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On solutions of quasi-linear partial differential equations $-\mathrm{div}\mathcal{A}(x,\nabla u)+\mathcal{B}(x,u)=0$

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§0. Introduction

Recently, a nonlinear potential theory has been developed in [1] for quasi-linear elliptic partial differential equations of second order of the form

$$-\mathrm{div}\mathcal{A}(x,\nabla u)=0,$$

where $\mathcal{A}$ is a mapping of $R^n\times R^n$ to $R^n (n\geq 2)$ satisfying a growth condition $\mathcal{A}(x,h)\cdot h\approx w(x)|h|^p$ $(1<p<\infty)$ with a "weight" $w(x)$, which is a nonnegative locally integrable function in R^n. A prototype is the so-called weighted p-Laplace equations

$$-\mathrm{div}(w(x)|\nabla u|^{p-2}\nabla u)=0,$$

This purpose of this paper is to extend some of the results in [1] to the equation

$$(*)\qquad -\mathrm{div}\mathcal{A}(x,\nabla u)+\mathcal{B}(x,u)=0,$$

where $\mathcal{B}(x,t)$ is a mapping of $R^n\times R$ to R, which is non-decreasing in t. A prototype equation may be given by

$$-\mathrm{div}(w(x)|\nabla u|^{p-2}\nabla u)+w(x)|u|^{p-2}u=0.$$

As a matter of fact, we treat the following three topics: (i) Existence and uniqness of solutions of Dirichlet problems for equation $(*)$ with Sobolev boundary values, or more generally of obstacle problems (section 3); (ii) Harnack inequality and Hölder continuity for solutions of $(*)$ (section 4); (iii) Regularity at the boundary for solutions of $(*)$ (section 5).

We can discuss (i) in the same way as in [1, Appendix I], using a general result of monotone operators. For (ii) and (iii), the methods in [1] are no longer applicable. We follow the discussion in [2] (for (ii)) and those in [4] (for (iii)), in which the unweighted case, namely the case $w=1$, is treated.

§1. Weighted Sobolev space

We recall the weighted Sobolev spaces $H^{1,p}(\Omega;\mu)$ which are adopted in [1].

Throughout this paper Ω will denote an open subset of $R^n (n\geq 2)$ and $1<p<\infty$. We denote $B(x,r)=\{y\in R^n : |x-y|<r\}$, and $\lambda B=B(x,\lambda r)$ if $B=B(x,r)$ and $\lambda>0$.

Let w be a locally integrable, nonnegative function in R^n. Then a Radon measure μ is canonically associated with the *weight* w :

$$(1)\qquad \mu(E)=\int_E w(x)dx.$$

Thus $d\mu(x) = w(x)dx$, where dx is the n-dimensional Lebesgue measure. We say that w (or μ) is *p-admissible* if the following four conditions are satisfied:

I. $0 < w < \infty$ almost everywhere in R^n and the measure μ is *doubling* , i.e. there is a constant $C_I > 0$ such that

$$\mu(2B) \leq C_I \mu(B)$$

whenever B is a ball in R^n .

II. If D is an open set and $\varphi_i \in C_0^\infty(D)$ is a sequence of functions such that $\int_D |\varphi_i|^p d\mu \to 0$ and $\int_D |\nabla\varphi_i - v|^p d\mu \to 0 (i \to \infty)$, where v is a vector-valued measurable function in $L^p(D;\mu;R^n)$, then $v = 0$.

III.(Sobolev inequality) There are constants $k > 1$ and $C_{III} > 0$ such that

$$\Big(\frac{1}{\mu(B)}\int_B |\varphi|^{kp} d\mu\Big)^{1/kp} \leq C_{III} r \Big(\frac{1}{\mu(B)}\int_B |\nabla\varphi|^p d\mu\Big)^{1/p}$$

whenever $B = B(x_0, r)$ is a ball in R^n and $\varphi \in C_0^\infty(B)$.

IV. There is a constant $C_{IV} > 0$ such that

$$\int_B |\varphi - \varphi_B|^p d\mu \leq C_{IV} r^p \int_B |\nabla\varphi|^p d\mu$$

whenever $B = B(x_0, r)$ is a ball in R^n and $\varphi \in C^\infty(B)$ is bounded. Here

$$\varphi_B = \frac{1}{\mu(B)}\int_B \varphi d\mu.$$

¿From now on, unless otherwise stated, we assume that μ is a p-admissible measure and $d\mu(x) = w(x)dx$.

In this paper, both condition IV and the following inequality are called the *Poincaré inequality*.

Poincaré inequality ([1, p.9])
If Ω is bounded, then

$$\int_\Omega |\varphi|^p d\mu \leq C_{III}^p (\text{diam } \Omega)^p \int_\Omega |\nabla\varphi|^p d\mu$$

for $\varphi \in C_0^\infty(\Omega)$.

Throughout this paper, let c_μ denote constants depending on C_I, C_{II}, C_{III}, k and C_{IV}.

For a μ-measurable function f defined on an open set Ω, L^p-norm of f is defined by

$$\|f\|_{p,\Omega} = \Big(\int_\Omega |f|^p d\mu\Big)^{1/p}.$$

For a function $\varphi \in C^\infty(\Omega)$ we let

$$\|\varphi\|_{1,p;\Omega} = \Big(\int_\Omega |\varphi|^p d\mu\Big)^{1/p} + \Big(\int_\Omega |\nabla\varphi|^p d\mu\Big)^{1/p},$$

where, we recall, $\nabla\varphi = (\partial_1\varphi, \cdots, \partial_n\varphi)$ is the gradient of φ. The *Sobolev space* $H^{1,p}(\Omega;\mu)$ is defined to be the completion of

$$\{\varphi \in C^\infty(\Omega) \,:\, \|\varphi\|_{1,p;\Omega} < \infty\}$$

with respect to norm $\|\cdot\|_{1,p;\Omega}$. In other words, a function u is in $H^{1,p}(\Omega;\mu)$ if and only if u is in $L^p(\Omega;\mu)$ and there is a vector-valued function v in $L^p(\Omega;\mu;R^n)$ such that for some sequence $\varphi_i \in C^\infty(\Omega)$

$$\int_\Omega |\varphi_i - u|^p d\mu \to 0$$

and

$$\int_\Omega |\nabla\varphi_i - v|^p d\mu \to 0$$

as $i \to \infty$. The function v is called the *gradient of u in* $H^{1,p}(\Omega;\mu)$ and denoted by ∇u.

The space $H_0^{1,p}(\Omega;\mu)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega;\mu)$. The corresponding local space $H_{loc}^{1,p}(\Omega;\mu)$ is defined in the obvious manner.

§2. Quasilinear PDE's

$\mathcal{A}$ is a mapping of $R^n \times R^n$ to R^n satisfying the following assumptions for some constants $0 < \alpha_1 \le \alpha_2 < \infty$:

(a1) the mapping $x \mapsto \mathcal{A}(x,h)$ is measurable for all $h \in R^n$ and

the mapping $h \mapsto \mathcal{A}(x,h)$ is continuous for a.e. $x \in R^n$;

for all $h \in R^n$ and a.e. $x \in R^n$

(a2) $$\mathcal{A}(x,h)\cdot h \ge \alpha_1 w(x)|h|^p,$$

(a3) $$|\mathcal{A}(x,h)| \le \alpha_2 w(x)|h|^{p-1},$$

(a4) $$(\mathcal{A}(x,h_1) - \mathcal{A}(x,h_2))\cdot(h_1 - h_2) > 0$$

whenever $h_1, h_2 \in R^n$, $h_1 \ne h_2$.

$\mathcal{B}$ is a mapping of $R^n \times R$ to R satisfying the following assumptions for a constant $0 < \alpha_3 < \infty$:

(b1) the mapping $x \mapsto \mathcal{B}(x,t)$ is measurable for all $t \in R$ and

the mapping $t \mapsto \mathcal{B}(x,t)$ is continuous for a.e. $x \in R^n$;

for all $t \in R$ and a.e. $x \in R^n$

(b2) $$|\mathcal{B}(x,t)| \le \alpha_3 w(x)(|t|^{p-1} + 1),$$

(b3) $$(\mathcal{B}(x,t_1) - \mathcal{B}(x,t_2))(t_1 - t_2) \ge 0.$$

whenever $t_1, t_2 \in R^n$. Using $\mathcal{A}$ and $\mathcal{B}$ we consider the quasilinear elliptic equation

(2) $$-\mathrm{div}\mathcal{A}(x,\nabla u) + \mathcal{B}(x,u) = 0.$$

A function $u \in H^{1,p}_{loc}(\Omega;\mu)$ is a (weak) solution of (2) if

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0 \tag{3}$$

whenever $\varphi \in C_0^\infty(\Omega)$. A function $u \in H^{1,p}_{loc}(\Omega;\mu)$ is a *supersolution* of (2) in Ω if

$$-\mathrm{div}\mathcal{A}(x,\nabla u) + \mathcal{B}(x,u) \geq 0$$

weakly in Ω , i.e.

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx \geq 0 \tag{4}$$

whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative. A function $u \in H^{1,p}_{loc}(\Omega;\mu)$ is a subsolution in Ω if (4) holds for all nonpositive $\varphi \in C_0^\infty(\Omega)$.

Lemma 2.1 *If $u \in H^{1,p}(\Omega;\mu)$ is a solution (respectively, a supersolustion) of (2) in Ω, then*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0 \quad (\textit{respectively}, \geq 0) \tag{5}$$

for all $\varphi \in H_0^{1,p}(\Omega;\mu)$ (respectively, for all nonnegative $\varphi \in H_0^{1,p}(\Omega;\mu)$) with compact support.

Proof : Let Ω' be an open set such that $\mathrm{spt}\varphi \subset \Omega' \subset\subset \Omega$. Since $\varphi \in H_0^{1,p}(\Omega';\mu)$, we can choose a sequence of functions $\varphi_i \in C_0^\infty(\Omega')$ such that $\varphi_i \to \varphi$ in $H^{1,p}(\Omega';\mu)$. If φ is nonnegative, pick nonnegative functions φ_i ([1, Lemma 1.23, p.21]). Then by (a3)

$$\begin{aligned}
&\Big|\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx - \Big(\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi_i dx + \int_\Omega \mathcal{B}(x,u)\varphi_i dx\Big)\Big| \\
&\leq \alpha_2\int_{\Omega'} |\nabla u|^{p-1}|\nabla\varphi - \nabla\varphi_i| d\mu + \alpha_3\int_{\Omega'}(|u|^{p-1}+1)|\varphi-\varphi_i| d\mu \\
&\leq \alpha_2\Big(\int_{\Omega'}|\nabla u|^p d\mu\Big)^{(p-1)/p}\Big(\int_{\Omega'}|\nabla\varphi-\nabla\varphi_i|^p d\mu\Big)^{1/p} \\
&\quad +2\alpha_3\Big(\int_{\Omega'}(|u|+1)^p d\mu\Big)^{(p-1)/p}\Big(\int_{\Omega'}|\varphi-\varphi_i|^p d\mu\Big)^{1/p}.
\end{aligned}$$

Because the last integral tends to zero as $i \to 0$, we have

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = \lim_{i\to\infty}\Big(\int_{\Omega'} \mathcal{A}(x,\nabla u)\cdot\nabla\varphi_i dx + \int_{\Omega'} \mathcal{B}(x,u)\varphi_i dx\Big) = (\geq)0,$$

and the lemma follows. □

The proof of Lemma 2.1 implies that (5) holds for all (nonnegative) $\varphi \in H_0^{1,p}(\Omega;\mu)$ if Ω is bounded.

A function u is a solution of (2) if and only if u is a supersolution and a subsolution. Indeed, if u is a supersolution and a subsolution of (2), since the positive part φ^+ of a test function $\varphi \in C_0^\infty(\Omega)$, belongs $H_0^{1,p}(\Omega;\mu)$ and has compact support, u satisfies (3) for φ^+. Similarly, u satisfies (3) for the negative part of φ. Hence u is a solution of (2).

§3. The existence of solutions

In this section, The existence of solutions of Dirichlet problems for equation (2) with Sobolev boundary values will be proved, using a general result in the theory of monotone operators.

Let X be a reflexive Banach space with dual X' and let $\langle\cdot,\cdot\rangle$ denote a pairing between X' and X. If $K \subset X$ is a closed convex set, then a mapping $\Im : K \to X'$ is called *monotone* if

$$\langle \Im u - \Im v \ , \ u - v\rangle \geq 0$$

for all u, v in K. Futher, $\Im$ is called *coercive on* K if there exists $\varphi \in K$ such that

$$\frac{\langle \Im u_j - \Im\varphi \ , \ u_j - \varphi\rangle}{\|u_j - \varphi\|} \to \infty$$

whenever u_j is a sequence in K with $\|u_j\| \to \infty$.

We recall the following proposition. ([3, Corollary III.1.8, p.87]).

Proposition 3.1 *Let K be a nonempty closed convex subset of X and let $\Im : K \to X'$ be monotone, coercive, and weakly continuous on K. Then there exists an element u in K such that*

$$\langle \Im u \ , \ v - u\rangle \geq 0$$

whenever $v \in K$.

Throughout this section, we assume that Ω is bounded.

Suppose that ψ is any function in Ω with values in the extended reals $[-\infty, \infty]$, and that $\theta \in H^{1,p}(\Omega;\mu)$. Let

$$\mathcal{K}_{\psi,\theta} = \mathcal{K}_{\psi,\theta}(\Omega) = \{v \in H^{1,p}(\Omega;\mu) : v \geq \psi \text{ a.e in } \Omega \ , \ v - \theta \in H_0^{1,p}(\Omega;\mu)\}.$$

Set $X = L^p(\Omega;\mu;R^n) \times L^p(\Omega;\mu;R)$ and $K = \{(\nabla v, v) \ : \ v \in \mathcal{K}_{\psi,\theta}(\Omega)\}$.

Lemma 3.2 *K is a closed convex set in X.*

Proof : K is clearly convex. To show the closedness, let $(\nabla v_i, v_i) \in K$ be a sequence converging to (f, u) in X. By $\nabla v_i \to f$ in $L^p(\Omega;\mu;R^n)$ and $v_i \to u$ in $L^p(\Omega;\mu;R)$, v_i is a bounded sequence in $H^{1,p}(\Omega;\mu)$. Since $\mathcal{K}_{\psi,\theta}$ is a convex and closed subset of $H^{1,p}(\Omega;\mu)$, there is a function $v \in \mathcal{K}_{\psi,\theta}$ such that $v = u$ and $\nabla v = f$ ([1, Theorem 1.31, p.25]). Thus $(f, u) \in K$. The lemma is proved. □

Let $\langle\cdot,\cdot\rangle$ be the pairing between X and X',

$$\langle (f,u),(g,v)\rangle = \int_\Omega f \cdot g d\mu + \int_\Omega uv d\mu,$$

where (f, u) is in X and (g, v) in $X' = L^{p/(p-1)}(\Omega;\mu;R^n) \times L^{p/(p-1)}(\Omega;\mu;R)$.

A mapping $\Im : K \to X'$ is well defined by the formula

$$\langle \Im(\nabla v, v),(f,u)\rangle = \int_\Omega \mathcal{A}(x, \nabla v(x)) \cdot f(x)dx + \int_\Omega \mathcal{B}(x, v(x))u(x)dx$$

for $(f,u) \in X$; indeed, by (a3) and (b2),

$$|\int_\Omega \mathcal{A}(x,\nabla v)\cdot f dx| \le \alpha_2\Big(\int_\Omega |\nabla v|^p d\mu\Big)^{(p-1)/p}\Big(\int_\Omega |f|^p d\mu\Big)^{1/p}$$
$$|\int_\Omega \mathcal{B}(x,v) u dx| \le 2\alpha_3\Big(\int_\Omega (|v|+1)^p d\mu\Big)^{(p-1)/p}\Big(\int_\Omega |u|^p d\mu\Big)^{1/p}.$$

Lemma 3.3 $\Im$ *is monotone, coercive, and weakly continuous on* K.

Proof : By (a4) and (b3), $\Im$ is monotone.

Next we show that $\Im$ is coercive on K. Fix $(\nabla\varphi,\varphi) \in K$. Hereafter, for simplicity, we shall write $\|\cdot\|$ for $\|\cdot\|_{p,\Omega}$. By (a2), (a3) and (b3)

$$\begin{aligned}
&\langle \Im(\nabla u,u) - \Im(\nabla\varphi,\varphi), (\nabla u,u) - (\nabla\varphi,\varphi)\rangle \\
&= \int_\Omega (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla\varphi))\cdot(\nabla u - \nabla\varphi)dx + \int_\Omega (\mathcal{B}(x,u) - \mathcal{B}(x,\varphi))(u-\varphi)dx \\
(6)\quad &\ge \alpha_1(\|\nabla u\|^p + \|\nabla\varphi\|^p) - \alpha_2(\|\nabla u\|^{p-1}\|\nabla\varphi\| + \|\nabla u\|\|\nabla\varphi\|^{p-1}) \\
&\ge \|\nabla u - \nabla\varphi\|\alpha_1 2^{-p}\|\nabla u - \nabla\varphi\|^{p-1} - \alpha_2 2^{p-1}\|\nabla\varphi\|(\|\nabla\varphi\|^{p-1} + \|\nabla u - \nabla\varphi\|^{p-1}) \\
&\qquad -\alpha_2\|\nabla\varphi\|^{p-1}(\|\nabla\varphi\| + \|\nabla u - \nabla\varphi\|).
\end{aligned}$$

Since $u - \varphi \in H_0^{1,p}(\Omega;\mu)$,

(7) $$\|u-\varphi\| \le c\|\nabla u - \nabla\varphi\|.$$

By (6) and (7), $\Im$ is coercive on K.

Finally, to show that $\Im$ is weakly continuous on K, let $(\nabla u_i, u_i) \in K$ be a sequence that converges to an element $(\nabla u, u) \in K$ in X. For any subsequence $(\nabla u_{i_j}, u_{i_j})$ of $(\nabla u_i, u_i)$, there is a subsequence $(\nabla u'_{i_j}, u'_{i_j})$ of $(\nabla u_{i_j}, u_{i_j})$ such that $(\nabla u'_{i_j}, u'_{i_j}) \to (\nabla u, u)$ a.e. in Ω. By (a1) and (b1), we have

$$\mathcal{A}(x,\nabla u'_{i_j}(x))w^{-1/p}(x) \to \mathcal{A}(x,\nabla u(x))w^{-1/p}(x)$$
$$\mathcal{B}(x,u'_{i_j}(x))w^{-1/p}(x) \to \mathcal{B}(x,u(x))w^{-1/p}(x)$$

a.e. in Ω. Since

$$\int_\Omega |\mathcal{A}(x,\nabla u_i)w^{-1/p}|^{p/(p-1)}dx \le \alpha_2^{p/(p-1)}\int_\Omega |\nabla u_i|^p d\mu$$
$$\int_\Omega |\mathcal{B}(x,u_i)w^{-1/p}|^{p/(p-1)}dx \le 2\alpha_3^{p/(p-1)}\int_\Omega (|u_i|+1)^p d\mu,$$

$L^{p/(p-1)}(\Omega;dx)$-norms of $\mathcal{A}(x,\nabla u_i)w^{-1/p}$ and $\mathcal{B}(x,u_i)w^{-1/p}$ are uniformly bounded. Therefore

$$\mathcal{A}(x,\nabla u'_{i_j})w^{-1/p} \to \mathcal{A}(x,\nabla u)w^{-1/p}$$
$$\mathcal{B}(x,u'_{i_j})w^{-1/p} \to \mathcal{B}(x,u)w^{-1/p}$$

weakly in $L^{p/(p-1)}(\Omega;dx)$. Since the weak limit is independent of $(\nabla u_{i_j}, u_{i_j})$,

$$\mathcal{A}(x,\nabla u_i)w^{-1/p} \to \mathcal{A}(x,\nabla u)w^{-1/p}$$
$$\mathcal{B}(x,u_i)w^{-1/p} \to \mathcal{B}(x,u)w^{-1/p}.$$

weakly in $L^{p/(p-1)}(\Omega; dx)$. Hence we have for all $(f,g) \in X$ that

$$\begin{aligned}\langle \Im(\nabla u_i, u_i), (f,g)\rangle &= \int_\Omega \mathcal{A}(x, \nabla u_i)\cdot f dx + \int_\Omega \mathcal{B}(x, u_i) g dx \\ &= \int_\Omega \mathcal{A}(x, \nabla u_i) w^{-1/p} \cdot f w^{1/p} dx + \int_\Omega \mathcal{B}(x, u_i) w^{-1/p} g w^{1/p} dx \\ &\to \int_\Omega \mathcal{A}(x, \nabla u) w^{-1/p} \cdot f w^{1/p} dx + \int_\Omega \mathcal{B}(x, u) w^{-1/p} g w^{1/p} dx \\ &= \langle \Im(\nabla u, u), (f,g)\rangle.\end{aligned}$$

Therefore the lemma follows. □

Now the following theorem follows form Proposotion 3.1, Lemma 3.2 and Lemma 3.3.
Theorem 3.4 *Suppose that $\mathcal{K}_{\psi,\theta}(\Omega) \neq \emptyset$, then there is a function u in $\mathcal{K}_{\psi,\theta}$ such that*

$$\int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla(v-u) dx + \int_\Omega \mathcal{B}(x,u)(v-u) dx \geq 0 \tag{8}$$

whenever $v \in \mathcal{K}_{\psi,\theta}$.

A function u in $\mathcal{K}_{\psi,\theta}(\Omega)$ that satisfies (8) for all $v \in \mathcal{K}_{\psi,\theta}(\Omega)$ is called a *solution to the obstacle problem in $\mathcal{K}_{\psi,\theta}(\Omega)$.*

As a corollaly to this theorem, we have the existence of solutions of Dirichlet problems with Sobolev boundary values.
Corollaly 3.5 *Suppose that $\theta \in H^{1,p}(\Omega;\mu)$. Then, there is a function $u \in H^{1,p}(\Omega;\mu)$ with $u - \theta \in H_0^{1,p}(\Omega;\mu)$ such that*

$$-\mathrm{div}\mathcal{A}(x, \nabla u) + \mathcal{B}(x,u) = 0$$

weakly in Ω, that is

$$\int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0$$

whenever $\varphi \in H_0^{1,p}(\Omega;\mu)$.

Proof: Choose $\psi \equiv -\infty$. Let u be the solution to the obstacle problem in $\mathcal{K}_{\psi,\theta}$ and $\varphi \in H_0^{1,p}(\Omega;\mu)$. Since $u+\varphi$, $u-\varphi \in \mathcal{K}_{\psi,\theta}$, we have

$$\int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx \geq 0$$

and

$$-\int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\varphi dx - \int_\Omega \mathcal{B}(x,u)\varphi dx \geq 0.$$

Then

$$\int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0.$$

Hence Corollary 3.5 follows. □

The uniqueness of solutions of Dirichlet problems for equation (2) and obstacle problems in $\mathcal{K}_{\psi,\theta}$ follows from the following comparison principle Lemma 3.6 and Lemma 3.7 respectively.

Lemma 3.6 *Let $u \in H^{1,p}(\Omega;\mu)$ be a supersolution and $v \in H^{1,p}(\Omega;\mu)$ a subsolution of (2) in Ω. If $\eta = \min(u-v,0) \in H_0^{1,p}(\Omega;\mu)$, then $u \geq v$ a.e. in Ω.*

Proof: By (a4) and (b3),

$$\int_\Omega (\mathcal{A}(x,\nabla v) - \mathcal{A}(x,\nabla u)) \cdot \nabla \eta dx \leq -\int_{\{u<v\}} (\mathcal{A}(x,\nabla v) - \mathcal{A}(x,\nabla u)) \cdot (\nabla v - \nabla u) dx \leq 0,$$

$$\int_\Omega (\mathcal{B}(x,v) - \mathcal{B}(x,u))\eta dx \leq -\int_{\{u<v\}} (\mathcal{B}(x,v) - \mathcal{B}(x,u))(v-u) dx \leq 0.$$

From this we have

$$0 \leq \int_\Omega \mathcal{A}(x,\nabla v) \cdot \nabla \eta dx + \int_\Omega \mathcal{B}(x,v)\eta dx - \Big(\int_\Omega \mathcal{A}(x,\nabla u) \cdot \nabla \eta dx + \int_\Omega \mathcal{B}(x,u)\eta dx\Big) \leq 0.$$

and, hence

$$\int_\Omega (\mathcal{A}(x,\nabla v) - \mathcal{A}(x,\nabla u)) \cdot \nabla \eta dx = 0$$

and

$$\int_\Omega (\mathcal{B}(x,v) - \mathcal{B}(x,u))\eta dx = 0.$$

Therefore $\nabla \eta = 0$ a.e. in Ω. Because $\eta \in H_0^{1,p}(\Omega;\mu)$, $\eta = 0$ a.e. in Ω ([1, Lemma 1.17, p.18]). The lemma follows. □

Lemma 3.7 *Suppose that u is a solution to the obstacle problem in $\mathcal{K}_{\psi,\theta}(\Omega)$. If $v \in H^{1,p}(\Omega;\mu)$ is a supersolution of (2) in Ω such that $\min(u,v) \in \mathcal{K}_{\psi,\theta}(\Omega)$, then $v \geq u$ a.e. in Ω.*

Proof: Since $u - \min(u,v) \in H_0^{1,p}(\Omega;\mu)$ and is nonnegative, the lemma is proved in the same manner as in the proof of Lemma 3.6. □

§4. The local behavior of solutions

In this section, we study the local behavior of solutions of (2).

The next theorem can be shown in the same manner as [2, Theorem 1].

Theorem 4.1 *Each solution of (2) in Ω is locally bounded.*

We obtain, using the Moser iteration technique, the following Harnack inequality.

Let $B(R)$ denote an open ball of radius R.

Theorem 4.2 *Let u be a nonnegative solution of equation (2) in Ω. Given $R_0 > 0$ there is a constant $c > 0$ such that*

$$\operatorname*{ess\,sup}_{B(R)} u \leq c \operatorname*{ess\,inf}_{B(R)} (u + R)$$

whenever $B(R)$ is a ball in Ω such that $3B(R) \subset \Omega$ and $R \leq R_0$. Here c depends only on n, p, α_1, α_2, α_3, c_μ and R_0.

We require some lemmas to prove Theorem 4.2.

Lemma 4.3 ([2, Lemma 2, p.252]) *Let a be a positive exponent, and let a_i, b_i $(i = 1, \cdots, N)$, be two sets of N real numbers such that $0 < a_i < \infty$ and $0 \le b_i < a$. Suppose that z is a positive number satisfying*

$$z^a \le \sum a_i z^{b_i}.$$

Then

$$z \le c \sum (a_i)^{\gamma_i}$$

where c depends only on N, a, and b_i, and where $\gamma_i = (a - b_i)^{-1}$.

Lemma 4.4 (John-Nirenberg lemma) ([1, Appendix II]) *Suppose that v is a locally μ-integrable function in Ω with*

$$\sup \frac{1}{\mu(B)} \int_B |v - v_B| d\mu \le c_o,$$

where

$$v_B = \frac{1}{\mu(B)} \int_B v d\mu$$

and the supremum is taken over all balls $B \subset\subset \Omega$. Then there are positive constants c_1 and c_2 depending on c_0, n, and c_μ such that

$$\sup \frac{1}{\mu(B)} \int_B e^{c_1 |v - v_B|} d\mu \le c_2,$$

where the supremum is taken over all balls $B \subset\subset \Omega$.

Let u be a nonnegative solution of equation (2) in Ω and $B = B(R)$ is a ball in Ω. We set $\bar{u} = u + R$. Thus, by Theorem 4.1, if $\eta \in C_0^\infty(B)$ is nonnegative, then $\varphi(x) = \eta^p \bar{u}^\beta \in H_0^{1,p}(B; \mu)$ for any real value of β . Moreover,

$$|\mathcal{B}(x, u)| \le 2\alpha_3 w \max(1, 1/R^{p-1}) \bar{u}^{p-1}.$$

We set $\alpha_3' = 2\alpha_3 \max(1, 1/R^{p-1})$.

Next lemma guarantees that $v = \log \bar{u}$ satisfies the hypothesis of John-Nirenberg lemma.

Lemma 4.5 *Suppose that u is a nonnegative solution of equation* (2) *in Ω* and $B = B(R)$ *is a ball in Ω such $3B \subset \Omega$. Then there is a constant $c > 0$ such that*

$$\int_{B_1} |v - v_{B_1}| d\mu \le c\mu(B_1) \quad (v = \log \bar{u}),$$

whenever B_1 is a ball with $B_1 \subset 2B$. Here c depends on p, α_1, α_2, $\alpha_3' R^p$ and c_μ.

Proof : Setting $\varphi = \eta^p \bar{u}^{1-p}$, we have

$$0 = \int_{3B} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{3B} \mathcal{B}(x, u) \varphi dx$$

$$\begin{aligned}
&= \int_{3B} \mathcal{A}(x, \nabla u) \cdot \{p(\eta/\bar{u})^{p-1}\nabla\eta + (1-p)(\eta/\bar{u})^p \nabla u\} dx + \int_{3B} \mathcal{B}(x, u)\eta^p \bar{u}^{1-p} dx \\
&\leq -\alpha_1(p-1)\int_{3B}(\eta/\bar{u})^p |\nabla u|^p d\mu + \alpha_2 p \int_{3B}(\eta/\bar{u})^{p-1}|\nabla\eta||\nabla u|^{p-1} d\mu \\
&\qquad + \alpha_3' \int_{3B} \eta^p \bar{u}^{1-p}|\bar{u}|^{p-1} d\mu \\
&= -\alpha_1(p-1)\int_{3B}|\eta\nabla v|^p d\mu + \alpha_2 p\int_{3B}|\nabla\eta||\eta\nabla v|^{p-1}d\mu + \alpha_3'\int_{3B}\eta^p d\mu,
\end{aligned}$$

where $v = \log \bar{u}$. Hence

$$\alpha_1(p-1)\|\eta\nabla v\|_{p,3B}^p \leq \alpha_2 p \int_{3B}|\nabla\eta||\eta\nabla v|^{p-1}d\mu + \alpha_3'\int_{3B}\eta^p d\mu. \tag{9}$$

Let $B_1 \subset 2B$ be any open ball of radius h. Let η be so chosen that $\eta = 1$ in B_1, $0 \leq \eta \leq 1$ in $3B \backslash B_1$, the support of η is contained in $(3/2)B_1$, and $|\nabla\eta| \leq 3/h$. Then by Hölder's inequality we obtain

$$\begin{aligned}
\int_{3B}|\nabla\eta||\eta\nabla v|^{p-1}d\mu &\leq \Big(\int_{(3/2)B_1}|\nabla\eta|^p d\mu\Big)^{1/p}\Big(\int_{(3/2)B_1}|\eta\nabla v|^p d\mu\Big)^{(p-1)/p} \\
&\leq \frac{3}{h}\{\mu((3/2)B_1)\}^{1/p}\|\eta\nabla v\|_{p,3B}^{p-1},
\end{aligned}$$

$$\int_{3B}\eta^p d\mu \leq \mu((3/2)B_1).$$

By the above inequalities and (9) we have

$$\alpha_1(p-1)\|\eta\nabla v\|_{p,3B}^p \leq \frac{3\alpha_2 p}{h}\{\mu((3/2)B_1)\}^{1/p}\|\eta\nabla v\|_{p,3B}^{p-1} + \frac{\alpha_3'(3R)^p}{h^p}\mu((3/2)B_1).$$

Application of Lemma 4.3 yields,

$$\|\nabla v\|_{p,B_1} \leq ch^{-1}\mu((3/2)B_1)^{1/p},$$

where $\eta = 1$ in B_1 have been used. Finally by the the doubling property, Hölder's inequality and Poincaré inequality we have

$$\int_{B_1}|v - v_{B_1}|d\mu \leq c\{\mu((3/2)B_1)\}^{(p-1)/p} h\Big(\int_{B_1}|\nabla v|^p d\mu\Big)^{1/p} \leq c\mu(B_1) \quad (v = \log\bar{u}),$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3' R^p, c_\mu)$. □

The following estimates will be used when we apply to the Moser iteration technique.

Lemma 4.6 *Suppose that u is a nonnegative solution of equation* (2) *in* Ω and $B = B(R)$ *is a ball in* Ω. *For* $\beta \neq 0$, $p-1$, *let* q *satisfying* $pq = p + \beta - 1$ and $v = \bar{u}^q$. *Then there is a constant* $c > 0$ *such that*

(i)if $\beta > 0$,

$$\|\eta v\|_{kp,B} \leq c\{\mu(B)\}^{(1-k)/kp} R(1+\beta^{-1})(1+q)^p(\|v\nabla\eta\|_{p,B} + \|\eta v\|_{p,B}),$$

(ii)if $1-p<\beta<0$,

$$\|\eta v\|_{kp,B} \le c\{\mu(B)\}^{(1-k)/kp} R(1-\beta^{-1})(\|v\nabla\eta\|_{p,B}+\|\eta v\|_{p,B}),$$

(iii)if $\beta<1-p$,

$$\|\eta v\|_{kp,B} \le c\{\mu(B)\}^{(1-k)/kp} R(1+|q|)^p(\|v\nabla\eta\|_{p,B}+\|\eta v\|_{p,B}),$$

where c depends only on p, α_1, α_2, c_μ and $\alpha_3' R^{p-1}$.

Proof : We prove only (i), the proofs of (ii) and (iii) being similar. For $\varphi=\eta^p\bar{u}^\beta$, we have

$$\begin{aligned} 0 &= \int_B \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_B \mathcal{B}(x,u)\varphi dx \\ &= \int_B \mathcal{A}(x,\nabla u)\cdot(p\eta^{p-1}\bar{u}^\beta\nabla\eta+\beta\eta^p\bar{u}^{\beta-1}\nabla u)dx + \int_B \mathcal{B}(x,u)\eta^p\bar{u}^\beta dx \\ &\ge \alpha_1\beta\int_B \eta^p\bar{u}^{\beta-1}|\nabla u|^p d\mu - p\alpha_2\int_B |\nabla u|^{p-1}\eta^{p-1}|\nabla\eta|\bar{u}^\beta d\mu - \alpha_3'\int_B \eta^p\bar{u}^\beta\bar{u}^{p-1}d\mu. \end{aligned}$$

Since $pq=p+\beta-1$ and $v=\bar{u}^q$,

$$\text{(10)}\qquad \frac{\alpha_1\beta}{q^p}\|\eta\nabla v\|_p^p \le \frac{p\alpha_2}{q^{p-1}}\int_B |v\nabla\eta||\eta\nabla v|^{p-1}d\mu + \alpha_3'\int_B(\eta v)^p d\mu.$$

Here for simplicity we have written $\|\cdot\|_p$ for $\|\cdot\|_{p,B}$.

By Hölder's inequality,

$$\begin{aligned} \int_B |v\nabla\eta||\eta\nabla v|^{p-1}d\mu &\le \|v\nabla\eta\|_p\|\eta\nabla v\|_p^{p-1}, \\ \int_B(\eta v)^p d\mu &= \|\eta v\|_p\Big(\int_B(\eta v)^p d\mu\Big)^{(p-1)/p} \\ &\le \|\eta v\|_p\Big\{\Big(\int_B(\eta v)^{kp}d\mu\Big)^{1/k}\Big(\int_B d\mu\Big)^{(k-1)/k}\Big\}^{(p-1)/p} \\ &= \mu(B)^{(k-1)(p-1)/(kp)}\|\eta v\|_p\|\eta v\|_{kp}^{p-1} \\ &\le c_\mu R^{p-1}\|\eta v\|_p(\|v\nabla\eta\|_p^{p-1}+\|\eta\nabla v\|_p^{p-1}), \end{aligned}$$

where we have used Sobolev inequality. By the above inequalities, if we set

$$z=\frac{\|\eta\nabla v\|_p}{\|v\nabla\eta\|_p},\qquad \zeta=\frac{\|\eta v\|_p}{\|v\nabla\eta\|_p},$$

then (10) can be written as

$$\beta z^p \le c\{qz^{p-1}+q^p\zeta(1+z^{p-1})\},$$

where $c=c(p,\alpha_1,\alpha_2,\alpha_3'R^{p-1},c_\mu)$. Application of Lemma 4.3 yields

$$z\le c(1+\beta^{-1})(1+q)^p(1+\zeta),$$

that is,

$$\text{(11)}\qquad \|\eta\nabla v\|_p \le c(1+\beta^{-1})(1+q)^p(\|v\nabla\eta\|_p+\|\eta v\|_p).$$

Finally using Sobolev inequality again, from (11) we obtain the desired estimate. □

Proof of Theorem 4.2 : Set $v = \log \bar{u}$. By Lemma 4.4 and Lemma 4.5, there are positive constants r_0 and c_0 such that

$$\begin{aligned}\Big(\int_{B_1} e^{r_0 v} d\mu\Big)\Big(\int_{B_1} e^{-r_0 v} d\mu\Big) &= \Big(\int_{B_1} e^{r_0(v-v_{B_1})} d\mu\Big)\Big(\int_{B_1} e^{r_0(v_{B_1}-v)} d\mu\Big) \\ &\leq \Big(\int_{B_1} e^{r_0|v-v_{B_1}|} d\mu\Big) \leq c_0^2\{\mu(B_1)\}^2.\end{aligned}$$

Because B_1 is any ball contained in $2B$,

$$\Big(\int_{2B} e^{r_0 v} d\mu\Big)\Big(\int_{2B} e^{-r_0 v} d\mu\Big) \leq c_0^2\{\mu(2B)\}^2.$$

Hence

$$(12) \qquad \Big(\int_{2B} \bar{u}^{r_0} d\mu\Big)^{1/r_0} \leq c\{\mu(B)\}^{2/r_0}\Big(\int_{2B} \bar{u}^{-r_0} d\mu\Big)^{-1/r_0}.$$

Next, let $0 < h' < h \leq 3R$. Let the function $\eta \in C_0^\infty(B(h))$ be so chosen that $\eta = 1$ in $B(h')$, $0 \leq \eta \leq 1$ in $B(h)$ and $|\nabla\eta| \leq 3(h-h')^{-1}$. Then Lemma 4.6 yields
(i) if $\beta > 0$,

$$(13) \qquad \|\bar{u}^q\|_{kp,B(h')} \leq c\{\mu(B)\}^{(1-k)/kp} R(1+q)^p (h-h')^{-1}(1+\beta^{-1})\|\bar{u}^q\|_{p,B(h)},$$

(ii) if $1-p < \beta < 0$,

$$(14) \qquad \|\bar{u}^q\|_{kp,B(h')} \leq c\{\mu(B)\}^{(1-k)/kp} R(h-h')^{-1}(1-\beta^{-1})\|\bar{u}^q\|_{p,B(h)},$$

(iii) if $\beta < 1-p$,

$$(15) \qquad \|\bar{u}^q\|_{kp,B(h')} \leq c\{\mu(B)\}^{(1-k)/kp} R(h-h')^{-1}(1+|q|)^p\|\bar{u}^q\|_{p,B(h)},$$

where c depends only on p, α_1, α_2, c_μ and $\alpha_3' R^{p-1}$.

Putting $r = pq = p+\beta-1$ in (13) and (14), combining the result in a single inequality, we obtain

$$(16) \qquad \begin{aligned}\Big(\int_{B(h')} \bar{u}^{kr} d\mu\Big)^{1/kr} &\leq \Big\{c\{\mu(3B)\}^{(1-k)/kp} R(h-h')(1+|\beta|^{-1})(1+r)^p\Big\}^{p/r} \\ &\times\Big(\int_{B(h)} \bar{u}^{r} d\mu\Big)^{1/r},\end{aligned}$$

for all $0 < r \neq p-1$. Let

$$r_\nu = k^\nu r_0' \quad \nu = 0, 1, 2, \cdots,$$

and $h_\nu = R(1+2^{-\nu})$, $h_\nu' = h_{\nu+1}$, where $r_0' \leq r_0$ is so chosen that $r_\nu \neq p-1$ for any $\nu = 0, 1, 2, \cdots$. Thus

$$|\beta| = |r-(p-1)| \geq c > 0,$$

whenever $r = r_\nu$, where c depends only on p, k, r_0. The term $(1+|\beta|^{-1})$ in (16) can thus be absorbed into the general constant c. Hence from(16) we have that

$$\Big(\int_{B(h_\nu')} \bar{u}^{r_{\nu+1}} d\mu\Big)^{1/r_{\nu+1}} \leq \Big\{c\{\mu(3B)\}^{(1-k)/kp} 2^{\nu+1}(1+r_\nu)^p\Big\}^{p/r_\nu}\Big(\int_{B(h_\nu)} \bar{u}^{r_\nu} d\mu\Big)^{1/r_\nu}$$

$$= c^{1/k^\nu}\{\mu(3B)\}^{(1-k)/kr_0'k^\nu} 2^{p\nu/r_0'k^\nu}\{(1+r_0'k^\nu)^{p^2/r_0'}\}^{1/k^\nu}\Big(\int_{B(h_\nu)} \bar{u}^{r_\nu}\,d\mu\Big)^{1/r_\nu}$$
$$\le c_1^{1/k^\nu} c_2^{\nu/k^\nu}\{\mu(3B)\}^{(1-k)/kr_0'k^\nu}\Big(\int_{B(h_\nu)} \bar{u}^{r_\nu}\,d\mu\Big)^{1/r_\nu}.$$

By iterating, it follows that

$$\operatorname*{ess\,sup}_B \bar{u} \le c\{\mu(3B)\}^{-1/r_0'}\Big(\int_{2B} \bar{u}^{r_0'}\,d\mu\Big)^{1/r_0'}. \tag{17}$$

Setting $s = pq$ in (15), since s and q are negative, we obtain

$$\Big(\int_{B(h')} \bar{u}^{ks}\,d\mu\Big)^{1/ks} \ge \Big\{c\{\mu(3B)\}^{(1-k)/kp} R(h-h')^{-1}(1+|s|)^p\Big\}^{p/s}\Big(\int_{B(h)} \bar{u}^{s}\,d\mu\Big)^{1/s}.$$

Let $s_\nu = -k^\nu r_0$, $h_\nu = R(1+2^{-\nu})$ and $h_\nu' = h_{\nu+1}$. Then

$$\Big(\int_{B(h_\nu')} \bar{u}^{s_{\nu+1}}\,d\mu\Big)^{1/s_{\nu+1}} \ge c_1^{-1/k^\nu} c_2^{-\nu/k^\nu}\{\mu(3B)\}^{-(1-k)/kr_0k^\nu}\Big(\int_{B(h_\nu)} \bar{u}^{s_\nu}\,d\mu\Big)^{1/s_\nu}.$$

By iterating, we obtain

$$\operatorname*{ess\,inf}_B \bar{u} \ge c^{-1}\{\mu(3B)\}^{1/r_0}\Big(\int_{2B} \bar{u}^{-r_0}\,d\mu\Big)^{-1/r_0}. \tag{18}$$

Finally, by (12), (17), (18), and a simple application of Hölder's inequality, we have

$$\begin{aligned}\operatorname*{ess\,sup}_B \bar{u} &\le c\{\mu(3B)\}^{-1/r_0'}\Big(\int_{2B} \bar{u}^{r_0'}\,d\mu\Big)^{1/r_0'} \le c\{\mu(3B)\}^{-1/r_0}\Big(\int_{2B} \bar{u}^{r_0}\,d\mu\Big)^{1/r_0} \\ &\le c\{\mu(3B)\}^{1/r_0}\Big(\int_{2B} \bar{u}^{-r_0}\,d\mu\Big)^{-1/r_0} \le c \operatorname*{ess\,inf}_B \bar{u}.\end{aligned}$$

Since $\bar{u} = u + R$, this concludes the proof of Theorem 4.2. □

We apply Theorem 4.4 to show that any solutions of (2) has Hölder continuous representative.

Theorem 4.7 *Let u be a solution of* (2) *in Ω and x_0 be any point of Ω. If $0 < R < \infty$ is such that $\bar{B}(x_0, R) \subset \Omega$ and if $|u| \le L$ a.e in $B(x_0, R)$, then there are constants c and $0 < \lambda < 1$ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u - \operatorname*{ess\,inf}_{B(x_0,\rho)} u \le c\Big(\frac{\rho}{R}\Big)^\lambda,$$

whenever $0 < \rho < R$. Here c and λ depend only on $n, p, \alpha_1, \alpha_2, \alpha_3, c_\mu, R$ and L.
Proof : We write $B(r) = B(x_0, r)$ and

$$M(r) = \operatorname*{ess\,sup}_{B(r)} u, \quad m(r) = \operatorname*{ess\,inf}_{B(r)} u.$$

Then $M(r)$ and $m(r)$ are well defined for $0 < r \le R$, and

$$\bar{u} = M(r) - u, \quad \bar{\bar{u}} = u - m(r)$$

are non-negative in $B(r)$. Obviously $\bar{u}$ is a solution of

$$-\text{div}\bar{\mathcal{A}}(x,\nabla\bar{u})+\bar{\mathcal{B}}(x,\bar{u})=0$$

where $\bar{\mathcal{A}}(x,\bar{h})=-\mathcal{A}(x,-\bar{h})$ and $\bar{\mathcal{B}}(x,\bar{t})=-\mathcal{B}(x,M(r)-\bar{t})$. Thus

$$|\bar{\mathcal{B}}(x,\bar{t})|\leq\alpha_3'w(x)(|\bar{t}|^{p-1}+1),$$

where α_3' is a constant depending only on α_3, p and L. By applying Harnack inequality to $\bar{u}$, we have

(19) $M(r)-m(r/3)=\text{ess}\sup_{B(r/3)}\bar{u}\leq c(\text{ess}\inf_{B(r/3)}\bar{u}+r)=c\{M(r)-M(r/3)+r\}.$

Similarly we have

(20) $M(r/3)-m(r)=\text{ess}\sup_{B(r/3)}\bar{\bar{u}}\leq c(\text{ess}\inf_{B(r/3)}\bar{\bar{u}}+r)=c\{m(r/3)-m(r)+r\}.$

Here $c>1$ depends on $n,p,\alpha_1,\alpha_2,\alpha_3,c_\mu,R$ and L. By (19) and (20),

$$M(r/3)-m(r/3)\leq\frac{c-1}{c+1}\{M(r)-m(r)\}+\frac{2c}{c+1}r. \tag{21}$$

Thus setting

$$\theta=\frac{c-1}{c+1},\quad\tau=\frac{2cR}{c-1}$$

and

$$\omega=M(r)-m(r),$$

(21) can be written as

$$\omega(r/3)\leq\theta\{\omega(r)+\tau(r/R)\}.$$

Since $\omega(r)$ is an increasing function, for any number $s\geq3$ we have also

$$\omega(r/s)\leq\theta\{\omega(r)+\tau(r/R)\},\quad 0<r\leq R.$$

By iterating, we obtain

$$\omega(R/s^\nu)\leq\theta^\nu\{\omega(R)+\tau\{1+(\theta s)^{-1}+\cdots+(\theta s)^{-\nu+1}\}\}, \tag{22}$$

for $\nu=1,2,3,\cdots$. Let s be so chosen that $\theta s=3$. Then (22) implies

$$\omega(R/s^\nu)\leq\theta^\nu\{\omega(R)+2\tau\}. \tag{23}$$

For any ρ such that $0<\rho\leq R/s$ choose ν such that $R/s^{\nu+1}<\rho\leq R/s^\nu$. Then from (23) we have

$$\omega(\rho)\leq\omega(R/s^\nu)\leq\theta^\nu(\omega(R)+2\tau). \tag{24}$$

If we set $\gamma=-\log_3\theta$, then we have $\theta=s^{-\lambda}$ where $\lambda=\gamma/(\gamma+1)>0$. Thus

$$\theta^\nu=\Big(\frac{R}{s^{\nu+1}}\frac{s}{R}\Big)^\lambda\leq c\Big(\frac{\rho}{R}\Big)^\lambda.$$

Hence, since $\omega(R) + 2\tau \le c(L+R)$, (22) implies

$$\omega(\rho) \le c(L+R)\Big(\frac{\rho}{R}\Big)^{\lambda}, \quad (\rho < R),$$

as desired. $\square$

§5. A regularity at the boundary for solutions

In this section, we are concerned with the continuity of solutions at the boundary.

First, we recall the definition of the (p,μ)-capacity which is adopted in [1]. Suppose that K is a compact subset of Ω. Let

$$W(K,\Omega) = \{u \in C_0^\infty(\Omega) \,:\, u \ge 1 \text{ on } K\}$$

and define

$$\mathrm{cap}_{p,\mu}(K,\Omega) = \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p d\mu.$$

Further, if $U \subset \Omega$ is open, set

$$\mathrm{cap}_{p,\mu}(U,\Omega) = \sup_{K \subset U \text{ compact}} \mathrm{cap}_{p,\mu}(K,\Omega),$$

and, finally, for an arbitary set $E \subset \Omega$

$$\mathrm{cap}_{p,\mu}(E,\Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \mathrm{cap}_{p,\mu}(U,\Omega).$$

The number $\mathrm{cap}_{p,\mu}(E,\Omega) \in [0,\infty]$ is called the *(p,μ)-capacity* of the *condenser* (E,Ω).

If $u \in H^{1,p}_{loc}(\Omega;\mu)$, $x_0 \in \partial\Omega$, and $l \in R$ we say that

$$u(x_0) \le l \quad \textit{weakly} \tag{25}$$

if for every $k > l$ there is an $r > 0$ such that $\eta(u-k)^+ \in H_0^{1,p}(\Omega;\mu)$ whenever $\eta \in C_0^\infty(B(x_0,r))$. The condition

$$u(x_0) \ge l \quad \textit{weakly} \tag{26}$$

is defined analogously and $u(x_0) = l$ *weakly* if both (25) and (26) hold. Observe that if f is a continuous function on $R^n \backslash \Omega$, $f \in H^{1,p}_{loc}(R^n;\mu)$, and $u - f \in H_0^{1,p}(\Omega;\mu)$, then $u(x) = f(x)$ weakly for every $x \in \partial\Omega$.

Lemma 5.1 *Suppose that $u \in H^{1,p}_{loc}(\Omega;\mu)$ is a subsolution of* (2) *in Ω, that $u \le L$ a.e. in Ω, and that $u(x_0) \le l$ weakly for $x_0 \in \partial\Omega$. For $k > l$, let*

$$u_k = \begin{cases} (u-k)^+ & \text{on } \Omega \\ 0 & \text{otherwise} \end{cases}$$

and define

$$M(r) = \operatorname*{ess\,sup}_{B(x_0,r)} u_k.$$

Choose $r_0 > 0$ so small that $\eta u_k \in H_0^{1,p}(\Omega;\mu)$ whenever $\eta \in C_0^\infty(B(x_0,r_0))$.

Then there is a constant c depending only on n, p, l, r_0, α_1, α_2, α_3, c_μ and L such that

$$\int_{B(x_0,r/2)} |\nabla(\eta v^{-1})|^p d\mu \le c\,(M(r)+r)(M(r)-M(r/2)+r)^{p-1}\mu(B(x_0,r))r^{-p}$$

where $0 < r \le r_0/2$, $v^{-1} = M(r) + r - u_k$ *and* $\eta \in C_0^\infty(B(x_0, r/2))$ *with* $0 \le \eta \le 1$ *and* $|\nabla\eta| \le 5/r$.

Before proving Lemma 5.1, we will state its implication.

Theorem 5.2 *Let* $u \in H^{1,p}_{loc}(\Omega;\mu)$ *be a subsolution of* (2) *which is bounded above on* Ω, $x_0 \in \partial\Omega$, *and* $u(x_0) \le l$ *weakly. If*

$$\int_0^1 \left(\frac{\mathrm{cap}_{p,\mu}(B(x_0,t)\backslash\Omega, B(x_0,2t))}{\mathrm{cap}_{p,\mu}(B(x_0,t), B(x_0,2t))}\right)^{1/(p-1)} \frac{dt}{t} = \infty, \tag{27}$$

then

$$\operatorname{ess\,lim\,sup}_{x\to x_0} u(x) \le l.$$

Proof : Since, for any $k > l$, it follows immediately from Theorem 5.1, the definition of (p,μ)-capacity and [1, Lemma 2.14] that

$$(M(r)+r)\left(\frac{\mathrm{cap}_{p,\mu}(B(x_0,r/4)\cap\{u_k=0\}, B(x_0,r/2))}{\mathrm{cap}_{p,\mu}(B(x_0,r/4), B(x_0,r/2))}\right)^{1/(p-1)}$$
$$\le c(M(r) - M(r/2) + r),$$

the thorem is proved in the same manner as in the proof of [4, Theorem 2.2]. □

If u is a supersolution of (2), then $-u$ is a subsolution of

$$-\mathrm{div}\bar{\mathcal{A}}(x,\nabla v) + \bar{\mathcal{B}}(x,v) = 0,$$

where $\bar{\mathcal{A}}(x,h) = -\mathcal{A}(x,-h)$ and $\bar{\mathcal{B}}(x,t) = -\mathcal{B}(x,-t)$. Consequently, Theorem 5.2 has the obvious counterpart for supersolutions of (2). These results yield

Theorem 5.3 *Let* $u \in H^{1,p}_{loc}(\Omega;\mu)$ *be a bounded solution of* (2), *that* $x_0 \in \partial\Omega$, *and that* $u(x_0) = l$ *weakly. If* (27) holds, then

$$\lim_{x\to x_0} u(x) = l.$$

Proof of Lemma 5.1 : Fix $r > 0$ so that $0 < r \le r_0/2$, let $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \le \eta \le 1$ and $|\nabla\eta| \le 5/r$. Set

$$I(r) = (M(r)+r)(M(r)-M(r/2)+r)^{p-1}\mu(B(x_0,r))r^{-p}.$$

Since

$$\int |\nabla(\eta v^{-1})|^p d\mu \le c\left(\int \eta^p |\nabla u_k|^p d\mu + \int v^{-p}|\nabla\eta|^p d\mu\right),$$

we will show that

$$\int \eta^p |\nabla u_k|^p d\mu \le c\, I(r) \quad \text{and} \quad \int v^{-p}|\nabla \eta|^p d\mu \le c\, I(r),$$

by using following two estimates.

Estimate 1 For $(1-p)/p < \alpha \neq 0$

$$(c\, m(\alpha))^{-1}\int_{B(x_0,r)} |\nabla(\omega v^{\alpha})|^p d\mu \le \int_{B(x_0,r)} v^{p\alpha}\{(\omega v)^p + |\nabla\omega|^p\} d\mu,$$

whenever $\omega \in C_0^\infty(B(x_0,r))$ with $0 \le \omega \le 1$, where c is a constant depending on p, α_1, α_2, α_3, l, r_0, and L, and

$$\begin{aligned} &0 < m(\alpha) < 1 + \alpha^p \quad \text{if } \alpha > 0, \\ &m(\alpha) > 0 \text{ and a decreasing function of } \alpha \quad \text{if } (1-p)/p < \alpha < 0. \end{aligned}$$

Estimate 2 For $0 < \sigma < p-1$,

$$\mu(B(x_0,r))^{-1}\|v^{-\sigma k}\|_{1,B(x_0,r/2)} \le c(M(r) - M(r/2) + r)^{\sigma k},$$

where c is a constant depending on p, n, α_1, α_2, α_3, l, r_0, L and σ.

Let us suppose that Estimate 1 and Estimate 2 are true. Fix $\alpha < 0$ such that $1 < (1+\alpha)p < k$, then putting $B = B(x_0, r/2)$, we have

$$\begin{aligned} \int_B \eta^{p-1}|\nabla u_k|^{p-1}|\nabla\eta| d\mu &= \int_B (\eta v^{1+\alpha}|\nabla u_k|)^{p-1}(v^{-(1+\alpha)(p-1)}|\nabla\eta|) d\mu \\ &= c\int_B (\eta|\nabla v^{\alpha}|)^{p-1}(v^{-(1+\alpha)(p-1)}|\nabla\eta|) d\mu \qquad (28)\\ &\le c\left(\int_B (\eta|\nabla v^{\alpha}|)^p d\mu\right)^{(p-1)/p}\left(\int_B (v^{-(1+\alpha)(p-1)}|\nabla\eta|)^p d\mu\right)^{1/p} \\ &\le c\left\{\left(\int_B |\nabla(\eta v^{\alpha})|^p d\mu\right)^{1/p} + \left(\int_B |v^{\alpha}\nabla\eta|^p d\mu\right)^{1/p}\right\}^{p-1} \\ &\qquad \times\left(\int_B (v^{-(1+\alpha)(p-1)}|\nabla\eta|)^p d\mu\right)^{1/p} \\ &\le c\left(r^{-p}\int_B v^{\alpha p} d\mu\right)^{(p-1)/p}\left(\int_B (v^{-(1+\alpha)(p-1)}|\nabla\eta|)^p d\mu\right)^{1/p} \\ &\le c\left\{(M(r) - M(r/2) + r)^{-\alpha p}\mu(B(x_0,r))r^{-p}\right\}^{(p-1)/p} \\ &\qquad \times\left\{(M(r) - M(r/2) + r)^{(1+\alpha)(p-1)p}\mu(B(x_0,r))r^{-p}\right\}^{1/p} \\ &= c\,(M(r) - M(r/2) + r)^{(p-1)}\mu(B(x_0,r))r^{-p}, \end{aligned}$$

in the last inequality we have used Estimate 2 with $\sigma = -\alpha p/k$ and $\sigma = (1+\alpha)(p-1)p/k$ respectively. Also since $\eta \le 1$,

$$\int_B \eta^p d\mu \le \mu(B(x_0,r)) \le c\, I(r). \qquad (29)$$

Hence, by (28) and (29),

$$\int_B \eta^p |\nabla u_k|^p d\mu \leq c\Big(\int_B \eta^p d\mu + M(r)\int_B \eta^{p-1}|\nabla u_k|^{p-1}|\nabla\eta| d\mu\Big) \leq c\, I(r). \tag{30}$$

Here the first inequality has been obtained by using the facts that $\varphi = \eta^p u_k \in H_0^{1,p}(\Omega;\mu)$, φ is nonnegative, u is a subsolution and the structure of $\mathcal{A}$ and $\mathcal{B}$. From Estimate 2 with $\sigma = (p-1)/k$ again

$$\int_B |v^{-1}\nabla\eta|^p d\mu \leq c\, r^{-p}\, (M(r)+r) \int_B v^{-p+1} d\mu \leq c\, I(r). \tag{31}$$

Therefore we obtain from (30) and (31)

$$\int_B |\nabla(\eta v^{-1})|^p d\mu \leq c\, I(r).$$

Finally, we will prove Estimate 1 and Estimate 2. For $\beta > 0$, let

$$\psi = v^\beta - (M(r)+r)^{-\beta}$$

and

$$\varphi = \omega^p \psi,$$

where $\omega \in C_0^\infty(B(x_0,r))$. Then $\varphi \in H_0^{1,p}(\Omega;\mu)$. Since $\varphi = 0$ on $\{u_k = 0\}$ and $\varphi \geq 0$ on Ω,

$$\int \beta\omega^p v^{\beta+1}\mathcal{A}(x,\nabla u)\cdot\nabla u_k dx + \int p\omega^{p-1}\psi\mathcal{A}(x,\nabla u)\cdot\nabla\omega dx + \int \mathcal{B}(x,u)\varphi dx \leq 0,$$

where the integrals are taken over $B(x_0,r)\cap\{u_k > 0\}$. Hereafter we will suppress explicit indication of this domain of integration.

Using (a2), (a3) and (b2) we have

$$\alpha_1\beta\int \omega^p v^{\beta+1}|\nabla u_k|^p d\mu \leq p\alpha_2 \int \omega^{p-1}\psi|\nabla u_k|^{p-1}|\nabla\omega| d\mu + \alpha_3\int \omega^p\psi(|u|^{p-1}+1)d\mu.$$

Since $\psi \leq v^\beta$, $v^{-1} \leq M(r_0)+r_0$ and $l \leq u \leq L$, we obtain

$$c^{-1}\beta\int \omega^p v^{\beta+1}|\nabla u_k|^p d\mu \leq \int \omega^{p-1}v^\beta|\nabla u_k|^{p-1}|\nabla\omega| d\mu + \int \omega^p v^{\beta+1} d\mu, \tag{32}$$

where c depends on $p, \alpha_1, \alpha_2, \alpha_3, r_0, L$. Application of Young's inequality yields that

$$\int \omega^{p-1}v^\beta|\nabla u_k|^{p-1}|\nabla\omega| d\mu \leq \varepsilon^{p/(p-1)}(p-1)p^{-1}\int \omega^p v^{\beta+1}|\nabla u_k|^p d\mu$$
$$+\varepsilon^{-p}p^{-1}\int v^{\beta-p+1}|\nabla\omega|^p d\mu,$$

for any $\varepsilon > 0$. By the above inequality and (32), with an appropriate choice for ε, we have

$$c^{-1}\beta\int \omega^p v^{\beta+1}|\nabla u_k|^p d\mu \leq \int \omega^p v^{\beta+1} d\mu + \beta^{1-p}\int v^{\beta-p+1}|\nabla\omega|^p d\mu. \tag{33}$$

By letting $\beta = p\alpha + p - 1$ with $0 < \beta \neq p-1$, , we obtain Estimate 1.
Next we prove Estimate 2. In (33) letting $\beta = p - 1$,

$$\int \omega^p |\nabla(\log v)|^p \leq c\Big\{(p-1)^{-1} \int \omega^p v^p d\mu + (p-1)^{-p} \int |\nabla \omega|^p d\mu\Big\}.$$

Since, by using $v \leq 1/r$ and Sobolev inequality,

$$\int \omega^p v^p d\mu \leq r^{-p} \mu(B(x_0, r))^{(k-1)/k} \Big(\int \omega^{pk} d\mu\Big)^{1/k} \leq c \int |\nabla \omega|^p d\mu,$$

we have

$$\int \omega^p |\nabla(\log v)|^p \leq c \int |\nabla \omega|^p d\mu$$

whenever $0 \leq \omega \in C_0^\infty(B(x_0, r))$. Using Lemma 4.4(John-Nirenberg lemma) in the same manner as in the proof of Lemma 4.5 and Theorem 4.2, it follows that there are positive constants c and σ_0 such that

$$\int_{B(x_0,s)} v^{-\sigma} d\mu \int_{B(x_0,s)} v^{\sigma} d\mu \leq c\Big\{\mu(B(x_0, s))\Big\}^2, \tag{34}$$

whenever $\sigma \leq \sigma_0$ and $0 < s \leq 3r/4$.
Let $0 < s < t \leq r$ and let a function $\omega \in C_0^\infty(B(x_0, t))$ be chosen such that $0 \leq \omega \leq 1$, $\omega = 1$ on $B(x_0, s)$ and $|\nabla \omega| \leq 2(t-s)^{-1}$. Then $(\omega v)^p \leq v^p \leq r^{-p} \leq 2(t-s)^{-p}$. Hence, from Sobolev inequality and Estimate 1,

$$\Big(\int_{B(x_0,s)} |v^\alpha|^{kp} d\mu\Big)^{1/k} \leq c\, m(\alpha) \{\mu(B(x_0, r))\}^{(1-k)/k} r^p (t-s)^{-p} \int_{B(x_0,t)} v^{p\alpha} d\mu, \tag{35}$$

whenever $0 < s < t \leq r$ and $(1-p)p^{-1} < \alpha \neq 0$.
Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \cdots$. Then since $m(\alpha_0 k^j) \leq c\, (k^p)^j$ for $0 < \alpha_0 \leq \sigma_0 p^{-1}$, (35) yields that

$$\Big(\int_{B(x_0,r_{j+1})} |v^{\alpha_0 k^j}|^{kp} d\mu\Big)^{1/k} \leq c\, (k^p)^j \{\mu(B(x_0, r))\}^{(1-k)/k} (2^p)^j \int_{B(x_0,r_j)} v^{p\alpha_0 k^j} d\mu,$$

and hence

$$\|v^{p\alpha_0}\|_{k^{j+1},B(x_0,r_{j+1})} \leq \Big\{c\, \{\mu(B(x_0, r))\}^{(1-k)/k}\Big\}^{k^{-j}} (2^p k^p)^{jk^{-j}} \|v^{p\alpha_0}\|_{k^j,B(x_0,r_j)}$$

for $j = 0, 1, \cdots$. Hereafter, for simplicity, we shall write $\|\cdot\|_{p,r}$ for $\|\cdot\|_{p,B(x_0,r)}$. By iterating, we have

$$(M(r) - M(r/2) + r)^{-p\alpha_0} \leq c\, \{\mu(B(x_0, r))\}^{-1} \|v^{p\alpha_0}\|_{1,3r/4}, \tag{36}$$

whenever $0 < p\alpha_0 \leq \sigma_0$. From (34) and (36), we obtain that

$$\mu(B(x_0, r))^{-1} \|v^{-p\alpha_0}\|_{1,3r/4} \leq c\, (M(r) - M(r/2) + r)^{p\alpha_0} \tag{37}$$

whenever $0 < p\alpha_0 \leq \sigma_0$.
Return to (35) with $1 - p < p\alpha < 0$. Let $0 < \sigma < p - 1$ and let j_0 is a positive integer such that $p - 1 \leq \sigma_0 k^{j_0}$. Put $\sigma_1 = \sigma k^{-j_0}$. Since $0 < \sigma_1 k^j \leq \sigma < p - 1$ for $0 \leq j \leq j_0$, $m(-\sigma_1 k^j p^{-1}) \leq m(-\sigma p^{-1})$ for $0 \leq j \leq j_0$.

Let $r_j = (r/4)\{3 - j/(j_0+1)\}$ for $0 \le j \le j_0 + 1$. Then (35) yields that

$$\|v^{-\sigma_1}\|_{k^{j+1},r_{j+1}} \le \left[c\, m(-\sigma p^{-1})\{\mu(B(x_0,r))\}^{(1-k)/k}\{4(j_0+1)\}^p\right]^{k^{-j}} \|v^{-\sigma_1}\|_{k^j,r_j}.$$

By iterating for $0 \le j \le j_0$, we have

$$\mu(B(x_0,r))^{-1}\|v^{-\sigma_1}\|^{k^{j_0+1}}_{k^{j_0+1},r/2} \le \left[c\, m(-\sigma p^{-1})\{4(j_0+1)\}^p\right]^{\frac{k(k^{j_0+1}-1)}{k-1}} \times\left[\{\mu(B(x_0,r))\}^{-1}\|v^{-\sigma_1}\|_{1,3r/4}\right]^{k^{j_0+1}}.$$

Since $0 < \sigma_1 < \sigma_0$, from (37) we obtain Estimate 2.

Hence Lemma 5.1 follows. □

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