Title	Fixed Points of Multivalued Nonexpansive Mappings in Certain Convex Metric Spaces(NONLINEAR ANALYSIS AND CONVEX ANALYSIS)
Author(s)	Shimizu, Tomoo
Citation	数理解析研究所講究録 (1998), 1031: 75-84
Issue Date	1998-04
URL	http://hdl.handle.net/2433/61858
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1031 巻 1998 年 75-84

Fixed Points of Multivalued Nonexpansive Mappings in Certain Convex Metric Spaces

Tokyo Inst. Tech. 清水朝雄 (Tomoo Shimizu)

1. Introduction. The investigation concerning convexity in metric spaces was initiated by Menger [11] in 1928. This investigation was developed by several authors [1]. The terms "metrically convex" and "convex metric space" are due to Blumenthal[1]. Throughout this report, let X be a metric space with metric d.

Definition 1 *$z \in X$ is said to be a between-point of x, y if*

$$z \neq x \quad z \neq y, \quad \textit{and} \quad d(x,y) = d(x,z) + d(z,y).$$

Definition 2 *X is metrically convex if for each pair $x, y \in X$ such that $x \neq y$, there exists $z \in X$ that is a between-point of x, y. Then X is said to be a convex metric space.*

Let T be a mapping of X into itself. T is said to be nonexpansive [2], if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x,y).$$

In 1970, W.Takahashi [14] introduced a notion of convexity into metric spaces, studied properties of such spaces and proved several fixed point theorems for nonexpansive mappings.

Definition 3 *Put $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y).$$

X is called a convex metric space, if it has a convex structure.

Such kind of convex metric space seems to be often called w-convex metric space.

In 1981/82, Kirk [7] introduced a notion of a metric space of hyperbolic type and showed that it is a w-convex metric space. As a consequence of the proof of theorem 1 [7], we have the following result.

Theorem 1 *Let X be a bounded w-convex metric space that has a unique convex structure and T be a nonexpansive mapping of X into itself. Then* $\inf_{x \in X} d(x, Tx) = 0$. *(i.e., X has the almost fixed point property for nonexpansive mappings)*

On the other hand, in 1987, Kijima [5] generalized, in certain sense [cf. 15], the notion of w-convex metric spaces.

Definition 4 *X is said to be a convex metric space if for each pair $x, y \in X$ there exists $z \in X$ such that*

$$d(z,u) \leq \frac{d(x,u) + d(y,u)}{2} \quad \textit{for all} \quad u \in X. \qquad (*)$$

We shall call such X a metric space with property (S).

Example 1 *A dyadic cube in R^n.*

$$X = \left\{ \left(\frac{k_1}{2^{m_1}}, \cdots, \frac{k_n}{2^{m_n}}\right) \in R^n \ : \ k_i = 0,1,2,\cdots,2^{m_i}, \ m_i = 1,2,\cdots, \ i = 1,\cdots,n \right\}.$$

Recently, Kijima[6] proved the following result and generalized theorem 1.

Let X be a bounded metric space with property (S). Then $\inf_{x \in X} d(x, Tx) = 0$. *(i.e., X has the almost fixed point property for nonexpansive mappings of X into itself)*

This result is proved for the case of Banach space, using the Banach contraction principle; for instance, see [2]. However, the proof dose not carry over to the case of metric space with property (S). Kijima [6] proved the result by introducing an (ϵ, n)-sequence without using the Banach contraction principle.

Let $K(X)$ be the class of all nonempty compact subsets of X. A mapping T of X into $K(X)$ is said to be nonexpansive, if for each pair $x, y \in X$,

$$\mathcal{H}(Tx, Ty) \leq d(x,y).$$

where $\mathcal{H}$ is the Hausdorff metric on $K(X)$.

In 1992, Shimizu and Takahashi[12] generalized Kijima's result in the case of multivalued nonexpansive mappings with nonempty compact-values.

Theorem 2 *Let X be a bounded metric space with property (S) and T be a multivalued nonexpansive mapping of X into $K(X)$. Then* $\inf_{x\in X} d(x,Tx) = 0$, *where* $d(x,Tx) = \inf_{y\in Tx} d(x,y)$. *(i.e., X has the almost fixed point property for multivalued nonexpansive mappings of X into $K(X)$)*

We sketch the outline of the proof.
Suppose that $\inf_{x\in X} d(x,Tx) = 2\delta > 0$. $\forall \epsilon > 0$, $\exists x_0 \in X$ s.t.

$$d(x_0, Tx_0) \leq 2\delta + \epsilon.$$

Since Tx_0 is nonempty compact, $\exists y_0 \in X$ s.t.

$$d(x_0, y_0) \leq 2(\delta + \epsilon).$$

Define $\{x_n\}$ and $\{y_0\}$ inductivery. Assume that x_k and y_0 s.t. $y_k \in Tx_k$ are known. Choose $x_{k+1} \in X$ form $(*)$ such that

$$d(x_{k+1}, u) \leq \frac{d(x_k, u) + d(y_k, u)}{2}$$

for all $u \in X$.
Since Tx_{k+1} is nonempty compact, we can choose $y_{k+1} \in X$ such that

$$y_{k+1} \in Tx_{k+1} \text{ and } d(y_k, y_{k+1}) = d(y_k, Tx_{k+1}).$$

$$\begin{aligned} d(y_k, y_{k+1}) &= d(y_k, Tx_{k+1}) \\ &\leq \sup_{y\in Tx_k} d(y, Tx_{k+1}) \\ &\leq \mathcal{H}(Tx_k, Tx_{k+1}) \\ &\leq d(x_k, x_{k+1}). \end{aligned}$$

By this inequality and induction using (ϵ, n)-sequences, we have

$$d(x_k, x_{k+1}) \leq \delta + \epsilon$$

and

$$d(x_k, y_k) \leq 2(\delta + \epsilon)$$

for all nonnegative integer k. And by these inequalities and induction using (ϵ, n)-sequences, we have

$$d(x_k, y_{k+n}) \geq (n+2)(\delta + \epsilon) - 2^{n+1}\epsilon$$

for all nonnegative integer k and n.
By this inequality, we can choose $\{x_n^m\}$, $\{y_n^m\} \subseteq X$ such that

$$d(x_0^m, Tx_0^m) \leq 2\delta + \frac{\delta}{2^m}$$

and

$$d(x_0^m, y_m^m) \geq (m+2)(\delta+\epsilon) - 2^{m+1}\frac{\delta}{2^m} > m\delta.$$

Hence we have

$$\lim_{m\to\infty} d(x_0^m, y_m^m) = \infty .$$

This contradicts the boundedness of X. Therefore we have

$$\inf_{x\in X} d(x, Tx) = 0.$$

By theorem 2, we have

Theorem 3 *Let X be a nonempty compact metric space with property (S) and T be a multivalued nonexpansive mapping of X into $K(X)$. Then T has a fixed point, i.e., there exists $x_0 \in X$ such that $x_0 \in Tx_0$.*

Concerning fixed point theorems for multivalued nonexpansive mappings, in 1968, Markin [10] proved the first fixed point theorem.

Theorem 4 *Let H be a Hilbert space and C be a nonemty bounded closed convex subset of H and T be a multivalued nonexpansive mapping of C into $K(C)$ such that Tx is convex for each $x \in C$. Then T has a fixed point.*

He proved this theorem by proving that $(I - T)(C)$ is a closed subset of C. This theorem was generalized by several authors[3,16].

In 1974, Lim[8] generalized Markin's result to uniformly convex Banach spaces by transfinite induction as follows.

Theorem 5 *Let C be a nonempty bounded closed convex subset of uniformly convex Banach space E and T be a multivalued nonexpansive mapping of C into $K(C)$. Then T has a fixed point.*

We introduce a notion of uniformly convexity into convex metric spaces and prove a fixed point theorem for multivalued nonexpansive mappings in such spaces. Our theorem generalizes Lim's result and we can prove the theorem smartly by virture the filter theory.

2. Main results [13]. Let X be a w-convex metric space and W be its convex structure.

Definition 5 *X is said to be uniformly convex if for any $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon)$ such that, for all $r > 0$ and $x, y, z \in X$ with $d(z,x) \leq r$, $d(z,y) \leq r$ and $d(x,y) \geq r\epsilon$,*

$$d(z, W(x,y,1/2)) \leq r(1-\alpha) < r.$$

Example 2 *Uniformly convex Banach spaces.*

Example 3 *Let H be a Hilbert space and X be a nonempty closed subset of $\{x \in X : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ then $(\alpha x + \beta y)/\|\alpha x + \beta y\| \in X$ $\delta(X) \leq \sqrt{2}/2$. Let $d(x,y) = \cos^{-1}\{(x,y)\}$ for all $x, y \in X$, where $(\cdot,\cdot)$ is the inner product of H. When we define a convex structure W for (X,d) adiquately, it is easily seen that (X,d) becomes a complete and uniformly convex metric space[9].*

A convex metric space X is said to have a property(C) if every decreasing sequence of nonempty bounded closed convex subsets of X has a nonempty intersection. The authors proved the following results.

Theorem 6 *Let X be a complete and uniformly convex metric space. Then X has the property(C).*

We sckech the outline of the proof. Let $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty bounded closed convex subsets of X. Suppose that for each $n \geq 1$, $\delta(K_n) > 0$. Then for each $n \geq 1$, there exists $x, y \in K_n$ such that

$$d(x,y) \geq \frac{\delta(K_n)}{2} \quad \text{and} \quad d(z,x) \leq \delta(K_n) \ , \ d(z,y) \leq \delta(K_n) \quad \text{for all } z \in K_n.$$

Since X is uniformly convex, for each $n \geq 1$, there exists $u_n^1 \in K_n$ such that

$$d\left(z, u_n^1\right) \leq \delta(K_n)(1-\alpha) \quad \text{for all} \quad z \in K_n.$$

Put

$$K_n^1 = \left\{u_n^1, u_{n+1}^1, \cdots\right\}.$$

Then we have for each $n \geq 1$,

$$K_n^1 \neq \phi, K_n^1 \subseteq K_n, \ and\ K_{n+1}^1 \supseteq K_n^1.$$

Suppose that for each $n \geq 1$, $\delta\left(K_n^1\right) > 0$. Put for each $n \geq 1$,

$$B_n^1 = \bigcap\nolimits_{k=0}^{\infty} B\left[u_{n+k}^1, \delta\left(K_n^1\right)\right].$$

Note that for each $n \geq 1$, $\overline{co}K_n^1 \subseteq B_n^1$, $\delta\left(K_n^1\right) \leq \delta\left(K_n\right)(1-\alpha)$ and

$$\delta\left(\overline{co}K_n^1\right) \leq \delta\left(B_n^1\right) \leq \delta\left(B\left[u_n^1, \delta\left(K_n^1\right)\right]\right) \leq 2\delta\left(K_n\right)(1-\alpha).$$

And we have for each $n \geq 1$, there exist $x, y \in K_n^1$ such that

$$d(x,y) \geq \frac{\delta\left(K_n^1\right)}{2}, d(z,x) \leq \delta\left(K_n^1\right) \ and\ d(z,y) \leq \delta\left(K_n^1\right) \ for\ all\ z \in \overline{co}K_n^1.$$

Since X is uniformly convex, there exists $u_n^2 \in \overline{co}K_n^1$ such that

$$d\left(z, u_n^2\right) \leq \delta\left(K_n\right)(1-\alpha)^2$$

for all $z \in \overline{co}K_n^1$. Put $K_n^2 = \left\{u_n^2, u_{n+1}^2, \cdots\right\}$. Then we have for each $n \geq 1$,

$$\delta\left(\overline{co}K_n^2\right) \leq 2\delta\left(K_n\right)(1-\alpha)^2.$$

By the same method as above, we obtain for each $n \geq 1$,

$$\overline{co}K_n^3, \overline{co}K_n^4, \cdots, \ and\ u_n^3, u_n^4, \cdots.$$

And we have for each $n \geq 1$,

$$K_n \supseteq \overline{co}K_n^1 \supseteq \overline{co}K_n^2 \supseteq \cdots$$

and

$$\delta\left(\overline{co}K_n^m\right) \leq 2\delta\left(K_n\right)(1-\alpha)^m \to 0, \ as\ m \to \infty.$$

Since X is complete, for each $n \geq 1$, there exists $u_n \in K_n$ such that

$$\bigcap\nolimits_{m=1}^{\infty} \overline{co}K_n^m = \{u_n\}.$$

Since for each $n \geq 1$, $\bigcap_{m=1}^{\infty} \overline{co} K_n^m \supseteq \bigcap_{m=1}^{\infty} \overline{co} K_{n+1}^m$, we have

$$u_1 = u_2 = \cdots \ .$$

Hence we have, for each $n \geq 1$, there exists $u \in X$ such that

$$u \in \bigcap_{m=1}^{\infty} \overline{co} K_n^m \subseteq K_n.$$

So we have

$$\bigcap_{n=1}^{\infty} K_n \neq \phi.$$

To prove our main theorem, we need a lemma about filters on X. Concerning the filter theory, for instance, see[4]. Let $\mathcal{B}$ be a filterbase on X that contains at least one nonempty bounded subset in $\mathcal{B}$. Put for each $x \in X$

$$r(x, \mathcal{B}) \underset{def.}{=} \inf_{A \in \mathcal{B}} \sup_{y \in A} d(x, y).$$

We denote by $\lim_{A \in \mathcal{B}} \sup_{y \in A} d(x, y)$ the righthand side of above definition.

Lemma 1 *Let X be a complete and uniformly convex metric space. Let K be a nonempty closed convex subset of X and $\mathcal{F}$ be a filter on X that contains at least one nonempty bounded set of $\mathcal{F}$. Then, there exists a unique $u_0 \in K$ such that*

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

We sketch the outline of the proof. Put $r = \inf_{x \in K} r(x, \mathcal{F})$ and

$$K_n = \left\{ z \in K \ : \ r(z, \mathcal{F}) \leq r + \frac{1}{n} \right\}.$$

Then $\{K_n\}$ is a decreasing sequence of bounded closed convex subsets of K. By the previous theorem, we have

$$\bigcap K_n \neq \phi.$$

So there exists $u_0 \in K$ such that

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

The uniqueness of u_0 follows from uniformly convexity of X.

Our main theorem is as follows.

Theorem 7 *Let X be a bounded, complete and uniformly convex metric space. If T is a multivalued nonexpansive mapping of X into $K(X)$. Then T has a fixed point.*

We sketch the outline of the proof. By theorem 2, there exists $\{x_n\}$ such that

$$\lim_n d(x_n, Tx_n) = 0.$$

Put $A_n = \{x_n, x_{n+1}, \cdots\}$ for every $n \geq 1$. Since $\{A_n\}$ is a filterbase on X, it generates the filter $\mathcal{F}$ on X. Hence there exists an ultrafilter $\mathcal{U}$ on X and $\inf_{A\in\mathcal{U}}\sup_{x\in A} d(x, Tx) = 0$. On the other hand, by lemma 1, there exists a unique $u_0 \in X$ such that

$$r(u_0, \mathcal{U}) = \inf_{x\in X} r(x, \mathcal{U}).$$

Since Tx is nonempty compact for all $x \in X$, there exist $Sx \in Tx$ and $Px \in Tu_0$ such that

$$d(x, Sx) = d(x, Tx) \quad \textit{and} \quad d(Sx, Px) = d(Sx, Tu_0).$$

Since P is a mapping of X into Tu_0, $P(\mathcal{U})$ is a filterbase on Tu_0 and generates an ultrafilter on Tu_0. Since Tu_0 is compact, $P(\mathcal{U})$ converges to a point $p_0 \in Tu_0$.

$$\begin{aligned}
r(p_0, \mathcal{U}) &= \inf_{A\in\mathcal{U}}\sup_{x\in A} d(p_0, x) \\
&\leq \inf_{A\in\mathcal{U}}\sup_{xinA} \{d(p_0, Px) + d(Px, Sx) + d(Sx, x)\} \\
&= \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + d(Sx, Tu_0) + d(x, Tx)\} \\
&\leq \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + \mathcal{H}(Tx, Tu_0) + d(x, Tx)\} \\
&\leq \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + d(x, u_0) + d(x, Tx)\} \\
&= \inf_{A\in\mathcal{U}}\sup_{x\in A} d(x, u_0) \\
&= r(u_0, \mathcal{U}).
\end{aligned}$$

By lemma 1, we have

$$u_0 = p_0 \in Tu_0.$$

References

[1] L.M. Blumenthal, *Theory and Applications of Distance Geometry,* Oxford Univ. Press, London, 1953.

[2] F.E. Browder, *Nonlinear operators and nonlinear mappings in Banach spaces,* Proc. Symp. Pure. Math. 18, pt. 2, Amer. Math. Soc., Providence, R. I., (1976).

[3] D. Downing and W.O. Ray, *Some remarks on set valued mappings,* Nonlinear Analysis 5 (1981), 1367-1377.

[4] N. Dunford and J.T. Schwartz, *Linear Operators,* Part I, Interscience, New York, 1958.

[5] Y. Kijima, *Fixed points of nonexpansive self-maps of a compact metric space,* J. Math. Anal. Appl. 123 (1987), 114-116.

[6] Y. Kijima, *A fixed point theorem for nonexpansive self-maps of a metric space with some convexity,* Math. Japon. 37 (1992), 707-709.

[7] W.A. Kirk, *Krasnoselskiĭ's itration process in hyperbolic space,* Numer. Funct. Anal. Optim. 4 (1981/82), 371-381.

[8] T.C. Lim, *A fixed point theorem for for multivalued nonexpansive mappings in a uniformly convex Banach space,* Bull. Amer. Math. Soc. 80 (1974), 1123-1126.

[9] H.V. Machado, *Fixed point theorems for nonexpansive mappings in metric spaces with normal structure,* Thesis, The University Chicago, 1971.

[10] J.T. Markin, *A fixed point theorem for set valued mappings,* Bull. Amer. Math. Soc. 74 (1968), 639-640.

[11] K. Menger, *Untersuchngen über allgemaine Metric,* Mathematische Annalen 100 (1928), 75-163.

[12] T. Shimizu and W. Takahashi, *Fixed point theorems in certain convex metric spaces,* Math. Japon. 37 (1992), 855-859.

[13] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces,* TMNA 8 (1996), 197-203.

[14] W. Takahashi, *A convexity in metric space and nonexpansive mappings, I,* Kōdai Math. Sem. Rep. 22 (1970), 142-149.

[15] L.A. Talman, *Fixed points for condensing multifunctions in metric spaces with convex structure,* Kōdai Math. Sem. Rep. 29 (1977), 62-70.

[16] K. Yanagi, *On some fixed point theorems for malutivalued mappings,* Pacific J. Math., 87 (1980), 233-240.