数理解析研究所講究録
1034 巻 1998 年 65-75

Radial symmetry of positive solutions for semilinear elliptic equations in $\boldsymbol{R}^n$

神戸大学工学部　　内藤　雄基　　(Yūki Naito)

1. Introduction and statement of the results. In this note we consider the symmetry properties of positive solutions for the equation of the form

$$\Delta u + \phi(|x|)f(u) = 0 \tag{1.1}$$

in $\boldsymbol{R}^n$, where $n \geq 3$, Δ is the n-dimentional Laplacian, and $|x|$ denotes the Euclidean length of $x \in \boldsymbol{R}^n$. In equation (1.1), we assume that $\phi \not\equiv 0$ is a locally Hölder continuous function on $[0, \infty)$ which satisfies

$$\phi(r) \geq 0 \text{ for } r \geq 0 \quad \text{and} \quad \phi(r) \text{ is nonincreasing in } r > 0,$$

and that $f \in C^1([0, \infty))$ with $f(u) > 0$ for $u > 0$.

The problem of existence of positive solutions of equation (1.1) has been studied extensively. It has been shown in [4, 5, 12] that if

$$\int_0^\infty r\phi(r)dr < \infty \tag{1.2}$$

then, under some additional conditions on f, (1.1) has infinitely many bounded positive solutions in $\boldsymbol{R}^n$.

Our main result is the following, which is a slight extension of [10, Theorem 5.16].

Theorem. *Assume that* (1.2) *holds. Then all bounded positive solutions of* (1.1) *in* $\boldsymbol{R}^n$ *are radially symmetric about the origin.*

We give some corollaries of the theorem. First assume that (1.1) has a bounded positive solution u in $\boldsymbol{R}^n$ satisfying

$$\liminf_{|x| \to \infty} u(x) > 0. \tag{1.3}$$

Then, by Lemma B.1 in Appendix B, we get (1.2). Thus we obtain the following

Corollary 1. *Assume that* (1.1) *has a bounded positive solution u in $\boldsymbol{R}^n$ satisfying* (1.3). *Then all bounded positive solutions are radially symmetric about the origin.*

Next, we consider the case where $f(0) > 0$. Assume that (1.1) has a bounded positive solution u in $\boldsymbol{R}^n$. Then, by Lemma B.2 in Appendix B, we get (1.2). Thus we obtain the following

Corollary 2. *Assume that $f(0) > 0$. Then all bounded positive solutions of* (1.1) *in $\boldsymbol{R}^n$ are radially symmetric about the origin.*

Remark. For the case $f(u) = e^{2u}$, precise existence and nonexistence criteria for positive solutions of (1.1) are obtained in [8, Theorems 1.4 and 1.5].

Symmetry properties of solutions of semilinear elliptic equations in $\boldsymbol{R}^n$ have been studied by several authors [1-3, 6-11, 16-18]. Their arguments are based on the moving plane method first developed by Serrin [16] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [2, 3]. In this note, we present an approach based on the maximum principle on unbounded domains together with the method of moving plane. This approach helps us to improve the previous results and simplify the proofs.

In Section 2, we investigate the asymptotic behavior of positive solutions of (1.1). In Section 3, we prove the main Theorem by using the method of moving planes. We give the maximum principle on unbounded domains in Appendix A, and show the conditions which are equivalent to (1.2) in Appendix B.

2. Asymptotic behavior of positive solutions.

We show the following proposition.

Proposition. *Assume that* (1.2) *holds. Let u be a bounded positive solution of* (1.1) *in $\boldsymbol{R}^n$. Then $\lim_{|x|\to\infty} u(x) = c$ and $u(x) > c$ in $\boldsymbol{R}^n$ for some constant $c \geq 0$.*

In order to prove this, we first prove the following lemma.

Lemma 1. *Let g be a continuous function in $\boldsymbol{R}^n$, and let w be the Newtonian potential of g, i.e.,*

$$w(x) = c_n \int_{R^n} \frac{g(y)}{|x-y|^{n-2}} dy,$$

where $c_n = [n(n-2)\omega_n]^{-1}$ and ω_n is the volume of the unit ball in $\boldsymbol{R}^n$. Assume that there is a nonnegative nonincreasing function G on $[0, \infty)$ satisfying

$$g(x) \leq G(|x|), \quad x \in R^n, \quad \int_0^\infty rG(r)dr < \infty. \tag{2.1}$$

Then w is well defined and satisfies

$$\lim_{|x|\to\infty} w(x) = 0. \tag{2.2}$$

Proof. By $(2.1)_2$ for any $\varepsilon > 0$ there exists $R > 0$ satisfying

$$c_n \int_R^\infty rG(r)dr < \frac{1}{3}\varepsilon \quad \text{and} \quad 3^{n-2} c_n \int_{3R}^\infty rG(r)dr < \frac{1}{3}\varepsilon. \tag{2.3}$$

From $(2.1)_1$, we have

$$|w(x)| \le c_n \int_{R^n} \frac{G(|y|)}{|x-y|^{n-2}} dy.$$

We decompose the integral as follows:

$$|w(x)| \le c_n \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) \frac{G(|y|)}{|x-y|^{n-2}} dy \equiv I_1 + I_2 + I_3,$$

where Ω_1, Ω_2, and Ω_3 are defined as

$$\Omega_1 = \{y \in \boldsymbol{R}^n : |y| \le 3R\}, \quad \Omega_2 = \{y \in \boldsymbol{R}^n : |y| \ge 3R,\ |x-y| \ge \frac{1}{3}|y|\},$$

$$\Omega_3 = \{y \in \boldsymbol{R}^n : |y| \ge 3R,\ |x-y| \le \frac{1}{3}|y|\}.$$

We estimate I_1, I_2, and I_3 as follows. Since $\lim_{|x|\to\infty} I_1 = 0$, there exists $R_1 > 3R$ so that

$$I_1 < \frac{1}{3}\varepsilon \quad \text{for } |x| > R_1. \tag{2.4}$$

From $(2.3)_2$ we obtain

$$I_2 \le 3^{n-2} c_n \int_{\Omega_2} \frac{G(|y|)}{|y|^{n-2}} dy \le 3^{n-2} c_n \int_{3R}^\infty rG(r)dr < \frac{1}{3}\varepsilon. \tag{2.5}$$

For $y \in \Omega_3$, since $|y| - |x| \le |y-x| \le \frac{1}{3}|y|$, we see that

$$\frac{2}{3}|y| \le |x|. \tag{2.6}$$

Then, for $y \in \Omega_3$ and $r \in [0, \frac{1}{3}|y|]$, we have

$$|x| - r \ge \frac{2}{3}|y| - \frac{1}{3}|y| = \frac{1}{3}|y| \ge r \quad \text{and} \quad |x| - \frac{1}{3}|y| \ge \frac{1}{3}|y| \ge R. \tag{2.7}$$

Since G is nonincreasing and $|y| \ge |x| - |x-y|$, it follows that

$$I_3 \le c_n \int_{\Omega_3} \frac{G(|x| - |x-y|)}{|x-y|^{n-2}} dy = c_n \int_0^{\frac{1}{3}|y|} rG(|x|-r)dr.$$

From (2.7) and $(2.3)_1$ we obtain

$$I_3 \le c_n \int_0^{\frac{1}{3}|y|} (|x| - r) G(|x| - r) dr = c_n \int_{|x|-\frac{1}{3}|y|}^{|x|} sG(s)ds \le c_n \int_R^{\infty} sG(s)ds < \frac{1}{3}\varepsilon. \tag{2.8}$$

Then by (2.4), (2.5), and (2.8), we have $|w(x)| < \varepsilon$ for $|x| > R_1$. Since $\varepsilon > 0$ is arbitrary, we conclude that (2.2) holds.

Proof of Proposition. Let v be the Newtonian potential of $\phi f(u)$, i.e.,

$$v(x) = c_n \int_{R^n} \frac{\phi(|y|) f(u(y))}{|x-y|^{n-2}} dy.$$

Define $f_\infty = \max\{f(s) : 0 \le s \le \|u\|_{L^\infty(R^n)}\}$. Then $\phi(|x|) f(u(x)) \le \phi(|x|) f_\infty$ in $\boldsymbol{R}^n$. Since ϕ is nonincreasing and (1.2) holds, we obtain

$$\lim_{|x|\to\infty} v(x) = 0 \tag{2.9}$$

by Lemma 1. It is easily seen that v satisfies $\Delta v + \phi f(u) = 0$ in $\boldsymbol{R}^n$. We have $\Delta(u - v) = 0$ in $\boldsymbol{R}^n$ while $u - v$ is bounded in $\boldsymbol{R}^n$ by (2.9). Then by Liouville's theorem we obtain

$$u(x) - v(x) \equiv c \quad \text{in } \boldsymbol{R}^n, \tag{2.10}$$

where c is a constant. From (2.9) we conclude that $u(x) \to c$ as $|x| \to \infty$. Observe that v satisfies $\Delta v = -\phi f(u) \le 0$ and $v \ge 0$ in $\boldsymbol{R}^n$. By the maximum principle, we have $v > 0$ in $\boldsymbol{R}^n$. From (2.10) we conclude that $u(x) > c$ in $\boldsymbol{R}^n$.

3. Proof of the theorem.

First, we introduce some notation. For $\lambda \in \boldsymbol{R}$, we define T_λ and Σ_λ as

$$T_\lambda = \{x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n : x_1 = \lambda\} \quad \text{and} \quad \Sigma_\lambda = \{x \in \boldsymbol{R}^n : x_1 < \lambda\}.$$

For $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and $\lambda \in \boldsymbol{R}$, let x^λ be the reflection of x with respect to the hyperplane T_λ, i.e., $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$. It is easy to see that, if $\lambda > 0$,

$$|x^\lambda| - |x| > 0 \quad \text{for } x \in \Sigma_\lambda. \tag{3.1}$$

Let u be a bounded positive solution of (1.1) in $\boldsymbol{R}^n$. By the propsition in Section 2, we have

$$\lim_{|x|\to\infty} u(x) = c \ge 0 \quad \text{and} \quad u(x) > c \quad \text{in } \boldsymbol{R}^n \tag{3.2}$$

for some constant c. We define

$$v_\lambda(x) = u(x) - u(x^\lambda) \quad \text{for } x \in \Sigma_\lambda.$$

Lemma 2. *Let $\lambda > 0$. Then v_λ satisfies*

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \leq 0 \quad in\ \Sigma_\lambda, \tag{3.3}$$

where

$$c_\lambda(x) = \phi(|x|) \int_0^1 f'\left(u(x^\lambda) + t(u(x) - u(x^\lambda))\right) dt. \tag{3.4}$$

We note that $c_\lambda(x)$ is well defined in $\boldsymbol{R}^n$.

Proof. Since ϕ in nonincreasing and (3.1) holds, it follows that

$$\begin{aligned} 0 &= \Delta u(x) + \phi(|x|) f(u(x)) - \Delta u(x^\lambda) - \phi(|x^\lambda|) f(u(x^\lambda)) \\ &\geq \Delta\left(u(x) - u(x^\lambda)\right) + \phi(|x|)\left(f(u(x)) - f(u(x^\lambda))\right) \\ &= \Delta v(x) + c_\lambda(x) v(x), \quad x \in \Sigma_\lambda, \end{aligned}$$

where $c_\lambda(x)$ is the function in (3.4).

Lemma 3. *Assume that* (1.2) *holds. Then there exsits a positive function $w(x)$ on $\{x \in \boldsymbol{R}^n : |x| \geq r_0\}$ satisfying for some $r_0 > 0$ and for any $\lambda > 0$*

$$\Delta w + c_\lambda(x) w \leq 0 \quad in\ |x| > r_0 \quad and \quad \liminf_{|x|\to\infty} w(x) > 0. \tag{3.5}$$

Proof. Define $g_\infty = \max\{|f'(s)| : 0 \leq s \leq \|u\|_{L^\infty(R^n)}\}$. Then from (3.4) we have

$$|c_\lambda(x)| \leq g_\infty \phi(|x|) \quad \text{in } \boldsymbol{R}^n \quad \text{for any } \lambda > 0. \tag{3.6}$$

Now consider the equation

$$\Delta w + g_\infty \phi(|x|) w = 0. \tag{3.7}$$

By applying Lemma B.1 in Appendix B to (3.7), we find that (3.7) has a positive solution w on $\{|x| \geq r_0\}$ for some $r_0 > 0$, satisfying $\liminf_{|x|\to\infty} w(x) > 0$. By (3.6), w satisfies (3.5).

Define $B_0 = \{x \in \boldsymbol{R}^n : |x| < r_0\}$, where r_0 is the constant appearing in Lemma 3.

Lemma 4. *Let $\lambda > 0$. Assume that $v_\lambda(x) > 0$ on $\partial B_0 \cap \Sigma_\lambda$. Then $v_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$.*

Proof. By Lemma 2 we obtain

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \leq 0 \quad \text{in } \Sigma_\lambda \setminus \overline{B_0}, \qquad v_\lambda > 0 \quad \text{on } \partial B_0 \cap \Sigma_\lambda.$$

By Lemma 3, there is a positive function w satisfying

$$\Delta w + c_\lambda(x)w \le 0 \quad \text{in } \Sigma_\lambda \setminus \overline{B_0}.$$

From (3.2) and (3.5) we see that

$$\frac{v_\lambda(x)}{w(x)} \le \frac{u(x)-c}{w(x)} \to 0 \quad \text{as } |x| \to \infty.$$

By applying Lemma A in Appendix A with $\Omega = \Sigma_\lambda \setminus \overline{B_0}$, we get $v_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$.

Define

$$\Lambda = \{\lambda \in (0,\infty) : v_\lambda(x) > 0 \text{ in } \Sigma_\lambda\}.$$

Lemma 5. *If $\lambda \notin \Lambda$, then there exists $x_0 \in \Sigma_\lambda \cap \overline{B_0}$ such that $v_\lambda(x_0) \le 0$.*

Proof. Assume to the contrary that $v_\lambda(x) > 0$ on $\Sigma_\lambda \cap \overline{B_0}$. Then by Lemma 4 we have $v_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$. Therefore, $v_\lambda(x) > 0$ in Σ_λ, which contradicts the assumption $\lambda \notin \Lambda$.

Lemma 6. *Let $\lambda \in \Lambda$. Then $\partial u/\partial x_1 < 0$ on T_λ.*

Proof. By Lemma 1, we have (3.3) and $v_\lambda > 0$ in Σ_λ. Since $v_\lambda = 0$ on T_λ, we obtain $\partial v_\lambda/\partial x_1 < 0$ on T_λ by the Hopf boundary lemma ([2, Lemma H]). Therefore

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial v_\lambda}{\partial x_1} < 0 \quad \text{on } T_\lambda.$$

Proof of the theorem. Since (3.2) holds, there exists $r_1 > r_0$ such that

$$\max\{u(x) : |x| \ge r_1\} < \min\{u(x) : |x| \le r_0\}, \tag{3.8}$$

where r_0 is the constant appearing in Lemma 3. We now divide the proof into several steps.

Step 1. $[r_1, \infty) \subset \Lambda$.

Let $\lambda \ge r_1$. We note that $\overline{B_0} \subset \Sigma_\lambda$. From (3.8), we have $v > 0$ in $\overline{B_0}$. Then by Lemma 4 we have $v_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$. Therefore $v > 0$ in Σ_λ, i.e., $\lambda \in \Lambda$. This implies that $[r_1, \infty) \subset \Lambda$.

Step 2. *Let $\lambda_0 \in \Lambda$. Then there exists $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$.*

Assume to the contrary that there exists an increasing sequence $\{\lambda_i\}$, $i = 1, 2, \ldots$, such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By Lemma 5 there exists a sequence $\{x_i\}$, $i = 1, 2, \ldots$, such that $x_i \in \Sigma_{\lambda_i} \cap \overline{B_0}$ and $v_{\lambda_i}(x_i) \leq 0$. Then there is a subsequence, which we again call $\{x_i\}$ which converges to some point $x_0 \in \overline{\Sigma_{\lambda_0}} \cap \overline{B_0}$. We have $v_{\lambda_0}(x_0) \leq 0$. Since $v_{\lambda_0} > 0$ in Σ_{λ_0}, we must have $x_0 \in T_{\lambda_0}$.

By the mean value theorem, there exists a point y_i satisfying $(\partial u/\partial x_1)(y_i) \geq 0$ on the straight segment joining x_i to $x_i^{\lambda_i}$, for each $i = 1, 2, \ldots$. Since $y_i \to x_0$ as $i \to \infty$, we have $(\partial u/\partial x_1)(x_0) \geq 0$. On the other hand, since $x_0 \in T_{\lambda_0}$ we have $(\partial u/\partial x_1)u(x_0) < 0$ by Lemma 6. This is a contradiction, and Step 2 is established.

Step 3. *We have*

$$u(x) \geq u(x^0) \quad \text{in } \Sigma_0. \tag{3.9}$$

Let $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$. We show that $\lambda_1 = 0$. Assume to the contrary that $\lambda_1 > 0$. From the continuity of u, we have $v_{\lambda_1}(x) = u(x) - u(x^{\lambda_1}) \geq 0$ in Σ_{λ_1}. By Lemma 2, we obtain (3.3) with $\lambda = \lambda_1$. Hence, by the maximum principle ([2]), we have either

$$v_{\lambda_1} \equiv 0 \ \text{ in } \Sigma_{\lambda_1}, \ \text{i.e.,} \ \ u(x) \equiv u(x^{\lambda_1}) \ \text{ in } \Sigma_{\lambda_1}, \quad \text{or} \tag{3.10}$$

$$v_{\lambda_1} > 0 \ \text{ in } \Sigma_{\lambda_1}, \ \text{i.e.,} \ \ u(x) > u(x^{\lambda_1}) \ \text{ in } \Sigma_{\lambda_1}. \tag{3.11}$$

If (3.10) occurs, by (1.1) we have $\phi(|x|)f(u(x)) \equiv \phi(|x^{\lambda_1}|)f(u(x))$ for $x \in \Sigma_{\lambda_1}$. Because $f(u(x)) > 0$, we have $\phi(|x|) \equiv \phi(|x^{\lambda_1}|)$ in Σ_{λ_1}. Since ϕ is nonincreasing, we see that $\phi(r) \equiv \phi(0)$ for $r \geq 0$. By (1.2), $\phi(r) \equiv 0$ for $r \geq 0$. This contradicts the assumption $\phi \not\equiv 0$. Therefore (3.10) cannot happen.

On the other hand, if (3.11) occurs. Then, $\lambda_1 \in \Lambda$. From Step 2, there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1] \subset \Lambda$. This contradicts the definition of λ_1.

Therefore, we conclude that $\lambda_1 = 0$. Thus, $u(x) > u(x^\lambda)$ in Σ_λ for $\lambda > 0$. By the continuity of u, we obtain (3.9).

We can repeat the previous Steps 1-3 for the negative x_1-direction to conclude that $u(x) \leq u(x^0)$ for $x \in \Sigma_0$. Hence, from (3.9), u must be symmetric about the plane $x_1 = 0$. Since the equation in (1.1) is invariant under rotation, we may take any direction as the x_1-direction and conclude that u is symmetric in every direction. Therefore, u must be radially symmetric about the origin.

Appendix A. Let Ω be an unbounded domain in $\boldsymbol{R}^n$, and let $Lu \equiv \Delta u + c(x)u$, where $c \in \mathrm{L}^\infty(\Omega)$.

Lemma A. *Suppose that u satisfies $Lu \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$. Suppose, furthermore,* *that there exists a function w such that $w > 0$ on $\Omega \cup \partial\Omega$ and $Lw \le 0$ in Ω. If*

$$\frac{u(x)}{w(x)} \to 0 \quad as\ |x| \to \infty,\ x \in \Omega, \tag{A.1}$$

then $u > 0$ in Ω.

Remark. If Ω is bounded, we do not require the condition (A.1). See [15, Chap. 2, Theorem 10].

Proof. First we show that $u \ge 0$ in Ω. Assume to the contrary that $u(x_0) < 0$ for some $x_0 \in \Omega$. Choose $\delta > 0$ so that

$$u(x_0) + \delta w(x_0) = 0. \tag{A.2}$$

From (A.1), there exists $R > |x_0|$ satisfying $u(x) + \delta w(x) \ge 0$ on $\{|x| = R\} \cap \Omega$. Define $B_R = \{x \in \boldsymbol{R}^n : |x| < R\}$. Then $u + \delta w$ satisfies $L(u + \delta w) \le 0$ on $\Omega \cap B_R$ and $u + \delta w \ge 0$ on $\partial(\Omega \cap B_R)$. By [15, Chap.2, Theorem 10], $(u + \delta w)/w$ cannot attain a nonpositive minimum at an interior point of $\Omega \cap B_R$ unless it is a constant. This contradicts (A.2). Therefore, $u \ge 0$ in Ω. By the maximum principle ([2]), we conclude that $u > 0$ in Ω.

Appendix B. Conditions which are equivalent to (1.2).

Lemma B.1. *Equation* (1.1) *has a bounded positive solution u on $\{x \in \boldsymbol{R}^n : |x| \ge r_0\}$ for some $r_0 > 0$ satisfying*

$$\liminf_{|x| \to \infty} u(x) > 0 \tag{B.1}$$

if and only if (1.2) *holds.*

Proof. Assume that u is a bounded solution of (1.1) on $\{|x| \ge r_0\}$ satisfying (B.1). Let $\overline{u}$ be the spherical mean of u, i.e.,

$$\overline{u}(r) = \frac{1}{n\omega_n r^{n-1}} \int_{|x|=r} u(x)dS \quad \text{for } r \ge r_0,$$

where ω_n is the volume of the unit ball in $\boldsymbol{R}^n$. Then, $\overline{u}$ satisfies

$$(r^{n-1}\overline{u}')' + r^{n-1}\phi(r)h(r) = 0, \quad r > r_0, \tag{B.2}$$

where

$$h(r) = \frac{1}{n\omega_n r^{n-1}} \int_{|x|=r} f(u(x))dS \quad \text{for } r \ge r_0.$$

(See, e.g., [13, 14].) Since u is bounded, by integrating (B.2) we obtain

$$\int_{r_0}^{\infty} r^{1-n} \int_{r_0}^{r} s^{n-1}\phi(s)h(s)dsdr = \frac{1}{n-2} \int_{r_0}^{\infty} s\phi(s)h(s)ds < \infty. \tag{B.3}$$

From (B.1), there exists a constant $u_0 > 0$ satisfying $u(x) \geq u_0$ for $|x| \geq r_0$. Define u_∞ and f_0 as $u_\infty = \max\{u(x) : |x| \geq r_0\}$ and $f_0 = \min\{f(s) : 0 < u_0 \leq s \leq u_\infty\}$. We see that $f_0 > 0$ and $h(r) \geq f_0$ for $r \geq r_0$. By (B.3) we have (1.2).

Conversely, assume that (1.2) holds. Let $c > 0$. Define $f_c = \max\{f(s) : c \leq s \leq 2c\}$. Choose $r_0 > 0$ so large that

$$\int_{r_0}^{\infty} s\phi(s)ds < \frac{(n-2)c}{f_c}.$$

Let $C([r_0,\infty))$ denote the Fréchet space of continuous functions on $[r_0,\infty)$ with the topology of uniform convergence on any compact subinterval of $[r_0,\infty)$. Consider the set

$$U = \{u \in C([r_0,\infty)) : c \leq u(r) \leq 2c, \quad r \geq r_0\},$$

which is a closed convex subset of $C([r_0,\infty))$. We define the operator F on U by

$$Fu(r) = c + \int_r^{\infty} s^{1-n} \int_{r_0}^{s} t^{n-1}\phi(t)f(u(t))dtds, \quad r \geq r_0.$$

If $u \in U$, then $Fu(r) \geq c$ and

$$Fu(r) \leq c + \frac{f_c}{n-2}\int_{r_0}^{\infty} s\phi(s)ds \leq 2c, \quad r \geq r_0.$$

Thus the operator F maps U into itself. It is easy to see that F is continuous on U and FU is relatively compact in the topology of $C([r_0,\infty))$. By the Schauder-Tychonoff fixed point theorem, F has an element $u \in U$ such that $u = Fu$, i.e., $u(r) = Fu(r)$ for $r \geq r_0$. Then $u = u(|x|)$ is a positive solution of (1.1) on $\{|x| \geq r_0\}$ and satisfies $\lim_{|x|\to\infty} u(x) = c$. This completes the proof of Lemma B.1.

Lemma B.2. *Assume that* $f(0) > 0$. *Then,* (1.1) *has a bounded positive solution* u *on* $\{x \in \boldsymbol{R}^n : |x| \geq r_0\}$ *for some* $r_0 > 0$ *if and only if* (1.2) *holds.*

Proof. Assume that u is a bounded positive solution of (1.1) on $\{|x| \geq r_0\}$. Let $\overline{u}$ be the spherical mean of u. Then by the argument in the proof of Lemma B.1 we have (B.3). Define u_∞ and f_0 as $u_\infty = \max\{u(x) : |x| \geq r_0\}$ and $f_0 = \min\{f(s) : 0 \leq s \leq u_\infty\}$. We see that $f_0 > 0$ since $f(s) > 0$ for $s \geq 0$, and that $h(r) \geq f_0$ for $r \geq r_0$. By (B.3) we have (1.2).

Conversely, assume that (1.2) holds. Then, by the argument in the proof of Lemma B.1, we obtain a bounded positive solution of (1.1) on $\{|x| \geq r_0\}$.

REFERENCES

[1] L. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271-297.

[2] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[3] B. Gidas, W.-M. Ni, and L. Nirenberg, "Symmetry of positive solutions of nonlinear elliptic equations in $\boldsymbol{R}^n$" in Mathematical Analysis and Applications, Part A, ed. by L. Nachbin, Adv. Math. Suppl. Stud. **7**, Academic Press, New York, 1981, 369-402.

[4] N. Kawano, *On bounded entire solutions of semilinear elliptic equations*, Hiroshima Math. J. **14** (1984), 125-158.

[5] T. Kusano and S. Oharu, *Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem*, Indiana Univ. Math. J. **34** (1985), 85-95.

[6] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains*, Comm. Partial Differential Equations **16** (1991), 585-615.

[7] Y. Li, *On the positive solutions of the Matukuma equation*, Duke Math. J. **70** (1993), 575-589.

[8] Y. Li and W.-M. Ni, *On the existence and symmetry properties of finite total mass solutions of the Matukuma equation, the Eddington equation and their generalizations*, Arch. Rational Mech. Anal. **108** (1989), 175-194.

[9] Y. Li and W.-M. Ni, *On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in* $\boldsymbol{R}^n$*, Part I. Asymptotic behavior*, Arch. Rational Mech. Anal. **118** (1992), 195-222.

[10] Y. Li and W.-M. Ni, *On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in* $\boldsymbol{R}^n$*, Part II. Radial symmetry*, Arch. Rational Mech. Anal. **118** (1992), 223-244.

[11] Y. Li and W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\boldsymbol{R}^n$, Comm. Partial Differential Equations **18** (1993), 1043-1054.

[12] M. Naito, *A note on bounded positive entire solutions of semilinear elliptic equations*, Hiroshima Math. J. **14** (1984), 211-214.

[13] W. -M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, *its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), 493-529.

[14] E. S. Noussair and C. A. Swanson, *Oscillation theory for semilinear Schrödinger equations and inequalities*, Proc. Roy. Soc. Edinburgh, Sect. A **75** (1975/76), 67-81.

[15] M. Protter and H. Weinberger, "Maximal Principles in Differential Equations", Prentice-Hall, Englewood Cliffs, N.J. 1967.

[16] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304-318.

[17] H. Zou, *Symmetry of positive solutions of* $\Delta u + u^p = 0$ *in* $\boldsymbol{R}^n$, J. Differential Equations, **120** (1995), 46-88.

[18] H. Zou, *Symmetry of ground states of semilinear elliptic equations with mixed Sobolev growth*, Indiana Univ. Math. J. **45** (1996), 221-240.