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**Weierstrass type expression of curves of genus two and modular forms.**  
**KOMATSU Makoto<sup>1</sup>**

**Abstract**

The parameter space of versal deformation of curve singularity of type  $A_4$  has a structure of  $\mathbf{C}^*$ -bundle in a wider sense. The aim of this paper is to clarify the structure of the bundle by a period mapping.

**1 Introduction.**

For any positive integer  $n$ , zeros of a polynomial  $-y^2 + x^{n+1}$  in  $\mathbf{C}^2$  gives a curve singularity of type  $A_n$ . And the following polynomial

$$F_{A_n}(x, y, t) := -y^2 + x^{n+1} + t_2x^{n-1} + \dots + t_nx + t_{n+1}$$

expresses versal deformation of the above singularity. That is,

$$\Xi_{A_n} := \{(x, y, t) \in \mathbf{C}^2 \times \mathbf{C}^n \mid F_{A_n}(x, y, t) = 0\}$$

is called versal deformation of curve singularity of tipe  $A_n$ . The parameter space  $\mathbf{C}^n$  is denoted by  $S_{A_n}$ :

$$S_{A_n} := \mathbf{C}^n (\ni t = (t_2, \dots, t_{n+1})).$$

On  $\Xi_{A_n}$  and  $S_{A_n}$  we define a  $\mathbf{C}^*$ -action as

$$\lambda \cdot (x, y, t) := \begin{cases} (\lambda x, \lambda^{\frac{n+1}{2}} y, \lambda \cdot t) & (n : \text{odd}) \\ (\lambda^2 x, \lambda^{n+1} y, \lambda \cdot t) & (n : \text{even}) \end{cases}$$

where

$$\lambda \cdot t := \begin{cases} (\lambda^2 t_2, \dots, \lambda^{n+1} t_{n+1}) & (n : \text{odd}) \\ (\lambda^4 t_2, \dots, \lambda^{2n+2} t_{n+1}) & (n : \text{even}). \end{cases}$$

By the action, the parameter space  $S_{A_n}$  is regarded as the total space of a  $\mathbf{C}^*$ -bundle in a wider sense. here we think of the following problem.

**Problem 1.1** Clarify the structure of the above  $\mathbf{C}^*$ -bundle  $S_{A_n}$ .

At the present time, only for  $n = 1$  and  $n = 2$ , answer to the problem is already known. As for the case  $n = 1$ , the problem is trivial, because the base space  $\mathbf{C}^* \backslash S_{A_1}$  consists of one point. As for the case  $n = 2$ , answer to the problem is a classical result, which we will see later (in subsection 2.4). In the present paper we think of the problem for  $n = 4$ . Here we avoid the problem for  $n = 3$ . If  $n$  is an odd integer and  $n \geq 3$ , there are some circumstances, in which, the case of  $A_n$  is rather different from that of  $A_2$ . Therefore we cannot apply the way used in solving the problem of the case  $n = 2$ , simply, to the problem for the  $n$ . As for the problem for the  $n$ , we have no idea now. In the case  $n = 2$ , using a period mapping and applying a well-known frame of automorphic forms, we can see that the transition functions of the bundle  $S_{A_2}$  are given as a factor of automorphy. In the following section we review the frame of automorphic forms.

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## 2 A frame of automorphic forms.

In this section we review a well-known frame of automorphic forms.

### 2.1 Equivariant group action on a trivial bundle and a factor of automorphy.

Suppose  $X$  be a complex manifold, and  $G$  be a group acting on  $X$  discontinuously. Then the following (2-1-1), (2-1-2) are equivariant.

(2-1-1) To give a factor of automorphy  $j : G \times X \rightarrow \mathbb{C}^*$ .

(2-1-2) To give a  $G$ -action on  $\mathbb{C}^* \times X$  which satisfies the following (i),(ii).

(i) The  $G$ -action is commutative to the natural  $\mathbb{C}^*$ -action on  $\mathbb{C}^* \times X$ .

(ii) The  $G$ -action is equivariant to the natural projection  $\mathbb{C}^* \times X \rightarrow X$ .

In fact, if a factor of automorphy  $j$  is given, we can give a  $G$ -action on  $\mathbb{C}^* \times X$  using  $j$  as follows:

$$\mathbb{C}^* \times X \ni (\lambda, x) \xrightarrow{\sigma} (j(\sigma, x)^{-1}\lambda, \sigma(x)) \in \mathbb{C}^* \times X \quad (\sigma \in G). \quad (1)$$

It can be easily seen that this  $G$ -action satisfy the above (i) and (ii). On the other hand, suppose that a  $G$ -action on  $\mathbb{C}^* \times X$  satisfying (i) and (ii) is given. Then we define a map  $j : G \times X \rightarrow \mathbb{C}^*$  by the following relation:

$$(1, x) \xrightarrow{\sigma} (j(\sigma, x)^{-1}, \sigma(x)) \quad (\sigma \in G, x \in X). \quad (2)$$

Then this  $j$  is a factor of automorphy. Those two procedures now explained are inverse to each other.

### 2.2 Invariant ring and ring of automorphic forms.

In general, when a group  $G$  is acting on a ring  $R$ , we denote by  $R^G$  the  $G$ -invariant subring of  $R$ . And for any complex analytic space  $Y$ , we denote by  $\mathcal{O}(Y)$  the ring of all of holomorphic functions on  $Y$ .

When (2-1-1) (or, equivalently, (2-1-2)) is satisfied, there are well known relations of four rings as follows:

$$\begin{aligned} & \text{[the ring of } (G, j)\text{-automorphic forms on } X\text{]} \\ & \cong \mathcal{O}(X)[\lambda, \lambda^{-1}]^G \subseteq \mathcal{O}(\mathbb{C}^* \times X)^G \cong \mathcal{O}((\mathbb{C}^* \times X)/G). \end{aligned} \quad (3)$$

It seems that in (3), only the first isomorphism is non-trivial (at least, to me). Here we explain the correspondence which gives the first isomorphism of (3). Suppose  $f$  be an element of  $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$ . We express  $f$  as Laurent polynomial in  $\lambda$ :

$$f(\lambda, x) = \sum_{k \in \mathbb{Z}} \lambda^k f_k(x) \quad (\text{finite sum}) \quad (4)$$

where  $f_k \in \mathcal{O}(X)$ . From the expansion,  $f$  satisfies the equality

$$f(j(\sigma, x)^{-1}\lambda, \sigma(x)) = \sum_k j(\sigma, x)^{-k} \lambda^k f_k(\sigma(x)) \quad (5)$$

for any  $\sigma \in G$ . Because  $f$  is  $G$ -invariant, (1), (4) and (5) imply that

$$f_k(\sigma(x)) = j(\sigma, x)^k f_k(x) \quad (\forall \sigma \in G, \forall x \in X, \forall k \in \mathbf{Z}).$$

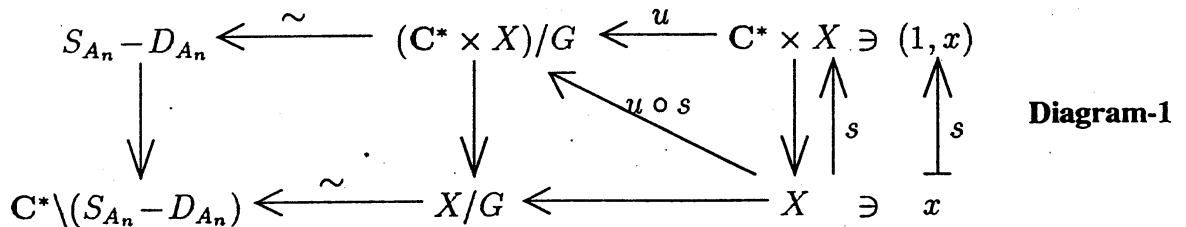
That is,  $f_k$  is a  $(G, j)$ -automorphic form of weight  $k$ . On the other hand, for given finite set  $\{f_k\}$  (where each  $f_k$  is a  $(G, j)$ -automorphic form of weight  $k$ ), if we define  $f$  by (4), we can easily see that  $f$  is an element of  $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$ .

### 2.3 Our plan.

We denote by  $D_{A_n}$  the discriminant set of  $S_{A_n}$ :

$$D_{A_n} := \{t \in S_{A_n} \mid F_{A_n}(x, 0, t) \text{ has multiple roots.}\} \tag{6}$$

We treat  $S_{A_n} - D_{A_n}$  rather than  $S_{A_n}$  itself. Suppose that there exist  $X$  and  $G$  which make the left hand side of the following diagram



commutative, where  $u$  is a natural projection, and  $s$  is a global section of the trivial bundle  $\mathbf{C}^* \times X \rightarrow X$  defined as in the above diagram. Then by (3), the ring  $\mathbf{C}[t_2, \dots, t_{n+1}]$  is regarded as a subring of  $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$ , and hence it is regarded as a subring of the ring of  $(G, j)$ -automorphic forms. Moreover, transition functions of the bundle  $S_{A_n} - D_{A_n}$  is given as a factor of automorphy  $j$ . By the way, the  $G$ -actions on the total space and on the base space of the bundle  $\mathbf{C}^* \times X \rightarrow X$  are equivariant to the projection. Hence, by the relation (2) the section  $s$  satisfies

$$s(\sigma(x)) = j(\sigma, x) \cdot \sigma(s(x)) \quad (\forall \sigma \in G, \forall x \in X).$$

Moreover, the  $\mathbf{C}^*$ -actions on  $(\mathbf{C}^* \times X)/G$  and on  $\mathbf{C}^* \times X$  are equivariant to the map  $u$ . And, in addition,  $u$  is  $G$ -invariant. Therefore, we have

$$(u \circ s)(\sigma(x)) = j(\sigma, x) \cdot (u \circ s)(x) \quad (\forall \sigma \in G, \forall x \in X).$$

Keeping the above frame in mind, we consider the problem 1.1 for  $n = 4$  according to the following plan.

(2-3-1) First we find  $X$  and  $G$  which give Diagram-1.

(2-3-2) Next we investigate the effect of  $G$ -action on the map  $u \circ s$  to obtain a factor of automorphy  $j$  explicitly.

## 2.4 Example. ( $A_2$ -type curve singularity.)

As an example, we review the answer to the problem 1.1 for  $n = 2$ . In order to adapt the problem to the theory of Weierstrass' elliptic function, we modify the definition of  $F_{A_2}$  as follows:

$$F_{A_2} := -y^2 + 4x^3 - g_2x - g_3.$$

Then  $S_{A_2} = \mathbf{C}^2$  and  $D_{A_2} = \{g \in S_{A_2} | g_2^3 - 27g_3^2 = 0\}$ . In this case, using the following multi-valued holomorphic mapping:

$$S_{A_2} - D_{A_2} \ni g \mapsto \left( \int_{A(g)} \frac{dx}{y}, \int_{B(g)} \frac{dx}{y} / \int_{A(g)} \frac{dx}{y} \right) \in \mathbf{C}^* \times \mathbf{H}, \quad (7)$$

we can apply the above frame to  $S_{A_2} - D_{A_2}$ , where  $G = SL(2, \mathbf{Z})$  and  $X = \mathbf{H} (= \{\tau \in \mathbf{C} | \Im \tau > 0\})$ . As a consequence, we obtain that  $S_{A_2} - D_{A_2} \cong \mathbf{C}^* \times \mathbf{H} / SL(2, \mathbf{Z})$ . Moreover, we have  $j \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) = c\tau + d$ , and obtain the expression of  $g_i$  ( $i = 2, 3$ ) as  $(G, j)$ -automorphic forms, which coincide to the well-known expressions as Eisenstein series.

## 3 Definition of period mapping.

We denote that  $S := S_{A_4}$ ,  $\Xi := \Xi_{A_4}$ , and  $D := D_{A_4}$ . Discriminant of the polynomial  $F_{A_4}(x, 0, t) \in (\mathbf{C}[t])[x]$  is as follows:

$$\begin{aligned} \Delta(t) := & 3125t_5^4 - 3750t_2t_3t_5^3 + 2000t_2t_4^2t_5^2 + 2250t_2^2t_4t_5^2 - 900t_2^3t_4t_5^2 + 825t_2^2t_3^2t_5^2 \\ & + 108t_2^5t_5^2 - 1600t_3t_4^3t_5 + 560t_2^2t_3t_4^2t_5 - 630t_2t_3^2t_4t_5 - 72t_2^4t_3t_4t_5 + 108t_3^5t_5 \\ & + 16t_2^3t_3^3t_5 + 256t_4^5 - 128t_2^2t_4^4 + 144t_2t_3^2t_4^3 + 16t_2^4t_4^3 - 27t_3^4t_4^2 - 4t_2^3t_3^2t_4^2. \end{aligned}$$

By (6), we have  $D = \{t \in S \mid \Delta(t) = 0\}$ . In addition,  $\pi$  denotes the natural projection  $\Xi \ni (x, y, t) \mapsto t \in S$ , and  $X_t$  denotes  $\pi^{-1}(t)$ . We take a point  $t_0 \in S - D$ .  $t_0$  is used as a base point of the fundamental group of  $S - D$ . Projection  $\pi : \Xi - \pi^{-1}(D) \rightarrow S - D$  has the property of local triviality. Hence  $\pi_1(S - D, t_0)$  acts on  $H_1(X_{t_0}, \mathbf{Z})$ . Moreover, this action preserves the intersection form  $\langle \cdot, \cdot \rangle$  on  $H_1(X_{t_0}, \mathbf{Z})$ . Therefore we have the following representation:

$$\rho : \pi_1(S - D, t_0) \longrightarrow \text{Aut}(H_1(X_{t_0}, \mathbf{Z}), \langle \cdot, \cdot \rangle)$$

(monodromy representation), where  $\text{Aut}(H_1(X_{t_0}, \mathbf{Z}), \langle \cdot, \cdot \rangle)$  denotes all of automorphisms of  $H_1(X_{t_0}, \mathbf{Z})$  which preserve the intersection form  $\langle \cdot, \cdot \rangle$ .  $\Gamma := \rho(\pi_1(S - D, t_0))$  is called as monodromy group. We take a symplectic basis of  $H_1(X_{t_0}, \mathbf{Z})$  as in Figure-1. Then by the basis, the following group isomorphism holds:

$$\text{Aut}(H_1(X_{t_0}, \mathbf{Z}), \langle \cdot, \cdot \rangle) \cong Sp(4, \mathbf{Z}), \quad (8)$$

and by the isomorphism,  $\Gamma$  is regarded as a subgroup of  $Sp(4, \mathbf{Z})$ . Now we define a covering space of  $S - D$  as follows:

$$\widehat{S - D} := (\text{universal covering space of } S - D) / (\text{kernel of } \rho).$$

$\widehat{S-D}$  is called as monodromy covering. Natural projection  $\widehat{S-D} \rightarrow S-D$  is denoted by  $\sigma$ . Here we can define a period mapping.

$$P : \widehat{S-D} \ni h \mapsto \begin{pmatrix} \omega_{11}(h) & \omega_{12}(h) & \omega_{13}(h) & \omega_{14}(h) \\ \omega_{21}(h) & \omega_{22}(h) & \omega_{23}(h) & \omega_{24}(h) \end{pmatrix} \in \mathbb{C}^{2 \times 4},$$

$$\omega_{ij}(h) := \int_{A_j(h)} \frac{x^{i-1} dx}{y},$$

where  $A_1(h), A_2(h), A_3(h) = B_1(h), A_4(h) = B_2(h)$  are symplectic basis of  $H_1(X_{\sigma(h)}, \mathbb{Z})$  and depend on  $h$  "continuously". That is, each  $A_j(h)$  is a local system. Note that, On some  $h_0 \in \sigma^{-1}(t_0)$ , we take  $A_j(h_0)$  ( $j = 1, 2, 3, 4$ ) as in the Figure-1.

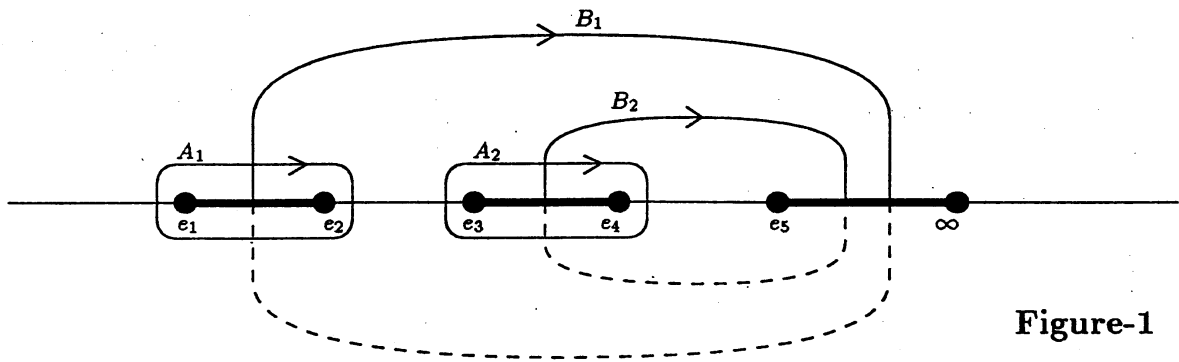


Figure-1

**Remark.** Each  $A_j(t)$  is multi-valued on  $S-D$ . But, on  $\widehat{S-D}$ , each  $A_j(t)$  is single-valued. In fact  $\widehat{S-D}$  is the minimal covering on which each  $A_j(t)$  is single-valued. Therefore the above period map  $P$  is single-valued.

By the definition of  $P$ , each  $P(h)$  ( $h \in \widehat{S-D}$ ) is a  $2 \times 4$  matrix. We define a map  $\varphi$  as

$$\varphi : \text{Image}(P) \ni (\Omega_A \ \Omega_B) \mapsto (\Omega_A^{-1} \Omega_B) \in \mathbb{H}_2$$

where  $\Omega_A, \Omega_B$  denote the left  $2 \times 2$  part, the right  $2 \times 2$  part of the  $2 \times 4$  matrix  $P(h)$ , respectively, and  $\mathbb{H}_2$  denotes the Siegel upper half space of genus two.

### 4 Results(1) — towards (2-3-1).

The  $\mathbb{C}^*$ -action on  $S-D$  can be lifted to an action on  $\widehat{S-D}$ , where the  $\mathbb{C}^*$ -action is fixed point free. Hence it can be easily seen that  $\widehat{S-D}$  is regarded as the total space of a  $\mathbb{C}^*$ -bundle in the strict sense. By the way, in the Problem for  $n=2$ , we can see that  $S_{A_2-\widehat{D}_{A_2}}$  is isomorphic to the trivial bundle  $\mathbb{C}^* \times \mathbb{H}$  via the period mapping (7). As for the Problem for  $n=4$ , we obtained the following theorem.

**Theorem 4.1** ([4]) *The above period mapping  $P$  gives an isomorphism*

$$\widehat{S-D} \cong \mathbb{C}^* \times \mathbb{H}_2^0 \tag{9}$$

as  $\mathbb{C}^*$ -bundles, where

$$\mathbb{H}_2^0 := \mathbb{H}_2 - A, \quad A := \{M \circ \tau \mid M \in Sp(4, \mathbb{Z}), \tau \in \mathbb{H}_2, \tau \text{ is a diagonal matrix}\}. \tag{10}$$

(Definition of  $M \circ \tau$  is given in subsection 5.2.)

(Outline of the proof.) To prove the theorem we give a global section of the bundle  $\widehat{S-D} \rightarrow \mathbf{H}_2^0$  by using Rosenhain's formula [7]. The formula says that, in our situation, the following equalities

$$\frac{e_3 - e_1}{e_2 - e_1} = \frac{\vartheta_{0000}^2 \vartheta_{0100}^2}{\vartheta_{1000}^2 \vartheta_{1100}^2}, \quad \frac{e_4 - e_1}{e_2 - e_1} = \frac{\vartheta_{0100}^2 \vartheta_{0001}^2}{\vartheta_{1100}^2 \vartheta_{1001}^2}, \quad \frac{e_5 - e_1}{e_2 - e_1} = \frac{\vartheta_{0000}^2 \vartheta_{0001}^2}{\vartheta_{1000}^2 \vartheta_{1001}^2} \quad (11)$$

hold, where

- $e_1, \dots, e_5$  are five roots of  $F(x, 0, \sigma(h))$ ,
- $\tau$  is a period matrix, which is obtained from  $X_{\sigma(h)}$  with a basis  $A_j(\sigma(h))$   $\{j = 1, 2, 3, 4\}$  of  $H_1(X_{\sigma(h)}, \mathbf{Z})$ ,

and for any  $\varepsilon = (\varepsilon' \varepsilon'') = (\varepsilon'_1 \dots \varepsilon'_g \varepsilon''_1 \dots \varepsilon''_g) \in \mathbf{Z}^{2g}$  and  $\tau \in \mathbf{H}_g$ ,

$$\vartheta_\varepsilon = \vartheta_\varepsilon(\tau) := \sum_{n \in \mathbf{Z}^g} \exp \left[ \pi i \left( n + \frac{\varepsilon'}{2} \right) \tau \left( n + \frac{\varepsilon'}{2} \right) + 2\pi i \left( n + \frac{\varepsilon'}{2} \right) \frac{\varepsilon''}{2} \right] \quad (12)$$

are theta constants of genus  $g$ , where  $\mathbf{H}_g$  denotes Siegel upper half space of genus  $g$ . Here we use only theta constants of genus two.

In our situation, the equality  $e_1 + \dots + e_5 = 0$  holds. Therefore, by using some formulas of theta constants, (11) imply that the following equality holds,

$$(e_1, \dots, e_5) = (\lambda^2 \alpha_1, \dots, \lambda^2 \alpha_5)$$

for some  $\lambda \in \mathbf{C}^*$ , where

$$\begin{aligned} \alpha_1 &:= \frac{1}{5} (-\vartheta_{1000}^2 \vartheta_{1100}^2 \vartheta_{1001}^2 - \vartheta_{0000}^2 \vartheta_{0100}^2 \vartheta_{1001}^2 - \vartheta_{1000}^2 \vartheta_{0100}^2 \vartheta_{0001}^2 - \vartheta_{0000}^2 \vartheta_{1100}^2 \vartheta_{0001}^2), \\ \alpha_2 &:= \frac{1}{5} (+\vartheta_{1000}^2 \vartheta_{1100}^2 \vartheta_{1001}^2 - \vartheta_{0010}^2 \vartheta_{0110}^2 \vartheta_{1001}^2 - \vartheta_{0011}^2 \vartheta_{0110}^2 \vartheta_{1000}^2 - \vartheta_{0011}^2 \vartheta_{0010}^2 \vartheta_{1100}^2), \\ \alpha_3 &:= \frac{1}{5} (+\vartheta_{0000}^2 \vartheta_{0100}^2 \vartheta_{1001}^2 + \vartheta_{0010}^2 \vartheta_{0110}^2 \vartheta_{1001}^2 - \vartheta_{0110}^2 \vartheta_{1111}^2 \vartheta_{0100}^2 - \vartheta_{0010}^2 \vartheta_{1111}^2 \vartheta_{0000}^2), \\ \alpha_4 &:= \frac{1}{5} (+\vartheta_{1000}^2 \vartheta_{0100}^2 \vartheta_{0001}^2 + \vartheta_{0011}^2 \vartheta_{0110}^2 \vartheta_{1000}^2 + \vartheta_{0110}^2 \vartheta_{1111}^2 \vartheta_{0100}^2 - \vartheta_{0011}^2 \vartheta_{1111}^2 \vartheta_{0001}^2), \\ \alpha_5 &:= \frac{1}{5} (+\vartheta_{0000}^2 \vartheta_{1100}^2 \vartheta_{0001}^2 + \vartheta_{0011}^2 \vartheta_{0010}^2 \vartheta_{1100}^2 + \vartheta_{0010}^2 \vartheta_{1111}^2 \vartheta_{0000}^2 + \vartheta_{0011}^2 \vartheta_{1111}^2 \vartheta_{0001}^2). \end{aligned}$$

Using those functions, we define a map  $F : \mathbf{H}_2^0 \ni \tau \mapsto (t_2, \dots, t_5) \in S-D$  by

$$t_i = (-1)^i \sum_{1 \leq \nu_1 < \dots < \nu_i \leq 5} \alpha_{\nu_1} \dots \alpha_{\nu_i} \quad (i = 2, 3, 4, 5). \quad (13)$$

Then, the map  $F$  is "lifted" to a map  $\hat{F} : \mathbf{H}_2^0 \rightarrow \widehat{S-D}$  and  $\hat{F}$  is a global section of the bundle  $\widehat{S-D} \rightarrow \mathbf{H}_2^0$ . Therefore, the isomorphism (9) is obtained.  $\square$

By the theorem, we can take  $\mathbf{H}_2^0$  as  $X$  and  $\Gamma$  as  $G$  in the **Diagram-1**.

## 5 Results(2) — towards (2-3-2).

### 5.1 Preliminary.

In the previous section, we used  $\hat{F}$  as a global section of the bundle  $\widehat{S-D} \rightarrow \mathbf{H}_2^0$  to give isomorphism (9). As a result, in our trivialization, we obtain a factor of automorphy  $j : \Gamma \times \mathbf{H}_2^0 \rightarrow \mathbf{C}^*$ . From now on, we obtain  $j$  explicitly by clarifying factors of automorphy of  $t_2, t_3, t_4, t_5$  (or  $\alpha_1, \dots, \alpha_5$ ) under the action of  $\Gamma$ , where we regard  $t_i$  as functions on  $\mathbf{H}_2^0$  by the equality (13). Here we note the following three points.

(5-1-1) Each  $t_i$  (or  $\alpha_i$ ) is a homogeneous polynomial of theta constants.

(5-1-2)  $\Gamma$  is regarded as a subgroup of  $Sp(4, \mathbf{Z})$  by the isomorphism (8).

(5-1-3) The transformation formula of theta constants under the action of the full modular group is well-known. (cf. [6])

So we investigate the effects of  $\Gamma$ -action on  $\alpha_1, \dots, \alpha_5$ .

### 5.2 Transformation formula of theta constants.

Following to [6], here we give short review of the transformation formula of theta constants defined in (12). It is well-known that, for  $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbf{Z})$ ,  $\tau \in \mathbf{H}_g$ ,  $\varepsilon = (\varepsilon' \varepsilon'') \in \mathbf{Z}^{2g}$ , the following equality holds:

$$\vartheta_{M\circ\varepsilon}(M \circ \tau) = \kappa(M) \exp(\pi i \phi(M, \varepsilon)) \sqrt{\det(C\tau + D)} \vartheta_\varepsilon(\tau), \quad (14)$$

where

$$\begin{aligned} M \circ \tau &:= (A\tau + B)(C\tau + D)^{-1}, & M \circ \varepsilon &:= \varepsilon M^{-1} + ((C^t D)_0 (A^t B)_0), \\ \phi(M, \varepsilon) &:= \frac{1}{4} \{-\varepsilon''^t D B^t \varepsilon' + 2\varepsilon''^t C B^t \varepsilon' - \varepsilon''^t C A^t \varepsilon'' + 2(\varepsilon''^t D - \varepsilon''^t C)^t (A^t B)_0\} \end{aligned}$$

where for  $g \times g$  matrix  $X = (x_{ij})$ , we write  $X_0 := (x_{11}, x_{22}, \dots, x_{gg})$ . In (14),  $\kappa(M)$  is a constant, which depends on  $M$  and is independent to  $\varepsilon$  and  $\tau$ . Moreover, It is well known that  $\kappa(M)^8 = 1$  for any  $M \in Sp(2g, \mathbf{Z})$ . As for  $\kappa(M)$ , more various properties are known.

### 5.3 Definition of a group $\Gamma'$ .

For any two positive integers  $g$  and  $n$ , the principal congruence subgroup of level  $n$ , genus  $g$  is defined as follows:

$$\Gamma_g(n) := \{M \in Sp(2g, \mathbf{Z}) \mid M \equiv I_{2g} \pmod{n}\}.$$

(Note that  $\Gamma_g(1) = Sp(2g, \mathbf{Z})$ .) It is well known that  $\Gamma_2(1)/\Gamma_2(2)$  is isomorphic to the 6-th Symmetric group  $S_6$ . (See, for example, [3].) Usually, the isomorphism is given by the action of  $\Gamma_2(1)$  over the following six odd theta characteristics of genus two:

$$\langle 1 \rangle := (0101), \langle 2 \rangle := (0111), \langle 3 \rangle := (1011), \langle 4 \rangle := (1010), \langle 5 \rangle := (1110), \langle 6 \rangle := (1101). \quad (15)$$

Here we give an isomorphism explicitly.



For any  $M \in \Gamma_2(1)$ , the following map:

$$\varepsilon \bmod (2\mathbf{Z})^4 \mapsto M \circ \varepsilon \bmod (2\mathbf{Z})^4 \quad (16)$$

gives rise to a permutation of the six elements in (15). Hence for any  $M \in \Gamma_2(1)$  and for any  $i \in \{1, \dots, 6\}$ , there is a  $M(i) \in \{1, \dots, 6\}$  such that

$$\langle M(i) \rangle \equiv M \circ \langle i \rangle \bmod (2\mathbf{Z})^4.$$

Then the map  $i \mapsto M(i)$  is a permutation of  $\{1, \dots, 6\}$ . Therefore we have a group homomorphism  $\Gamma_2(1) \rightarrow S_6$ . It can be easily seen that the homomorphism is surjective, and that its kernel is  $\Gamma_2(2)$ . So we obtain an isomorphism  $\Gamma_2(1)/\Gamma_2(2) \cong S_6$ .

Here we treat the following subgroup  $\Gamma'$ :

**Definition 5.1**  $\Gamma' := \{M \in \Gamma_2(1) \mid M \circ (1101) \equiv (1101) \bmod (2\mathbf{Z})^4\}$ .

This subgroup  $\Gamma'$  has the following property.

$$\Gamma_2(2) \subset \Gamma' \subset \Gamma_2(1) \quad \text{and} \quad \Gamma'/\Gamma_2(2) \cong S_5.$$

The following lemma brings the above subgroup  $\Gamma'$  to our notice.

**Lemma 5.2** (A'Campo [1])  $\Gamma' = \Gamma$ .

#### 5.4 Factors of automorphy of $\alpha_i$ .

We denote  $\chi(M) := \kappa(M)^2 \exp[2\pi i \phi(M, (1101))]$  for any  $M \in \Gamma_2(1)$ . We can easily obtain the following obvious lemma by using transformation formula of theta constants.

**Lemma 5.3**  $\Gamma' \ni M \mapsto \chi(M) \in \mathbf{C}^*$  is a group homomorphism.

Moreover, the transformation formula (14) implies the following lemma.

**Lemma 5.4** ([4]) For any  $i \in \{1, 2, 3, 4, 5\}$ , for any  $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma$ , and for any  $\tau \in \mathbf{H}_2$ , the following equality holds.

$$\alpha_{M(i)}(M \circ \tau) = \chi(M) \det(C\tau + D)^3 \alpha_i(\tau).$$

By the lemma, consequently, we obtain the following theorem.

**Theorem 5.5** ([4]) Under the trivialization of  $\widehat{S-D}$  by  $\hat{F}$ , we conclude that

$$j(M, \tau)^2 = \chi(M) \det(C\tau + D)^3 \quad (\forall M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma, \tau \in \mathbf{H}_2).$$

## 6 Comparison with the ring of Siegel modular forms.

In the frame given in section 2, if we take  $X = \mathbb{H}_2$ ,  $G = \Gamma_2(1)$  and  $j\left(\begin{pmatrix} D & C \\ B & A \end{pmatrix}, \tau\right) := \det(C\tau + D)$ , then the ring of  $(G, j)$ -automorphic forms on  $X$  is the ring of ordinary Siegel modular forms of genus two. Igusa showed (in [2], [3]) that the ring is

$$\mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}] \quad (\text{indices denote weights}) \quad (17)$$

where  $\psi_4, \psi_6, \chi_{10}, \chi_{12}$  are algebraically independent over  $\mathbb{C}$ , and  $\chi_{35}^2 \in \mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$ . On the other hand, the ring which we pay attention to is

$$\mathbb{C}[t_2, t_3, t_4, t_5] \quad (\text{indices denote half of weights})$$

where  $t_2, t_3, t_4, t_5$  are algebraically independent over  $\mathbb{C}$ . Therefore, though we have not yet found conditions which determine the ring  $\mathbb{C}[t_2, t_3, t_4, t_5]$  in the ring of  $(\Gamma, j)$ -automorphic forms, we can already see that algebraic structures of our ring  $\mathbb{C}[t_2, t_3, t_4, t_5]$  is different from that of the ring (17).

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