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Ozeki's method on Hölder's inequality

富山大教育　泉野佐一 (Saichi Izumino)

Abstract. Ozeki's method means applications of the following two facts at the same time : (1) A convex funtion on an n-dimensionl cube attatins its maximum at a vertex of the cube. (2) For given two n-tuples (or n-dimensional vectors) of nonnegative numbers, if they are rearranged mutually in an opposite ordering, then the corresponding inner product is not larger than the initial one. In the preceding paper, using Ozeki's method, we succeded in estimating the differennce on both sides of Cauchy's inequality. In this paper we again make use of the method and show a complementary inequality on the difference derived from Hölder's inequality. We also discuss some appli-cations and an operator version related to the complementary inequality.

1. Introduction. Let

$$a = (a_1, ..., a_n) \quad \text{and} \quad b = (b_1, ..., b_n)$$

be n-tuples of real numbers satisfying

$$0 < m_1 \leqq a_k \leqq M_1 \quad \text{and} \quad 0 < m_2 \leqq b_k \leqq M_2 \quad (k = 1, 2, ..., n).$$

Then as a complementary inequality connected to Cauchy's inequality, the following one holds [4]:

$$(1.1) \qquad (T:=)\ \Sigma a_k{}^2 \Sigma b_k{}^2 - (\Sigma a_k b_k)^2 \leqq (n^2/3)(M_1M_2 - m_1m_2)^2.$$

It is Ozeki [9] who initiated to try such an estimation. He prese-nted an inequality [7][9]:

$$(1.2) \qquad T \leqq (n^2/4)(M_1M_2 - m_1m_2)^2,$$

though it was somewhat incorrect. Essentially by the same technique as

Ozeki devised the revised inequality (1.1) was deduced.

Consider $T = T(a,b)$ as a function defined in the product $[m_1,M_1]^n \times [m_2,M_2]^n$ of n-dimensional cubes $[m_1,M_1]^n$ and $[m_2,M_2]^n$. Then Ozeki's original idea for the estimation was making good use of the following two properties of $T(a,b)$.

(i) $T(a,b)$ is a separately convex function with respect to a and b, so that the maximum of $T(a,b)$ is attained at (a,b), each of a and b being an extremal point, that is, a vertex of the corresponding n-dimensional cube [6].

(ii) Define $c_- = (c_{-1},\dots,c_{-n})$ and $c^- = (c^-_1,\dots,c^-_n)$ be the re-arrangements of $c = (c_1,\dots,c_n)$ in decreasing order and in increasing order, respectively. Then for $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ we have $\Sigma a_{-k}b^-_k \leqq \Sigma a_k b_k$ and $\Sigma a^-_k b_{-k} \leqq \Sigma a_k b_k$ [2], so that

(1.3) $\quad T(a_-,b^-) \geqq T(a,b) \quad$ and $\quad T(a^-,b_-) \geqq T(a,b)$.

Take the difference

$$S(a,b) := (\Sigma a_k^2)^{1/2}(\Sigma b_k^2)^{1/2} - \Sigma a_k b_k$$

obtained from the square-root type Cauchy's inequality,or more generally, the difference

(1.4) $$S_p(a,b) := (\Sigma a_k^p)^{1/p}(\Sigma b_k^q)^{1/q} - \Sigma a_k b_k,$$

$$p>1,\ q>1,\ 1/p + 1/q = 1,$$

which is derived from Hölder's inequality. Then we come to a new problem to estimate it. Fortunately, the function $S_p(a,b)$ is again separately convex with respect to a and b, and similar inequalities as in (1.3) hold for $S_p(a,b)$ instead of $T(a,b)$.

In this paper we show an inequality on the difference $S_p(a,b)$,which gives an upper bound of $S_p(a,b)$, applying Ozeki's method. We also discuss some applications and an operator version connected to the inequality.

Throughout this paper we assume that $p>1$, $q>1$, $1/p + 1/q = 1$.

2. Difference from Hölder's inequality. In this section we prove

Theorem 2.1. Let $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ be n-tuples satisfying

$$0 < m_1 \leqq a_k \leqq M_1 \quad \text{and} \quad 0 < m_2 \leqq b_k \leqq M_2 \quad (k = 1,2,\dots,n).$$

Suppose that $\alpha := m_1/M_1 < 1$ and $\beta := m_2/M_2 < 1$. Then we have

(2.1) $$(\Sigma a_k^p)^{1/p}(\Sigma b_k^q)^{1/q} - \Sigma a_k b_k$$
$$\leqq nM_1M_2[(1-\alpha)/(1-\alpha^p) + (1-\beta)/(1-\beta^q) - 1$$
$$- c(\alpha,\beta;p,q)\{1/(1-\alpha^p) + 1/(1-\beta^q) - 1\}].$$

Here $c = c(\alpha,\beta;p,q)$ is defined as follows; letting

$$K = K(\alpha,\beta;p,q) = \{(1-\alpha^p)/p(1-\alpha)\}^{1/p}\{(1-\beta^q)/q(1-\beta)\}^{1/q},$$

$$g(t) = (1-\alpha)(1-Kt^{1/q}) \quad \text{and} \quad h(t) = (1-\beta)(1-Kt^{-1/p}),$$

we obtain a (unique) positive solution $t = t_*$ of the equation $g(t)= h(t)$, that is,

(2.2) $$(1-\alpha)(1-Kt^{1/q}) = (1-\beta)(1-Kt^{-1/p}),$$

and then we put

$$c = (1-\alpha)(1-Kt_*^{1/q}) \ (= (1-\beta)(1-Kt_*^{-1/p})).$$

Before we begin the proof, we remark that the inequality (2.1) is still holds (or has a meaning by taking a limit) if we replace the restrictions $0 < m_1$, $0 < m_2$, $\alpha < 1$ and $\beta < 1$ by weaker ones $0 \leqq m_1$, $0 \leqq m_2$, $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1$ in the theorem.

Proof of the theorem. Write

$$S_p(a,b) := (\Sigma a_k^p)^{1/p}(\Sigma b_k^q)^{1/q} - \Sigma a_k b_k,$$

$$\|a\|_p = (\Sigma a_k^p)^{1/p}, \quad \|b\|_q = (\Sigma b_k^q)^{1/q} \quad \text{and} \quad \langle a,b\rangle = \Sigma a_k b_k.$$

Then $\|a\|_p$ and $\|b\|_q$ satisfy the usual norm conditions, and $\langle a,b\rangle$ is a bilinear function. Hence we see that

$$S_p(a,b) = \|a\|_p\|b\|_q - \langle a,b\rangle$$

is a separately convex function with respect to a and b. Furthermore, we easily see that

$$S_p(a_-,b^-) \geqq S_p(a,b) \quad \text{and} \quad S_p(a^-,b_-) \geqq S_p(a,b).$$

Hence $S_p(a,b)$ has the same properties (i) and (ii) as $T(a,b)$ for application of Ozeki's method.

Note that an extremal point of $[m,M]^n$ is precisely a point, each of whose components is m or M. Hence we may seek the maximum of $S = S_p(a,b)$ among (a,b) of the following two types:

Case I. $$a = (\overbrace{M_1,\ldots,M_1}^{s},\overbrace{m_1,\ldots,m_1}^{n-s}), \quad 0 \leqq s \leqq n,$$

$$b = (\overbrace{m_2,\ldots,m_2}^{t},\overbrace{M_2,\ldots,M_2}^{n-t}), \quad 0 \leqq t \leqq n,$$

and

Case II. $$a = (\overbrace{m_1,\ldots,m_1}^{s},\overbrace{M_1,\ldots,M_1}^{n-s}), \quad 0 \leqq s \leqq n,$$

$$b = (\overbrace{M_2,\ldots,M_2}^{t},\overbrace{m_2,\ldots,m_2}^{n-t}), \quad 0 \leqq t \leqq n.$$

Each of the Cases I and II is again divided into two cases according to $t \leqq s$ and $s \leqq t$. For convenience sake, assume $M_1 = M_2 = 1$, and write $\alpha = m_1$ and $\beta = m_2$ for simplicity. Now we begin with

Case I(i): Let $0 \leqq t \leqq s \leqq n$ and let

$$a = (\overbrace{1, \ldots,1}^{t},\overbrace{1,\ldots,1}^{s-t},\overbrace{\alpha,\ldots,\alpha}^{n-s}),$$

$$b = (\overbrace{\beta,\ldots,\beta}^{t},\overbrace{1,\ldots,1}^{s-t}, \overbrace{1, \ldots,1}^{n-s}).$$

Then

$$S = \{s + (n-s)\alpha^p\}^{1/p}\{t\beta^q + (n-t)\}^{1/q} - \{t\beta + (s-t) + (n-s)\alpha\}.$$

Putting $x = t$, $y = s - t$, $z = n - s$, we then have

(2.3) $S = (x + y + \alpha^p z)^{1/p}(\beta^q x + n - x)^{1/q} - (\beta x + y + \alpha z)$

$$= \{n - (1-\alpha^p)z\}^{1/p}\{n - (1-\beta^q)x\}^{1/q} + (1-\alpha)z + (1-\beta)x - n,$$

$$z \geqq 0, \quad x \geqq 0, \quad z + x \leqq n.$$

In order to estimate S, we note the inequality

$$u^{1/p}v^{1/q} \leqq u/p + v/q, \quad \text{for } u \geqq 0, \ v \geqq 0,$$

or its slight extension

(2.4) $u^{1/p}v^{1/q} = (1/(p\lambda))^{1/p}(1/(q\mu))^{1/q}\cdot(p\lambda u)^{1/p}(q\mu v)^{1/q}$

$$\leqq (1/(p\lambda))^{1/p}(1/(q\mu))^{1/q}(\lambda u + \mu v), \quad \lambda > 0, \ \mu > 0.$$

We remark that the equality sign in the above inequality holds when

(2.5) $p\lambda u = q\mu v.$

Now if we put

$$\lambda = (1-\alpha)t^{1/2}/(1-\alpha^p), \quad \mu = (1-\beta)t^{-1/2}/(1-\beta^q) \quad \text{for } t > 0,$$

and $u = n - (1-\alpha^p)z, \quad v = n - (1-\beta^q)x,$

then from (2.4) we have

(2.6) $\{n - (1-\alpha^p)z\}^{1/p}\{n - (1-\beta^q)x\}^{1/q}$

$$\leqq \{(1-\alpha^p)t^{-1/2}/p(1-\alpha)\}^{1/p}\{(1-\beta^q)t^{1/2}/q(1-\beta)\}^{1/q} \times$$

$$[\{(1-\alpha)n/(1-\alpha^p)-(1-\alpha)z\}t^{1/2} + \{(1-\beta)n/(1-\beta^q)-(1-\beta)x\}t^{-1/2}]$$

$$=\{(1-\alpha^p)/p(1-\alpha)\}^{1/p}\{(1-\beta^q)/q(1-\beta)\}^{1/q}\times$$

$$\{(1-\alpha)nt^{1/q}/(1-\alpha^p) + (1-\beta)nt^{-1/p}/(1-\beta^q) - (1-\alpha)zt^{1/q} - (1-\beta)xt^{-1/p}\}$$

$$= K\{nt^{1/q}/K_\alpha + nt^{-1/p}/K_\beta - (1-\alpha)zt^{1/q} - (1-\beta)xt^{-1/p}\}.$$

Here

$$K_\alpha = (1-\alpha^p)/(1-\alpha), \quad K_\beta = (1-\beta^q)/(1-\beta)$$

and $K = (K_\alpha/p)^{1/p}(K_\beta/q)^{1/q}.$

By the way we remark that from the mean value theorem

$$K_\alpha/p = \theta_\alpha^{\,p-1} \quad \text{and} \quad K_\beta/q = \theta_\beta^{\,q-1}$$

for some θ_α and θ_β such that $\alpha < \theta_\alpha < 1$ and $\beta < \theta_\alpha < 1$, so that $K < 1$, and furthermore

(2.7) $K = (\theta_\alpha^{\,p-1})^{1/p}(\theta_\beta^{\,q-1})^{1/q} > \alpha^{1/q}\beta^{1/p}.$

Now it follows from (2.3) and (2.6) that

(2.8) $S \leqq nK\{t^{1/q}/K_\alpha + t^{-1/p}/K_\beta\} - n$
$$+ (1-\alpha)(1-Kt^{1/q})z + (1-\beta)(1-Kt^{-1/p})x.$$

It is convenient to write $F(z,x;t)$ the right side of the above inequality, that is,

$$F(z,x;t) = nK\{t^{1/q}/K_\alpha + t^{-1/p}/K_\beta\} - n + (1-\alpha)(1-Kt^{1/q})z + (1-\beta)(1-Kt^{-1/p})x.$$

Though the variables z and x are discrete, for a moment we assume that they are continuous. Let

$$\triangle = \{(z,x);\ z \geqq 0,\ x \geqq 0,\ z + x \leqq n\},$$

and for a fixed $t>0$, let ϕ_t be the maximum of $F(z,x;t)$ for $(z,x) \in \triangle$. Then it is our task to seek the minimum ϕ of $\phi_t = \phi(t)$ for $t>0$.

Since $F_t(z,x) = F(z,x;t)$ is a linear (or precisely, affine) function on the triangle $\triangle$, it attains its maximum ϕ_t at one of the vertexes $(0,0)$, $(n,0)$ and $(0,n)$ of $\triangle$. Since $(0<)\ K < 1$, we see that at least one of

$$g(t) = (1-\alpha)(1-Kt^{1/q}) \quad \text{and} \quad h(t) = (1-\beta)(1-Kt^{-1/p})$$

is nonnegative (for any $t>0$), so that either

$$F_t(n,0) = F_t(0,0) + g(t) \quad \text{or} \quad F_t(0,n) = F_t(0,0) + h(t)$$

is not smaller than $F_t(0,0)$. Hence, putting $G(t) = F_t(n,0)/n$ and $H(t) = F_t(0,n)/n$, we have

$$\begin{aligned}\phi(t)(=\phi_t) &= \max\{F_t(z,x);\ (z,x) \in \triangle\} \\ &= \max\{F_t(n,0),\ F_t(0,n)\} \\ &= n\max\{G(t),\ H(t)\}.\end{aligned}$$

By definition we see that

(2.9) $$\begin{aligned}G(t) = F_t(n,0)/n &= K(t^{1/q}/K_\alpha + t^{-1/p}/K_\beta) - 1 + g(t) \\ &= K\{\alpha^p t^{1/q}/K_\alpha + t^{-1/p}/K_\beta\} - \alpha,\end{aligned}$$

and

(2.10) $$\begin{aligned}H(t) = F_t(0,n)/n &= K(t^{1/q}/K_\alpha + t^{-1/p}/K_\beta) - 1 + h(t) \\ &= K\{t^{1/q}/K_\alpha + \beta^q t^{-1/p}/K_\beta\} - \beta.\end{aligned}$$

Let $t_G = qK_\beta \alpha^{-p}/pK_\alpha$ and $t_H = qK_\alpha \beta^q/pK_\beta$. Then $t_G > t_H$, and by an elementary calculation we see following facts.

(i) $G(t)$ is decreasing for $(0<)t<t_G$, increasing for $t>t_G$ and $\min_{t>0} G(t) = G(t_G) = 0$,

(ii) $H(t)$ is decreasing for $(0<)t<t_H$, increasing for $t>t_H$ and $\min_{t>0} H(t) = H(t_H) = 0$,

(iii) $L(t) := G(t) - H(t)$ $(= g(t) - h(t))$ is decreasing, and $\lim_{t\to 0} L(t) = \infty$, $\lim_{t\to\infty} L(t) = -\infty$. Hence the equation $L(t) = 0$, that is,

(2.11) $\qquad (1 - \alpha)(1 - Kt^{1/q}) = (1 - \beta)(1 - Kt^{-1/p})$

has a unique solution, which we denote by t_*. (We can see that $t_H < t_* < t_G$.) Hence we see that the minimum ϕ_0 $(= \phi/n)$ of the function $\phi(t)/n = \max\{G(t), H(t)\}$ is attained at $t = t_*$.

In order to express the minimum ϕ_0 in terms of t_*, or a parameter close to it, put (cf.(2.11))

$$c = (1 - \alpha)(1 - Kt_*^{1/q}) \ (= g(t_*) = h(t_*) = (1 - \beta)(1 - Kt_*^{-1/p})).$$

Then since

$$Kt_*^{1/q} = 1 - c/(1 - \alpha) \quad \text{and} \quad Kt_*^{-1/p} = 1 - c/(1 - \beta),$$

we have, from (2.9) (or (2.10)),

$$\begin{aligned}\phi_0 &= G(t_*) \ (= H(t_*)) \\ &= K(t_*^{1/q}/K_\alpha + t_*^{-1/p}/K_\beta) - 1 + g(t_*) \\ &= (1/K_\alpha)\{1 - c/(1 - \alpha)\} + (1/K_\beta)\{1 - c/(1 - \beta)\} - 1 + c \\ &= 1/K_\alpha + 1/K_\beta - 1 - c/(1 - \alpha))K_\alpha - c/(1 - \beta)K_\beta + c \\ &= (1-\alpha)/(1-\alpha^p) + (1-\beta)/(1-\beta^q) - 1 \\ &\qquad - c\{1/(1-\alpha^p) + 1/(1-\beta^q) - 1\}.\end{aligned}$$

This yields the desired inequality (2.1) with the assumption $M_1 = M_2 = 1$.

Next we consider the following

Case I (ii): Let $0 \leqq s \leqq t \leqq n$ and let

$$a = (\overbrace{1, \ldots, 1}^{s}, \overbrace{\alpha, \ldots, \alpha}^{t-s}, \overbrace{\alpha, \ldots, \alpha}^{n-t}),$$

$$b = (\overbrace{\beta, \ldots, \beta}^{t}, \overbrace{\beta, \ldots, \beta}^{t-s}, \overbrace{1, \ldots, 1}^{n-t}).$$

Then

$$S = \{s + (n - s)\alpha^p\}^{1/p}\{(n - t) + t\beta^q\}^{1/q} - \{s\beta + (t - s)\alpha\beta + (n - t)\alpha\}.$$

Putting $x = s$, $(y = t - s)$ and $z = n - t$, we have

$$(2.12) \quad S = \{n\alpha^p + (1-\alpha^p)x\}^{1/p}\{n\beta^q + (1-\beta^q)z\}^{1/q} - \beta(1-\alpha)x - \alpha(1-\beta)z - n\alpha\beta,$$

$$z \geqq 0,\ x \geqq 0,\ z + x \leqq n.$$

As in the preceding case, using (2.6),we have

$$(2.13) \quad \{n\alpha^p + (1-\alpha^p)x\}^{1/p}\{n\beta^q + (1-\beta^q)z\}^{1/q}$$

$$\leqq \{(1-\alpha^p)t^{-1/2}/p(1-\alpha)\}^{1/p}\{(1-\beta^q)t^{1/2}/q(1-\beta)\}^{1/q} \times$$

$$[\{(1-\alpha)n\alpha^p/(1-\alpha^p)+(1-\alpha)x\}t^{1/2} + \{(1-\beta)n\beta^q/(1-\beta^q)+(1-\beta)z\}t^{-1/2}]$$

$$= K\{n\alpha^p t^{1/q}/K_\alpha + n\beta^q t^{-1/p}/K_\beta + (1-\alpha)xt^{1/q} + (1-\beta)zt^{-1/p}\}.$$

Hence from (2.12) and (2.13)

$$(2.14) \quad S \leqq nK\{\alpha^p t^{1/q}/K_\alpha + \beta^q t^{-1/p}/K_\beta\} - n\alpha\beta + (1-\alpha)(Kt^{1/q} - \beta)x + (1-\beta)(Kt^{-1/p} - \alpha)z.$$

Put $\tilde{F}(x,z;t)$ the right side of the above inequality. Then for fixed $t>0$, the function $\tilde{F}_t(x,z) = \tilde{F}(x,z;t)$ is linear on $\triangle$ $(= \{(x,z);\ x \geqq 0,\ z \geqq 0,\ x + z \leqq n\})$ as in Case I(i). Note here that at least one of

$$Kt^{1/q} - \beta \quad \text{and} \quad Kt^{-1/p} - \alpha$$

is nonnegative; in fact, if we assume that

$$Kt^{1/q} - \beta < 0 \quad \text{and} \quad Kt^{-1/p} - \alpha < 0 \quad \text{for some } t>0,$$

then $(K/\alpha)^p < (\beta/K)^q$, which is impossible, because $(K/\alpha)^p > (\beta/K)^q$, or equivalently, $(K/\alpha)^p/(\beta/K)^q = (K/(\alpha^{1/q}\beta^{1/p}))^{p+q} > 1$, by (2.7).

Consequently, the maximum of $\tilde{F}_t(x,z)$ on the triangle $\triangle$ is $\tilde{F}_t(n,0)$ or $\tilde{F}_t(0,n)$ as in the preceding case. Putting $\tilde{G}(t) =$

$\tilde{F}_t(n,0)/n$ and $\tilde{H}(t) = \tilde{F}_t(0,n)/n$, we have

(2.15) $$\max\{\tilde{F}_t(x,z);\ (x,z) \in \triangle\} = n \max\{\tilde{G}(t),\ \tilde{H}(t)\}.$$

By definition

$$\tilde{G}(t) = K\{\{\alpha^p t^{1/q}/K_\alpha + \beta^q t^{-1/p}/K_\beta\} - \alpha\beta + (1-\alpha)(Kt^{1/q} - \beta)$$
$$= Kt^{1/q}/K_\alpha + K\beta^q t^{-1/p}/K_\beta - \beta,$$

and

$$\tilde{H}(t) = K\{\alpha^p t^{1/q}/K_\alpha + \beta^q t^{-1/p}/K_\beta\} - \alpha\beta + (1-\beta)(Kt^{-1/p} - \alpha)$$
$$= K\alpha^p t^{1/q}/K_\alpha + Kt^{-1/p}/K_\beta - \alpha.$$

Hence from (2.9) and (2.10) we see $\tilde{G}(t) = H(t)$ and $\tilde{H}(t) = G(t)$, so that we can reduce the remaining discussion in this case to one in the preceding case. Hence we have the same value ϕ as the minimum of $\phi(t) = \max\{\tilde{F}_t(x,z);\ (x,z) \in \triangle\}$.

Case II(i): Let $0 \leqq t \leqq s \leqq n$ and let

$$a = (\overbrace{\alpha,\ldots,\alpha}^{t}, \overbrace{\alpha,\ldots,\alpha}^{s-t}, \overbrace{1,\ldots,1}^{n-s}),$$

$$b = (\overbrace{1,\ldots,1}^{t}, \overbrace{\beta,\ldots,\beta}^{s-t}, \overbrace{\beta,\ldots,\beta}^{n-s}).$$

Then

$$S = \{s\alpha^p + (n-s)\}^{1/p}\{t + (n-t)\beta^q\}^{1/q}$$
$$- \{t\alpha + (s-t)\alpha\beta + (n-s)\beta\}.$$

Put $x = t$, $(y = s - t,)$ $z = n - s$. Then

$$S = \{n\alpha^p + (1-\alpha^p)z\}^{1/p}\{n\beta^q + (1-\beta^q)x\}^{1/q}$$
$$-\beta(1-\alpha)z - \alpha(1-\beta)x - n\alpha\beta.$$

If we exchange x and z, then we have the same identity (2.12) in Case I (ii). Hence we have the same upper bound ϕ of S as before.

Case II(ii): Let $0 \leqq s \leqq t \leqq n$ and let

$$a = (\overbrace{\alpha,\ldots,\alpha}^{s}, \overbrace{1,\ldots,1}^{t-s}, \overbrace{1,\ldots,1}^{n-t}),$$

$$b = (\overbrace{1,\ldots,1}^{s},\overbrace{1,\ldots,1}^{t-s},\overbrace{\beta,\ldots,\beta}^{n-t}).$$

Then

$$S = \{s\alpha^p + (n-s)\}^{1/p}\{t + (n-t)\beta^q\}^{1/q} - \{s\alpha + (t-s) + (n-t)\beta\}.$$

Putting $x = s$, $(y = t - s,)$ $z = n - t$, we have

$$S = \{n - (1-\alpha^p)x\}^{1/p}\{n - (1-\beta^q)z\}^{1/q} + (1-\alpha)x + (1-\beta)z - n.$$

Hence if we exchange x and z, then we have the same identity (2.3) in Case I (i), so that we obtain the same upper bound ϕ of S as before. This completes the proof.

3. Special cases. In this section we deduce two theorems from Theorem 2.1

Theorem 3.1. Let $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ be n-tuples satisfying

$$0 \leqq m_1 \leqq a_k \leqq M_1 \quad \text{and} \quad 0 \leqq m_2 \leqq b_k \leqq M_2 \qquad (k = 1,2,\ldots,n).$$

Suppose $M_1M_2 > 0$, and put $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then

$$(3.1) \quad (S =)(\Sigma a_k^2)^{1/2}(\Sigma b_k^2)^{1/2} - \Sigma a_kb_k \leqq n\, M_1M_2(1 - \alpha\beta)^2/\{2(1 + \alpha)(1 + \beta)\}.$$

If both α and β are rational and n is sufficiently large, then

$$s = (1 + 2\alpha + \alpha\beta)n/\{2(1 + \alpha)(1 + \beta)\}$$

is an integer, $0 \leqq s \leqq n$, and for a, b such that

$$(3.2) \qquad a_1 = \ldots = a_s = M_1,\ a_{s+1} = \ldots = a_n = m_1,$$
$$b_1 = \ldots = b_s = m_2,\ b_{s+1} = \ldots = b_n = M_2,$$

the equality sign in (3.1) is valid.

Proof. We may assume that $0 < \alpha < 1$ and $0 < \beta < 1$. Let $p = q = 2$. Then from (2.1) of Theorem 2.1 we have

$$(3.3) \quad S \leqq n\, M_1M_2[1/(1 + \alpha) + 1/(1 + \beta)$$

$$- c(\alpha, \beta; 2,2)\{1/(1 + \alpha) + 1/(1 + \beta) - 1\}].$$

It is easy to see that $K = K(\alpha, \beta; 2,2) = (1 + \alpha)^{1/2}(1 + \beta)^{1/2}/2$. The equation (2.11) is then

$$(1 - \alpha)(1 - Kt^{1/2}) = (1 - \beta)(1 - Kt^{-1/2}),$$

and the solution is $t\ (= t_*) = (1 + \beta)/(1 + \alpha)$. Hence we have

$$c = (1 - \alpha)(1 - Kt_*^{1/2}) = (1 - \alpha)(1 - \beta)/2.$$

Hence, replacing $c(\alpha, \beta; 2,2)$ in (3.3) by $(1 - \alpha)(1 - \beta)/2$, we obtain the desired inequality (3.1).

To obtain the particular integer $s\ (= (1 + 2\alpha + \alpha\beta)n/\{2(1 + \alpha)(1 + \beta)\})$, recall the relation (2.5) in the previous section, $p\lambda u = q\mu v$, which yields the equality sign of (2.4). Put in the equation

$$p = q = 2, \quad \lambda = (1-\alpha)t_*^{1/2}/(1-\alpha^2) = (1+\beta)^{1/2}/(1+\alpha)^{3/2},$$

$$\mu = (1-\beta)t_*^{-1/2}/(1-\beta^2) = (1+\alpha)^{1/2}/(1+\beta)^{3/2},$$

$$u = n - (1 - \alpha^2)z, \quad v = n - (1 - \beta^2)x.$$

Then we have

(3.4) $$(1+\beta)^2\{n - (1-\alpha^2)z\} = (1+\alpha)^2\{n - (1-\beta^2)x\}.$$

If we add the assumption $x + z = n$, then as the solution of these equations we can obtain the desired $x = s$.

For the difference of the p-th power mean and the usual arithmetic mean we have:

Theorem 3.2. Let $a = (a_1, \ldots, a_n)$ satisfy $0 \leqq m \leqq a_k \leqq M$ $(k = 1,2,\ldots,n)$, and let $\alpha = m/M$ $(M > 0)$. Then we have

(3.3) $$(\Sigma a_k^p/n)^{1/p} - \Sigma a_k/n$$
$$\leqq M[\{(1 - \alpha^p)/p(1 - \alpha)\}^{q-1}/q - (\alpha - \alpha^p)/(1 - \alpha^p)]$$

Proof. Let $M_1 = M$, $m_1 = m$ and $b_1 = \ldots = b_n = \beta < 1$. Then from Theorem 2.1 (2.1) we have

(3.4) $$(\Sigma a_k^p)^{1/p}(n\beta^q)^{1/q} - \Sigma a_k\beta$$

$$\leqq n\, M\beta\,[(1-\alpha)/(1-\alpha^{p}) + (1-\beta)/(1-\beta^{q}) - 1$$
$$- c(\alpha,\beta;p,q)\{1/(1-\alpha^{p}) + 1/(1-\beta^{q}) - 1\}].$$

If $\beta \to 1$ then $K = K(\alpha,\beta;p,q) \to K_0 := \{(1-\alpha^{p})/p(1-\alpha)\}^{1/p}$, and the equation (2.2) becomes $(1 - \alpha)(1 - K_0 t^{1/q}) = 0$ (as $\beta \to 1$), so that $t_* \to (1/K_0)^{q}$ by continuity of the solution. Moreover,

$$c = c(\alpha,\beta;p,q) \to 0$$

and

$$c/(1-\beta^{q}) = (1-\beta)(1 - Kt_*^{-1/p})/(1-\beta^{q}) \to (1/q)\{1 - K_0(1/K_0)^{-q/p}\}$$
$$= (1 - K_0^{q})/q.$$

Hence, letting $\beta \to 1$ in (3.4), we have

$$n^{1/q}(\Sigma a_k^{p})^{1/p} - \Sigma a_k$$
$$\leqq n\, M[(1-\alpha)/(1-\alpha^{p}) + 1/q - 1 - (1 - K_0^{q})/q].$$
$$= n\, M[(1-\alpha)/(1-\alpha^{p}) - 1 + \{(1-\alpha^{p})/p(1-\alpha)\}^{q/p}/q],$$

from which we can obtain the desired inequality.

Corollary 3.3. $(\Sigma a_k^{2}/n)^{1/2} - \Sigma a_k/n \leqq n(M - m)^{2}/4(M + m)$.

4. Extensions and operator versions. In this section we extend Theorem 2.1 and furethermore show some operator inequalities connected to the theorem. First we show a weigted version of Theorem 2.1.

Theorem 4.1. Let $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ be n-tuples satisfying

$$0 < m_1 \leqq a_k \leqq M_1 \quad\text{and}\quad 0 < m_2 \leqq b_k \leqq M_2 \qquad (k = 1,2,\ldots,n).$$

Suppose that $\alpha := m_1/M_1 < 1$ and $\beta := m_2/M_2 < 1$. If $w = (w_1,\ldots,w_n)$ is an n-tuple of nonnegative numbers with $w = \Sigma w_k$. Then , we have

(4.1) $$(\Sigma w_k a_k^{p})^{1/p}(\Sigma w_k b_k^{q})^{1/q} - \Sigma w_k a_k b_k$$
$$\leqq w\, M_1 M_2[(1-\alpha)/(1-\alpha^{p}) + (1-\beta)/(1-\beta^{q}) - 1$$
$$- c(\alpha,\beta;p,q)\{1/(1-\alpha^{p}) + 1/(1-\beta^{q}) - 1\}].$$

Here $c = c(\alpha, \beta; p, q)$ is defined as in Theorem 2.1, that is,; letting

$$K = \{(1-\alpha^p)/p(1-\alpha)\}^{1/p}\{(1-\beta^q)/q(1-\beta)\}^{1/q},$$

$$g(t) = (1 - \alpha)(1 - Kt^{1/q}) \quad \text{and} \quad h(t) = (1 - \beta)(1 - Kt^{-1/p}),$$

we obtain a (unique) positive solution $t = t_*$ of the equation $g(t)= h(t)$, and then we put $c = g(t_*)(= h(t_*))$.

Proof. We may assume that all w_k and w are rational numbers. Then,multiplying the both sides of (4.1) by a sufficiently large integer, we may assume that all w_k and w are integers. Hence we can obtain (4.1) from Theorem 2.1.

For convenience sake, from now on we write

$$F(\alpha, \beta; p,q) = (1-\alpha)/(1-\alpha^p) + (1-\beta)/(1-\beta^q) - 1$$
$$- c(\alpha, \beta; p,q)\{1/(1-\alpha^p) + 1/(1-\beta^q) - 1\}.$$
$$(0 \leqq \alpha < 1,\ 0 \leqq \beta < 1).$$

Let X be a measure space with a probability measure $\mu, \mu(X) = 1$, and let $L^r(X)$ $(r > 1))$ be the set of functions f such that $|f|^r$ is integrable on X. Then we have the following :

Theorem 4.2. Let $f \in L^p(X)$ and $g \in L^q(X)$. Suppose that

$$0 \leqq m_1 \leqq f \leqq M_1 \quad \text{and} \quad 0 \leqq m_2 \leqq g \leqq M_2,$$

and that $\alpha := m_1/M_1 < 1$ and $\beta := m_2/M_2 < 1$. Then

$$(4.2) \quad (\int f^p d\mu)^{1/p} (\int g^q d\mu)^{1/q} - \int fg d\mu \leqq M_1M_2\ F(\alpha, \beta; p,q).$$

Proof. Let $\{X_1,\ldots,X_n\}$ be a decomposition of X, $x_k \in X_k$ $(k = 1,\ldots,n)$. Then from Theorem 4.1 (4.1)

$$[\Sigma\{f(x_k)\}^p\mu(X_k)]^{1/p}[\Sigma\{g(x_k)\}^q\mu(X_k)]^{1/q} - \Sigma f(x_k)g(x_k)\mu(X_k)$$
$$\leqq M_1M_2\ F(\alpha, \beta; p,q).$$

Taking the limit of the decomposition, we obtain (4.2).

Let H be a Hilbert space, and let A and B be commuting self-adjoint operators on H. Then there exist commuting spectral families $E^A(.)$ and $E^B(.)$ corresponding to A and B such that for a polynomial p(A,B) (or a uniform limit of polynomials) in A and B [10, p.287]

$$\langle p(A,B)x, x\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(s,t)d\langle E^A(s)E^B(t)x, x\rangle \qquad \text{for } x \in H.$$

From this fact we show an operator version of Theorem 2.1 or its extension Theorem 4.2.

Theorem 4.3. Let A and B be two commuting selfadjoint operators on a Hilbert space H satisfying

$$0 \leqq m_1 \leqq A \leqq M_1 \quad \text{and} \quad 0 \leqq m_2 \leqq B \leqq M_2.$$

Suppose that $\alpha := m_1/M_1 < 1$ and $\beta := m_2/M_2 < 1$. Then for a unit vector $x \in H$

$$\text{(4.2)} \qquad \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \langle ABx, x\rangle \leqq M_1M_2\, F(\alpha, \beta; p,q).$$

Proof. Let $d\mu = d\langle E^A(s)E^B(t)x, x\rangle = d\| E^A(s)E^B(t)x \|^2$. Then μ is a positive measure on the rectangle $X = [m_1,M_1]\times[m_2,M_2]$ with $\mu(X) = 1$. Hence from Theorem 4.2 we have

$$\begin{aligned}\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \langle ABx, x\rangle &= \left(\iint_X s^p d\mu\right)^{1/p} \left(\iint_X t^q d\mu\right)^{1/q} - \iint_X st\,d\mu \\ &\leqq M_1M_2\, F(\alpha, \beta; p,q).\end{aligned}$$

In particular, if we assume $p = q = 2$ then from Theorem 3.1 (3.1)

$$F(\alpha,\beta;2,2) = (1-\alpha\beta)^2/\{2(1+\alpha)(1+\beta)\}.$$

Hence we have the following:

Corollary 4.4. $\langle A^2x, x\rangle^{1/2}\langle B^2x, x\rangle^{1/2} - \langle ABx, x\rangle$

$$\leqq M_1M_2\,(1-\alpha\beta)^2/\{2(1+\alpha)(1+\beta)\}.$$

We can obtain the following theorem from Theorems 4.3 or 3.2.

Theorem 4.5 (cf [8]). Let A be a selfadjoint operator on H such that $0 \leqq m \leqq A \leqq M$. Suppose $\alpha := m_1/M_1 < 1$. Then for any unit vector $x \in H$

(4.3) $\langle A^px, x\rangle^{1/p} - \langle Ax, x\rangle$

$$\leqq M[\{(1-\alpha^p)/p(1-\alpha)\}^{q-1}/q - (\alpha-\alpha^p)/(1-\alpha^p)]$$

We remark that B. Mond and J. E. Pecaric [8] defined the r-th mean $M_n^{[r]}(A;w) = (\Sigma w_kA_k^r/\Sigma w_k)^{1/r}$ for an n-tuple $A = (A_1,\dots,A_n)$ of positive operators and for an n-tuple $w = (w_1,\dots,w_n)$, and that they established an estimation of the difference $M_n^{[p]}(A,w) - M_n^{[r]}(A,w)$. The inequality (4.4) is compared to the case $n = 1$, $r = 1$.

In [5], F. Kubo and T. Ando introduced the s-geometric mean $A \#_s B$ of two positive operators which is defined by

$$A \#_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^sA^{1/2} \qquad (0 < s < 1).$$

Now in terms of the geometric mean we want to show an operator version of Theorem 2.1, or a reformulation of Theorem 4.3 (4.2) without the assumption that A and B commute, which is to be compared with [4, Theorem4.6].

Theorem 4.6. Let A and B be positive operators satisfying

$$0 < m_1 \leqq A \leqq M_1 \quad \text{and} \quad 0 < m_2 \leqq B \leqq M_2.$$

Suppose that $m_1/M_1 < 1$ and $m_2/M_2 < 1$. Then for any unit vector $x \in H$

$$(4.4) \quad \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \langle B^q \#_{1/p} A^p x, x\rangle$$
$$\leqq M_2{}^q\ G(p;\ m_1/M_2{}^{q/p},\ M_1/m_2{}^{q/p}),$$

where $G(p;m,M) = (M/q)[\{1 - (m/M)^p\}/p(1 - m/M)]^{q-1}$
$$- m\{1 - (m/M)^{p-1}\}/\{1 - (m/M)^p\}.$$

Proof. If C is a positive operator such that $0 < m \leqq A \leqq M$, then from (4.3) ($\alpha = m/M$), we have

$$\langle C^p x, x\rangle^{1/p} - \langle Cx, x\rangle$$
$$\leqq M[\{(1 - \alpha^p)/p(1 - \alpha)\}^{q-1}/q - (\alpha - \alpha^p)/(1 - \alpha^p)]$$
$$= G(p;m,M).$$

Replacing x by $x/\|x\|$ in the above inequality, we have

$$\langle C^p x, x\rangle^{1/p} \langle x, x\rangle^{1/q} - \langle Cx, x\rangle \leqq G(p;m,M)\langle x, x\rangle$$

for any vector $x \in H$. Furthermore, replace C by $(B^{-q/2}A^pB^{-q/2})^{1/p}$ and x by $B^{q/2}x$. Then we have

$$(4.5) \quad \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \langle B^{q/2}(B^{-q/2}A^pB^{-q/2})^{1/p}B^{q/2}x, x\rangle$$
$$\leqq G(p;m',M')\langle B^q x, x\rangle \leqq M_2{}^q\ G(p;\ m',M')$$

for some constants m' and M' such that

$$0 < m' \leqq (B^{-q/2}A^pB^{-q/2})^{1/p} \leqq M'.$$

To settle m' and M'. note

$$m_1{}^p/M_2{}^q \leqq B^{-q/2}A^pB^{-q/2} \leqq M_1{}^p/m_2{}^q$$

or $$m_1/M_2{}^{q/p} \leqq (B^{-q/2}A^pB^{-q/2})^{1/p} \leqq M_1/m_2{}^{q/p}.$$

Hence, putting $m' = m_1/M_2{}^{q/p}$ and $M' = M_1/m_2{}^{q/p}$ in (4.5), we obtain the desired inequalty (4.4)

We remark that in [4] the difference of the p-th power type inequality connected to (4.4), that is ,

$$\langle A^p x, x\rangle \langle B^q x, x\rangle^{p/q} - \langle A \#_{1/p} Bx, x\rangle^p$$

was estimated by using a result in M. Fujii, T. Furuta, R. Nakamoto and S. Takahasi [1].

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