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# GENERALIZED STRONGLY NONLINEAR QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of variational inequalities, which are called the generalized strongly nonlinear quasi-variational inequalities. An algorithm for finding the approximate solution of generalized strongly nonlinear quasi-variational inequalities is also given. These variational inequalities include the previously known classes of variational inequalities as special cases.

## 1. Introduction

Variational inequality theory introduced by Stampacchia [12] has enjoyed vigorous growth for the last thirty years. Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, and engineering sciences [1,6].

In recent years, various extensions and generalizations of the variational inequalities have been proposed and analyzed. An important one is the quasi-variational inequality introduced and studied by Bensoussan and Lions [2]. For the recent applications, and numerical methods, see [4,5].

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In this paper, we obtain an existence theorem of solutions of a generalized strongly nonlinear quasi-variational inequality and construst a new iterative algorithm, which includes many known algorithms as special cases to slove variational inequalities and quasi-variational inequalities. Further, we prove the convergence of the iterative sequences generated by this algorithm. Our main results extend and improve the earlier and recent results of Noor[8,9,10], Siddiqi and Ansari[11].

## 2. Preliminaries

Let  $H$  be a Hilbert space. We denote by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  the inner product and norm on  $H$ , respectively. Let  $K \subset H$  be a closed convex subset of  $H$ . Given mappings  $m : H \rightarrow H$ ,  $A : H \rightarrow H$ ,  $g : H \rightarrow H$ ,  $T : H \rightarrow 2^H$ , and  $V : H \rightarrow 2^H$ , we consider the problem of finding  $u \in H$ ,  $y \in V(u)$ , and  $w \in T(u)$  such that  $g(u) \in K(u)$  and

$$\langle v - g(u), w + Ay \rangle \geq 0 \quad (2.1)$$

for all  $v \in K(u)$ , where  $K(u) = m(u) + K$ .

The problem (2.1) is known as the generalized strongly nonlinear quasi-variational inequality.

If  $g \equiv I$ , the identity operator, the problem (2.1) is equivalent to finding  $u \in K(u)$ ,  $y \in V(u)$ , and  $w \in T(u)$  such that

$$\langle v - u, w + Ay \rangle \geq 0 \quad (2.2)$$

for all  $v \in K(u)$ . The problem (2.2) is called the multivalued strongly nonlinear quasi-variational inequality(see Noor[10]).

If  $K(u) \equiv K$ , the problem (2.2) is equivalent to finding  $u \in K$ ,  $y \in V(u)$ , and  $w \in T(u)$  such that

$$\langle v - u, w + Ay \rangle \geq 0 \quad (2.3)$$

for all  $v \in K$ , which is called the multivalued strongly nonlinear variational inequality (see Noor[10]).

If  $T : H \rightarrow H$  is a single valued operator and  $V : H \rightarrow H$  is the identity operator, the problem (2.3) is equivalent to finding  $u \in K$  such that

$$\langle v - u, T(u) + A(u) \rangle \geq 0$$

for all  $v \in K$ , which is called the strongly nonlinear variational inequality (see Noor[10]).

LEMMA 2.1[6]. *If  $K \subset H$  is a closed convex set and  $z \in H$  is a given point, then  $u \in K$  satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0$$

for all  $v \in K$  if and only if

$$u = P_K z. \quad (2.4)$$

LEMMA 2.2[6]. *The mapping  $P_K$  defined by (2.4) is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\|$$

for all  $u, v \in H$ .

LEMMA 2.3[7]. If  $K(u) = m(u) + K$  and  $K \subset H$  is a closed convex set, then for any  $u, v \in H$ , we have

$$P_{K(u)}(v) = m(u) + P_K(v - m(u)).$$

Let  $(X, d)$  be a metric space,  $2^X$  be the family of all nonempty subsets of  $X$ . For any  $A, B \in 2^X$ , define

$$\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}.$$

Let  $P = \{d(x, y) : x, y \in X\}$ ,  $\bar{P}$  denotes the closure of  $P$ . A mapping  $F : X \rightarrow 2^X$  is said to be the  $\varphi$ -contraction mapping if

$$\delta(Fx, Fy) \leq \varphi(d(x, y))$$

for all  $x, y \in X$ , where  $\varphi : \bar{P} \rightarrow [0, \infty)$  satisfies  $\varphi(t) < t$  for  $t \in \bar{P} - \{0\}$ .

By the proof of Theorem 1 and 2 of Boyd and Wong [3], it is easy to see that the following theorem holds.

THEOREM 2.1. Let  $(X, d)$  be a complete metrically convex metric space and  $F : X \rightarrow 2^X$  be a  $\varphi$ -contractive mapping. Then  $F$  has a fixed point and for any  $x_0 \in X$ ,  $x_n \in F(x_{n-1})$ ,  $n \geq 1$ ,  $\{x_n\}$  converges to a fixed point of  $F$  in  $X$ .

DEFINITION 2.1. Let  $D$  be a nonempty subset of  $H$ ,  $T : D \rightarrow 2^H$  and  $\Phi, \Psi : [0, \infty) \rightarrow [0, \infty)$ . We call

(1)  $T$  is  $\Phi$ -Lipschitz continuous if

$$\delta(Tx, Ty) \leq \|x - y\| \Phi(\|x - y\|)$$

for all  $x, y \in D$ .

(2)  $T$  is  $\Psi$ -strongly monotone if

$$\langle u - v, x - y \rangle \geq \|x - y\|^2 \Psi(\|x - y\|)$$

for all  $x, y \in D$ ,  $u \in T(x)$ , and  $v \in T(y)$ .

DEFINITION 2.2. An operator  $g : H \rightarrow H$  is said to be  
 (i) strongly monotone if there exists a constant  $\delta > 0$   
 such that

$$(g(u) - g(v), u - v) \geq \alpha \|u - v\|^2 \quad \text{for all } u, v \in H;$$

(ii) Lipschitz continuous if there exists a constant  $\sigma > 0$   
 such that

$$\|g(u) - g(v)\| \leq \sigma \|u - v\| \quad \text{for all } u, v \in H.$$

### 3. Main Results

THEOREM 3.1. Let  $K$  be a nonempty closed convex subset of  $H$ . Then  $u \in H$ ,  $y \in V(u)$ , and  $w \in T(u)$  are a solution of problem (2.1) if and only if, for some given  $\rho > 0$ , the mapping  $F : H \rightarrow 2^H$  defined by

$$F(u) = \bigcup_{w \in T(u)} \bigcup_{y \in V(u)} [u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u))]$$

has a fixed point.

*Proof.* Let  $u \in H$ ,  $y \in V(u)$ , and  $w \in T(u)$  be a solution of problem (2.1). Then we have  $g(u) \in K(u)$  and

$$\langle w + Ay, v - g(u) \rangle \geq 0$$

for all  $v \in K(u)$ , and hence for any given  $\rho > 0$ ,

$$\langle g(u) - (g(u) - \rho(w + Ay)), v - g(u) \rangle \geq 0$$

for all  $v \in K(u)$ . By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} g(u) &= P_{K(u)}(g(u) - \rho(w + Ay)) \\ &= m(u) + P_K(g(u) - \rho(w + Ay) - m(u)). \end{aligned}$$

Hence we get

$$\begin{aligned} u &= u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u)) \\ &\in \cup_{w \in T(u)} \cup_{y \in V(u)} [u - g(u) + m(u) \\ &\quad + P_K(g(u) - \rho(w + Ay) - m(u))] \\ &= F(u), \end{aligned}$$

i.e.,  $u$  is a fixed point of  $F$ .

Now let  $u$  be a fixed point of  $F$ . By the definition of  $F$ , there exist  $y \in V(u)$  and  $w \in T(u)$  such that

$$u = u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u))$$

Therefore

$$\begin{aligned} g(u) &= m(u) + P_K(g(u) - \rho(w + Ay) - m(u)) \\ &= P_{K(u)}(g(u) - \rho(w + Ay)). \end{aligned}$$

Hence

$$g(u) \in K(u)$$

and by Lemma 2.1,

$$\langle g(u) - (g(u) - \rho(w + Ay)), v - g(u) \rangle \geq 0$$

for all  $v \in K(u)$ . Note  $\rho > 0$ , and we have

$$\langle w + Ay, v - g(u) \rangle \geq 0$$

for all  $v \in K(u)$ . i.e.,  $u \in H$ ,  $y \in V(u)$ , and  $w \in T(u)$  are a solution of problem (2.1).

**THEOREM 3.2.** *Let  $K$  be a closed convex subset of  $H$ ,  $T : H \rightarrow 2^H$  be  $\Phi$ -Lipschitz continuous and  $\Psi$ -strongly monotone, and  $V : H \rightarrow 2^H$  be  $\Gamma$ -Lipschitz continuous,  $g : H \rightarrow H$  be Lipschitz continuous and strongly monotone, and  $A, m : H \rightarrow H$  be Lipschitz continuous. Suppose that there exists a constant  $\rho > 0$  such that  $\rho\xi\Gamma(t) < 1 - k$  and for all  $t \in [0, \infty)$*

$$\frac{1}{\rho}\{1 - [1 - (k + \rho\xi\Gamma(t))]^2 + \rho^2\Phi^2(t)\} < 2\Psi(t) < \frac{1}{\rho} + \rho\Phi^2(t) \quad (3.1)$$

and

$$k = 2(\sqrt{1 - 2\delta + \sigma^2} + \mu) < 1,$$

where  $\delta$  is a strong monotonicity constant of  $g$  and  $\xi, \sigma, \mu$  are Lipschitz constants of  $A, g$ , and  $m$ , respectively. Then, (2.1) has a solution.

*Proof.* Define a mapping  $F : H \rightarrow 2^H$  as

$$F(u) = \cup_{w \in T(u)} \cup_{y \in V(u)} [u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u))]$$

for each  $u \in H$ . By Theorem 3.1, it suffices to prove that  $F$  has a fixed point in  $H$ . For any  $u_1, u_2 \in H$ ,  $w_1 \in T(u_1)$ ,  $w_2 \in T(u_2)$ ,  $y_1 \in V(u_1)$ , and  $y_2 \in V(u_2)$ , by Lemma 2.2,



we have

$$\begin{aligned}
& \| (u_1 - g(u_1) + m(u_1) + P_K(g(u_1) - \rho(w_1 + Ay_1) - m(u_1))) \\
& \quad - (u_2 - g(u_2) + m(u_2) + P_K(g(u_2) - \rho(w_2 + Ay_2) - m(u_2))) \| \\
& \leq \| u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2) \| \\
& \quad + \| P_K(g(u_1) - \rho(w_1 + Ay_1) - m(u_1)) \\
& \quad \quad - P_K(g(u_2) - \rho(w_2 + Ay_2) - m(u_2)) \| \\
& \leq 2 \| u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2) \| \\
& \quad + \| u_1 - u_2 - \rho(w_1 - w_2) \| + \rho \| Ay_1 - Ay_2 \|. \tag{3.2}
\end{aligned}$$

Since  $T$  is  $\Phi$ -Lipschitz continuous and  $\Psi$ -strongly monotone, it can be obtained that

$$\begin{aligned}
& \| u_1 - u_2 - \rho(w_1 - w_2) \|^2 \\
& = \| u_1 - u_2 \|^2 - 2\rho \langle w_1 - w_2, u_1 - u_2 \rangle + \rho^2 \| w_1 - w_2 \|^2 \\
& \leq \| u_1 - u_2 \|^2 - 2\rho \| u_1 - u_2 \|^2 \Psi(\| u_1 - u_2 \|) + \rho^2 \delta^2(T(u_1), T(u_2)) \\
& \leq \| u_1 - u_2 \|^2 - 2\rho \| u_1 - u_2 \|^2 \Psi(\| u_1 - u_2 \|) \\
& \quad + \rho^2 \| u_1 - u_2 \|^2 \Phi^2(\| u_1 - u_2 \|) \\
& = [1 - 2\rho \Psi(\| u_1 - u_2 \|) + \rho^2 \Phi^2(\| u_1 - u_2 \|)] \| u_1 - u_2 \|^2. \tag{3.3}
\end{aligned}$$

By using the Lipschitz continuity of  $g$  and  $m$ , and the strong monotonicity of  $g$ , we easily see that

$$\begin{aligned}
& \| u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2) \| \\
& \leq \| u_1 - u_2 - (g(u_1) - g(u_2)) \| + \| m(u_1) - m(u_2) \| \\
& \leq (\sqrt{1 - 2\delta + \sigma^2} + \mu) \| u_1 - u_2 \|. \tag{3.4}
\end{aligned}$$

Further, since  $A$  is Lipschitz continuous and  $V$  is  $\Gamma$ -Lipschitz continuous, we have

$$\begin{aligned} \|Ay_1 - Ay_2\| &\leq \xi \|y_1 - y_2\| \\ &\leq \xi \delta(V(u_1), V(u_2)) \\ &\leq \xi \|u_1 - u_2\| \Gamma(\|u_1 - u_2\|). \end{aligned} \tag{3.5}$$

From (3.2)-(3.5), it follows that

$$\begin{aligned} \delta(F(u_1), F(u_2)) &\leq [2(\sqrt{1 - 2\delta + \sigma^2 + \mu}) \\ &\quad + (1 - 2\rho\Psi(\|u_1 - u_2\|) + \rho^2\Phi^2(\|u_1 - u_2\|))^{1/2} \\ &\quad + \rho\xi\Gamma(\|u_1 - u_2\|)] \|u_1 - u_2\| \\ &\leq \varphi(\|u_1 - u_2\|) \end{aligned}$$

for all  $u_1, u_2 \in H$ , where

$$\varphi(t) = t[k + (1 - 2\rho\Psi(t) + \rho^2\Phi^2(t))^{1/2} + \rho\xi\Gamma(t)]$$

and  $k = 2(\sqrt{1 - 2\delta + \sigma^2 + \mu})$ .

Clearly, each Hilbert space is a metrically convex metric space and by (3.1),  $\varphi(t) < t$  for each  $t \in [0, \infty)$ . By Theorem 2.1,  $F$  has a fixed point  $u$  in  $H$  and hence (2.1) has a solution  $u \in H$ ,  $y \in V(u)$ , and  $w \in T(u)$ .

**THEOREM 3.3.** *Let  $K$  be a closed convex subset of  $H$ ,  $T : H \rightarrow 2^H$  be  $\Phi$ -Lipschitz continuous and  $\Psi$ -strongly monotone, and  $V : H \rightarrow 2^H$  be  $\Gamma$ -Lipschitz continuous,  $g : H \rightarrow H$  be Lipschitz continuous and strongly monotone,*

and  $A, m, : H \rightarrow H$  be Lipschitz continuous. Suppose that there exists  $\rho > 0$  and  $h \in [0, 1)$  such that for all  $t \in [0, \infty)$ ,

$$0 < [1 - 2\rho\Psi(t) + \rho^2\Phi^2(t)]^{\frac{1}{2}} \leq h - k - \rho\xi\Gamma(t), \quad (3.6)$$

$$\overline{\lim}_{t \rightarrow 0^+} \Phi(t) \neq \infty, \quad \overline{\lim}_{t \rightarrow 0^+} \Gamma(t) \neq \infty.$$

and

$$k = 2(\sqrt{1 - 2\delta + \sigma^2} + \mu) < h,$$

where  $\delta$  is a strong monotonicity constant of  $g$  and  $\xi, \sigma$ , and  $\mu$  are Lipschitz constants of  $A, g$ , and  $m$ , respectively. Then for any  $u_0 \in H$ , the iterative scheme defined by

$$\begin{aligned} u_{n+1} = & (1 - \alpha_n)u_n + \alpha_n[u_n - g(u_n) + m(u_n) \\ & + P_K(g(u_n) - \rho(w_n + Ay_n) - m(u_n))], \end{aligned} \quad (3.7)$$

$$w_n \in T(u_n), \quad y_n \in V(u_n),$$

$$0 \leq \alpha_n \leq 1 \quad \text{for each } n \leq 0, \quad \sum_{n=0}^{\infty} \alpha_n \quad \text{diverges,}$$

satisfies that  $\{u_n\}$  converges to  $u$  strongly in  $H$ ,  $\{w_n\}$  and  $\{y_n\}$  converge to  $w$  and  $y$  strongly in  $H$ , respectively, and  $u \in H, y \in V(u)$ , and  $w \in T(u)$  is a solution of the problem (2.1).

*Proof.* By the assumption (3.6), for each  $t \in [0, \infty)$ , we have

$$\frac{1}{\rho}\{1 - [1 - (k + \rho\xi\Gamma(t))]^2 + \rho^2\Phi^2(t)\} < 2\Psi(t) < \frac{1}{\rho} + \rho\Phi^2(t).$$

By Theorem 3.2, the problem (2.1) has a solution  $u \in H$ ,  $y \in V(u)$ ,  $w \in T(u)$ , and

$$u = u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u)).$$

Hence, by Lemma 2.2, we have

$$\begin{aligned} \|u_{n+1} - u\| &\leq (1 - \alpha_n)\|u_n - u\| \\ &\quad + \alpha_n \{ \|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| \\ &\quad + \|P_K(g(u_n) - \rho(w_n + Ay_n) - m(u_n)) \\ &\quad - P_K(g(u) - \rho(w + Ay) - m(u))\| \} \\ &\leq (1 - \alpha_n)\|u_n - u\| \\ &\quad + \alpha_n \{ 2\|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| \\ &\quad + \|u_n - u - \rho(w_n - w)\| + \rho\|Ay_n - Ay\| \}. \end{aligned} \tag{3.8}$$

Since  $T$  is  $\Phi$ -Lipschitz continuous and  $\Psi$ -strongly monotone, it can be obtained that

$$\begin{aligned} \|u_n - u - \rho(w_n - w)\|^2 &\leq (1 - 2\rho\Psi(\|u_n - u\|) + \rho^2\Phi^2(\|u_n - u\|))\|u_n - u\|^2. \end{aligned} \tag{3.9}$$

By using the Lipschitz continuity of  $g$  and  $m$ , and the strongly monotonicity of  $g$ , we easily see that

$$\begin{aligned} \|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| &\leq (\sqrt{1 - 2\delta + \sigma^2} + \mu)\|u_n - u\|. \end{aligned} \tag{3.10}$$

Further, since  $A$  is Lipschitz continuous and  $V$  is  $\Gamma$ -Lipschitz continuous, we have

$$\|Ay_n - Ay\| \leq \xi\Gamma(\|u_n - u\|)\|u_n - u\|. \quad (3.11)$$

It follows from (3.8)-(3.11) that

$$\begin{aligned} & \|u_{n+1} - u\| \\ & \leq (1 - \alpha_n)\|u_n - u\| + \alpha_n\{2(\sqrt{1 - 2\delta + \sigma^2} + \mu) \\ & \quad + [1 - 2\rho\Psi(\|u_n - u\|) + \rho^2\Phi^2(\|u_n - u\|)]^{\frac{1}{2}} \\ & \quad + \rho\xi\Gamma(\|u_n - u\|)\}\|u_n - u\| \\ & \leq (1 - \alpha_n)\|u_n - u\| + \alpha_n h\|u_n - u\| \\ & = (1 - (1 - h)\alpha_n)\|u_n - u\| \\ & \leq \prod_{j=0}^n (1 - (1 - h)\alpha_j)\|u_0 - u\|. \end{aligned}$$

Since  $\sum_{j=0}^{\infty} \alpha_j$  diverges and  $1 - h > 0$ ,

$$\prod_{j=0}^{\infty} (1 - (1 - h)\alpha_j) = 0,$$

and hence  $\{u_n\}$  converges  $u$  strongly. Since  $w_n \in T(u_n)$ ,  $w \in T(u)$ , and  $T$  is  $\Phi$ -Lipschitz continuous, we have

$$\begin{aligned} \|w_n - w\| & \leq \delta(T(u_n), T(u)) \\ & \leq \Phi(\|u_n - u\|)\|u_n - u\| \end{aligned}$$

and hence  $\{w_n\}$  converges to  $w$  strongly. Similarly, we can prove  $\{v_n\}$  converges to  $v$  strongly. This completes the proof.

REMARK. For a suitable choice of the operators  $T$ ,  $V$ ,  $A$ ,  $g$ , and  $m$ , we obtain several known results[8,9,11] as special cases of Theorem 3.3.

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