^t (|f_1(\sigma)-f_2(\sigma)|^2+|z'(\sigma)|^2)d\sigma, \end{aligned}$$

it follows by (3.2)

$$\begin{aligned} &\beta\|z(t)\|_V^2+|z'(t)|_H^2+2\alpha\int_0^t |z'(\sigma)|^2 d\sigma \\ \le\ &\beta\|z(0)\|_V^2+|z'(0)|_H^2+2|\gamma|\int_0^t\{c_1^2\|z(\sigma)\|^2+|z'(\sigma)|^2\}d\sigma \\ &+\int_0^t(|f_1(\sigma)-f_2(\sigma)|^2+|z'(\sigma)|^2)d\sigma. \end{aligned}$$

If we put $\beta_1=\min\{1,\beta\}$ and

$$C_1=\frac{2|\gamma|+1}{\beta_1}\max\{c_1^2,1\},$$

we have by (3.3)

$$\begin{aligned} \|z(t)\|_V^2+|z'(t)|^2 &\le \|z(0)\|_V^2+|z'(0)|^2+\int_0^t |f_1(\sigma)-f_2(\sigma)|^2 d\sigma \\ &+C_1\int_0^t\left[\|z(\sigma)\|_V^2+|z'(\sigma)|^2\right]d\sigma. \end{aligned} \tag{3.3}$$

Applying Bellman-Gronwall's lemma to (3.3), we obtain

$$\begin{aligned} &\|z(t)\|_V^2+|z'(t)| \\ \le\ &\|z(0)\|_V^2+|z'(0)|^2+\int_0^t |f_1(\sigma)-f_2(\sigma)|^2 d\sigma \\ &+\int_0^t C_1e^{C_1(t-s)}\left\{\|z(0)\|^2+|z'(0)|^2+\int_0^s|f_1(\sigma)-f_2(\sigma)|^2d\sigma\right\}ds \\ \le\ &(TC_1e^{C_1T}+1)(\|z(0)\|^2+|z'(0)|^2+\int_0^t|f_1(\sigma)-f_2(\sigma)|^2d\sigma) \\ &\qquad\text{for all } t\in[0,T]. \end{aligned} \tag{3.4}$$

This proves Theorem 4.

Let $f\in L^2(Q)$ and $y_0\in H_0^1(\Omega), y_1\in L^2(\Omega)$. Then by standard manupulations (cf. Lions and Magenes [6]) we can verify that the weak solution $y=y(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}+\alpha\dfrac{\partial y}{\partial t}-\beta\Delta y+\gamma\sin y=f \ \text{ in } \ Q, \\ y=0 \ \text{ on } \ \Sigma, \\ y(0,x)=y_0(x) \ \text{ in } \ \Omega \ \text{ and } \ \dfrac{\partial y}{\partial t}(0,x)=y_1(x) \ \text{ in } \ \Omega \end{cases} \tag{3.5}$$

in the sense of distribution $\mathcal{D}'(Q)$, and

$$y,\ \frac{\partial y}{\partial t},\ \frac{\partial y}{\partial x}\in L^2(Q).$$

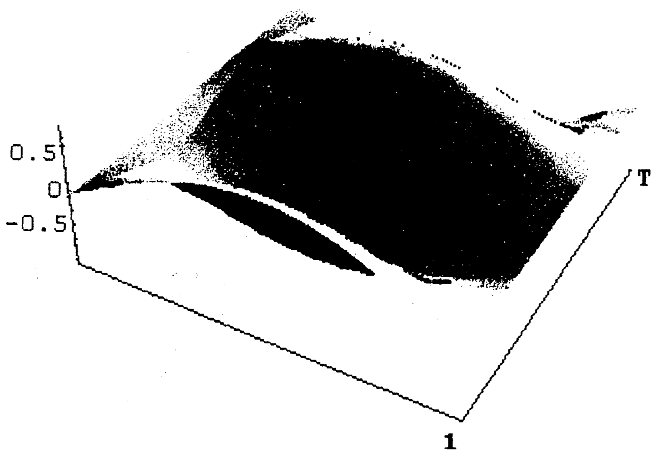
4 Correction of numerical simulations

In this section we give corrections of numerical simulations given in Section 5 of Elgamal and Nakagiri [3]. The program contains an error in constructing approximate solutions, and then many of the figures are incorrect. Here we give corrected numerical simulation results only for the damped sine-Gordon equations.

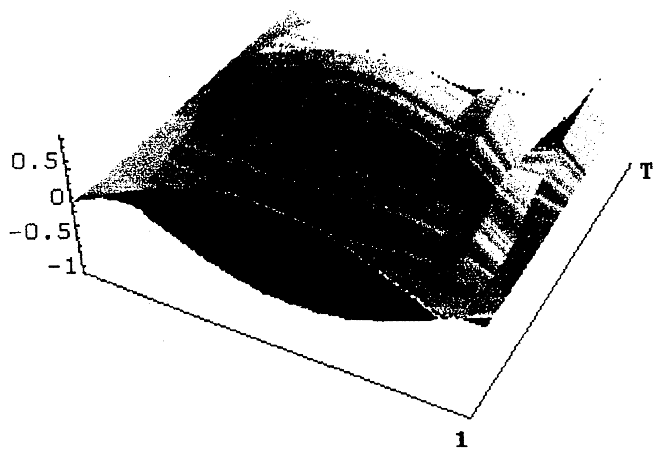
In all simulation results given below we set

$$f = 0, \quad y_0(x) = \sin \pi x, \quad y_1(x) = 0$$

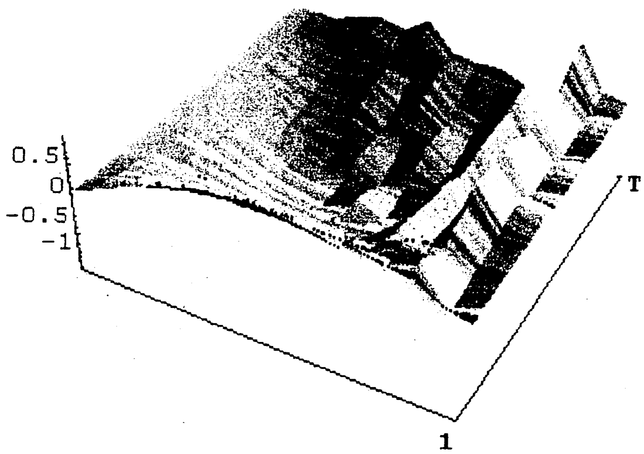
and these are normalized datum of those in [3].



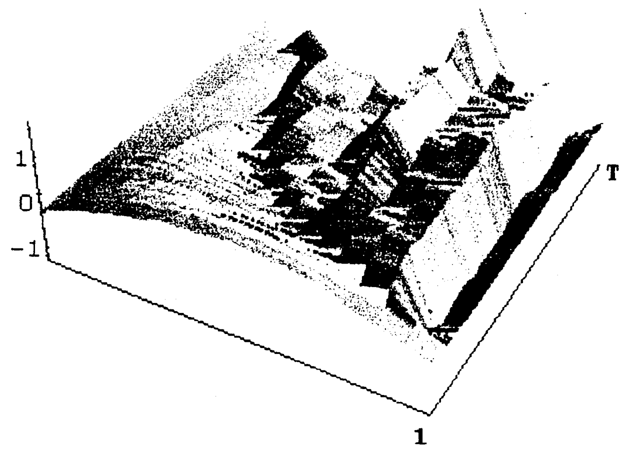
$\alpha=0, \beta=0.1, \gamma=0.1$



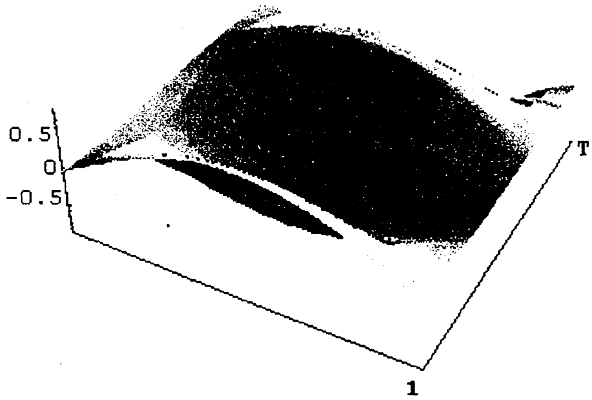
$\alpha=0, \beta=0.1, \gamma=1$



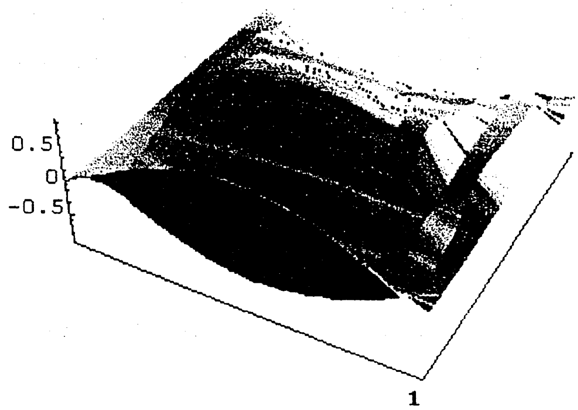
$\alpha=0, \beta=0.1, \gamma=10$



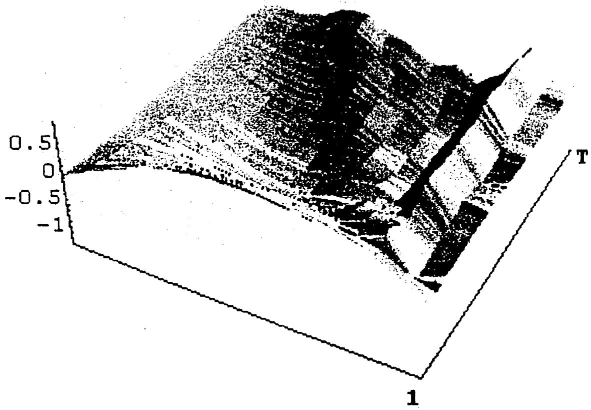
$\alpha=0, \beta=0.1, \gamma=100$



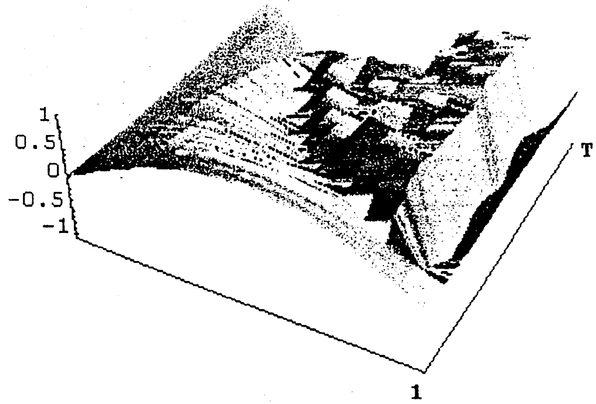
$\alpha=\beta=0.1,\gamma=0$



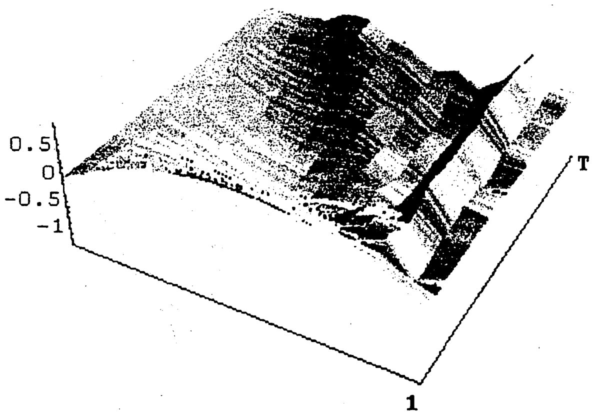
$\alpha=0.1,\beta=0.1,\gamma=1$



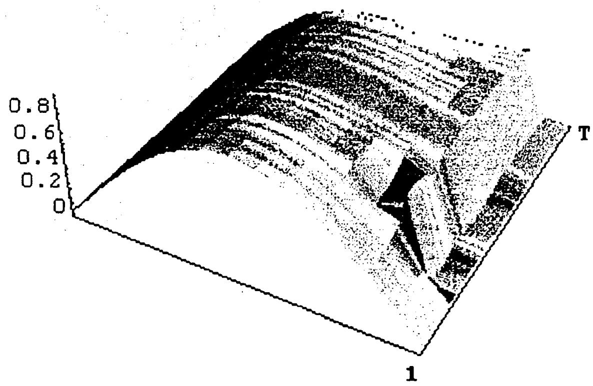
$\alpha=1,\beta=0.1,\gamma=10$



$\alpha=1,\beta=0.1,\gamma=100$



$\alpha=10,\beta=0.1,\gamma=10$



$\alpha=100,\beta=0.1,\gamma=100$

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