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A Noncooperative Analysis of Spatial Duopoly with Discriminatory Pricing

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Abstract

In this paper we extend the location problem of spatial competition given by Hotelling[2] to include the discriminatory pricing . We characterize various types of asymmetric equilibrium , in which the two players choose different location strategies.

1 Introduction

The first important contribution to the study of Location Problem of Spatial Competition was made by Hotelling[2]. He studied duopolistic competition in a liner market and claimed existence of a Nash equilibrium in prices. However his model was re-examined by d'Aspremont,Gabszewicz & Thisse[3]. They point to a flaw in the original paper of Hotelling because a pure strategy equilibrium in prices(given the 'location') does not always exist. Recently, Helmut Bester[1] show that Hotelling's model with quadratic consumer transportation cost possesses an infinity of equilibrium in which the duopolists randomize over locations.In our paper we consider a duopoly in which each firm selects a location simultaneously and then chooses to price discriminatory in a linear market where consumers are uniformly distributed. We can think of this model as a two-stage non-cooperative game between the two firms. We present a price equilibrium and characterize the location equilibrium. The paper is organized as follows. In section 2 we describe the model precisely, in section 3 we discuss our results on the location equilibrium.

2 The model

What is discriminatory price ? Discriminatory Price is a price which a firm can quote each customer a different mill price. In our paper, we assume that the firms set discriminatory price. The model is described as follows.

Let two firms, denoted A and B, be located on line segment $[0,1]$. The locations of the firms are denoted x and y $(0 \leq x \leq 1, 0 \leq y \leq 1)$ respectively. They sell the homogeneous product with zero cost(the identical production cost can be normalized to zero)and the customers are assumed to be uniformly distributed over the segment . Each customer purchases one unit of the product. Since the product is homogeneous, a customer will buy from the firm for which full price(mill price plus transportation cost) is lowest. It is supposed that the transport is under the customer's control, and the transportation cost are assumed to be linear in distance with coefficients t. We denote the transportation cost function by $C(d)$ (d is a Euclidean distance from customer's location to firm's location), i.e. $C(d) = td, C(0) = 0$. If the full prices are the same, the customer buys from the firm located closer to him, and if both prices and distances are equal , he selects the firm to buy randomly with probability $\frac{1}{2}$. We can think this model as a two-stage non-cooperative game. In the first stage, two firms select location (x, y) simultaneously, and then having observed the locations selected, choose discriminatory price (p_A, p_B) simultaneously. The customers choose according to the criterion above, and the firms receive their profits. We assume $p_A \in [0, \infty)$, $p_B \in [0, \infty)$.

Let $p_A(z), p_B(z)$ be the discriminatory price charged by the firm A, B respectively for the location z's customer. Let C_A, C_B be the region served by firm A, B respectively, then

$$\begin{aligned} C_A &= \{z \in [0,1]: p_A(z) + t|z-x| \leq p_B(z) + t|z-y| \ and \ |z-x| \leq |z-y|\}; \\ C_B &= \{z \in [0,1]: p_B(z) + t|z-y| \leq p_A(z) + t|z-x| \ and \ |z-y| \leq |z-x|\}. \end{aligned}$$

The payoff functions are :

$$\pi_A(p_A, p_B, x, y) = \int_{C_A} p_A(z) f(z) dz, \quad \pi_B(p_A, p_B, x, y) = \int_{C_B} p_B(z) f(z) dz. \tag{2.1}$$

where $f(z) = 1, z \in [0, 1]$.

At the first , for any given location (x, y), we present the price equilibrium pair $(p_A^*(z), p_B^*(z))$ for the location z's customer, to capture the location z's customer, firm A has to charge a price $p_A(z)$ satisfying

$$p_A(z) + t|z-x| \leq p_B(z) + t|z-y|, if x \leq y.$$

i.e. $p_A(z) \leq p_B(z) + t|z-y| - t|z-x|$. However, to capture the same customer, firm B will cutdown his price

$$p_B(z) \to 0, \quad p_A(z) \leq t|z-y| - t|z-x|.$$

Hence, $p_A^*(z) = max\{t|z-y| - t|z-x|, 0\}$.i.e.

$$p_A^*(z) = \begin{cases} t(y-x), & if \ \ 0 \leq z \leq x \\ t(y+x-2z), & if \ \ x \leq z \leq \dfrac{x+y}{2}; \\ 0, & o.w \end{cases} \tag{2.2}$$

similarly, $p_B^*(z) = max\{t|z-x| - t|z-y|, 0\}$. i.e.

$$p_B^*(z) = \begin{cases} t(y-x), & if \ \ y \leq z \leq 1 \\ t(2z-y-x), & if \ \ \dfrac{x+y}{2} \leq z \leq y. \\ 0, & o.w \end{cases} \tag{2.3}$$

If $x \geq y$, by symmetry we get the price equilibrium:

$$p_A^*(z) = \begin{cases} t(x-y), & if \ \ x \leq z \leq 1 \\ t(2z-x-y), & if \ \ \dfrac{x+y}{2} \leq z \leq x; \\ 0, & o.w \end{cases} \tag{2.4}$$

and

$$p_B^*(z) = \begin{cases} t(x-y), & if \ \ 0 \le z \le y \\ t(y+x-2z), & if \ \ y \le z \le \dfrac{x+y}{2}. \\ 0, & o.w \end{cases} \tag{2.5}$$

Proposition 1 : (p_A^*, p_B^*) is a pair of Nash equilibrium. i.e

$$\begin{cases} \pi_A(p_A^*, p_B^*, x, y) \ge \pi_A(p_A, p_B^*, x, y), & for \ \forall p_A \in [0,\infty); \\ \pi_B(p_A^*, p_B^*, x, y) \ge \pi_B(p_A^*, p_B, x, y), & for \ \forall p_B \in [0,\infty). \end{cases}$$

Proof: Without loss of generality, we only show:

$$\pi_A(p_A^*, p_B^*, x, y) \ge \pi_A(p_A, p_B^*, x, y), \qquad for \ \forall p_A \in [0,\infty), if x \le y.$$

Let

$$\begin{aligned} C_A^* &= \{z \in [0,1]: p_A^*(z) + t|z-x| \le p_B^*(z) + t|z-y| and |z-x| \le |z-y|\}; \\ C_A^{'} &= \{z \in [0,1]: p_A(z) + t|z-x| \le p_B^*(z) + t|z-y| and |z-x| \le |z-y|\}. \end{aligned}$$

Obviously, $C_A^* = C_A^{'}$.

$$\begin{aligned} \pi_A(p_A, p_B^*, x, y) &= \int_{C_A^{'}} p_A(z) f(z) dz \\ &\le \int_{C_A^{'}} [p_B^* + t|z-y| - t|z-x|] f(z) dz \\ &= \int_{C_A^{'}} [t|z-y| - t|z-x|] f(z) dz \\ &= \int_{C_A^*} p_A^*(z) f(z) dz \\ &= \pi_A(p_A^*, p_B^*, x, y). \end{aligned}$$

Q.E.D.

The payoffs under prices equilibrium for firms are :

$$\pi_A(p_A^*, p_B^*, x, y) = \frac{1}{4} t(y-x)(y+3x), \quad if x \le y, \tag{2.6}$$

$$\pi_A(p_A^*, p_B^*, x, y) = \frac{1}{4} t(x-y)(4-y-3x), if x \ge y \tag{2.7}$$

for firm A, and

$$\pi_B(p_A^*, p_B^*, x, y) = \frac{1}{4} t(x-y)(x+3y), \quad if x \ge y, \tag{2.8}$$

$$\pi_B(p_A^*, p_B^*, x, y) = \frac{1}{4} t(y-x)(4-x-3y), if x \le y \tag{2.9}$$

for firm B.
Define

$$\pi_A(x,y) \equiv \pi_A(p_A^*, p_B^*, x, y), \tag{2.10}$$

$$\pi_B(x,y) \equiv \pi_B(p_A^*, p_B^*, x, y). \tag{2.11}$$

Notice that payoffs are symmetric in the sense that

$$\pi_A(x,y) = \pi_B(1-x, 1-y), \tag{2.12}$$

$$\pi_A(x, 1-x) = \pi_B(x, 1-x). \tag{2.13}$$

3 Characterizations of the location equilibrium

In this section we investigate the location equilibrium of firms under price equilibrium.
Proposition 2 : In the above game two pure strategy location equilibria exist. They are $(x^*, y^*) = (\frac{1}{4}, \frac{3}{4})$ and$(x^*, y^*) = (\frac{3}{4}, \frac{1}{4})$.

Proof: Consider all (x, y) such that $x \leq y$ then

$$\begin{cases} \frac{\partial \pi_A}{\partial x} & = \frac{1}{2}t(y - 3x) = 0 \\ \frac{\partial \pi_B}{\partial y} & = \frac{1}{2}t(2 + x - 3y) = 0; \end{cases} \Rightarrow \begin{cases} x & = \frac{1}{4} \\ y & = \frac{3}{4} \end{cases} \tag{3.1}$$

and

$$\begin{cases} \frac{\partial^2 \pi_A}{\partial x^2} & = -\frac{3}{2}t < 0 \\ \frac{\partial^2 \pi_B}{\partial y^2} & = -\frac{3}{2}t < 0. \end{cases} \tag{3.2}$$

Therefore there is exactly one equilibrium such that $x^* \leq y^*$, namely $(x^*, y^*) = (\frac{1}{4}, \frac{3}{4})$. By symmetry of payoffs, there is exactly one equilibrium such that $x^* \geq y^*$, namely $(x^*, y^*) = (\frac{3}{4}, \frac{1}{4})$. Q.E.D.

Proposition 2 : In the above game there is a mixed equilibrium strategy in which firm B chooses $y^* = \frac{1}{6}$ with probability $\frac{1}{2}$ and $y^* = \frac{5}{6}$ with probability $\frac{1}{2}$ and firm A chooses $x^* = \frac{1}{2}$ with probability 1 . Symmetrically there is an equilibrium in which firm A chooses $x^* = \frac{1}{6}$ with probability $\frac{1}{2}$ and $x^* = \frac{5}{6}$ with probability $\frac{1}{2}$ and firm B chooses $y^* = \frac{1}{2}$ with probability 1 .

Proof: To prove the first part, we first show that , given the behaviors of firm B, firm A cannot gain by deviating from $x^* = \frac{1}{2}$. Indeed, firm A's payoff function is

$$\phi(x) \equiv \frac{1}{2}\pi_A(x, \frac{1}{6}) + \frac{1}{2}\pi_A(x, \frac{5}{6}). \tag{3.3}$$

We will prove $x^* = \frac{1}{2}$ is a global maximizer for $\phi(x)$ subject to $x \in [0, 1]$.
Case 1: $x \in [0, \frac{1}{6}]$

$$\begin{aligned} \phi(x) & \equiv \frac{1}{2}\pi_A(x, \frac{1}{6}) + \frac{1}{2}\pi_A(x, \frac{5}{6}) \\ & = \frac{1}{8}t(\frac{1}{6} - x)(\frac{1}{6} + 3x) + \frac{1}{8}t(\frac{5}{6} - x)(\frac{5}{6} + 3x) \\ & = \frac{1}{4}t(\frac{13}{36} + x - 3x^2). \\ \phi'(x) & = \frac{1}{4}t(1 - 6x) \Rightarrow x = \frac{1}{6}. \\ \phi''(x) & = -\frac{3}{2}t < 0, \phi(\frac{1}{6}) = \frac{1}{9}t. \end{aligned}$$

Case 2: $x \in [\frac{1}{6}, \frac{5}{6}]$

$$\begin{aligned}\phi(x) &\equiv \frac{1}{2}\pi_A(x, \frac{1}{6}) + \frac{1}{2}\pi_A(x, \frac{5}{6}) \\ &= \frac{1}{8}t(x - \frac{1}{6})(4 - \frac{1}{6} - 3x) + \frac{1}{8}t(\frac{5}{6} - x)(\frac{5}{6} + 3x) \\ &= \frac{1}{4}t(\frac{1}{36} + 3x - 3x^2). \\ \phi'(x) &= \frac{1}{4}t(3 - 6x) \Rightarrow x = \frac{1}{2}. \\ \phi''(x) &= -\frac{3}{2}t < 0, \phi(\frac{1}{2}) = \frac{7}{36}t.\end{aligned}$$

Case 3: $x \in [\frac{5}{6}, 1]$

$$\begin{aligned}\phi(x) &\equiv \frac{1}{2}\pi_A(x, \frac{1}{6}) + \frac{1}{2}\pi_A(x, \frac{5}{6}) \\ &= \frac{1}{8}t(x - \frac{1}{6})(4 - \frac{1}{6} - 3x) + \frac{1}{8}t(x - \frac{5}{6})(4 - \frac{5}{6} - 3x) \\ &= \frac{1}{4}t(-\frac{59}{36} + 5x - 3x^2). \\ \phi'(x) &= \frac{1}{4}t(5 - 6x) \Rightarrow x = \frac{5}{6}. \\ \phi''(x) &= -\frac{3}{2}t < 0, \phi(\frac{5}{6}) = \frac{1}{9}t.\end{aligned}$$

So $x^* = \frac{1}{2}$ is an optimal response of firm A to firm B's strategy.

Conversely, against firm A's strategy , firm B will select $y^* = \frac{1}{6}$ or $y^* = \frac{5}{6}$. In fact , firm A's payoff function is

$$\psi(y) \equiv \pi_B(\frac{1}{2}, y). \tag{3.4}$$

Case 1: $y \in [0, \frac{1}{2}]$

$$\begin{aligned}\psi(y) &\equiv \pi_B(\frac{1}{2}, y) \\ &= \frac{1}{4}t(\frac{1}{2} - y)(\frac{1}{2} + 3y) \\ &= \frac{1}{4}t(\frac{1}{4} + y - 3y^2). \\ \psi'(y) &= \frac{1}{4}t(1 - 6y) \Rightarrow y = \frac{1}{6}. \\ \psi''(y) &= -\frac{3}{2}t < 0, \psi(\frac{1}{6}) = \frac{1}{12}t.\end{aligned}$$

Case 2: $y \in [\frac{1}{2}, 1]$

$$\psi(y) \equiv \pi_B(\frac{1}{2}, y)$$

$$
\begin{aligned}
&= \frac{1}{4}t(y-\frac{1}{2})(4-\frac{1}{2}-3y) \\
&= \frac{1}{4}t(-\frac{7}{4}+5y-3y^2). \\
\psi'(y) &= \frac{1}{4}t(5-6y) \Rightarrow y=\frac{5}{6}. \\
\psi''(y) &= -\frac{3}{2}t<0, \psi(\frac{5}{6})=\frac{1}{12}t.
\end{aligned}
$$

As $\psi(\frac{1}{6}) = \psi(\frac{5}{6})$, this implies that both $y^* = \frac{1}{6}$ and $y^* = \frac{5}{6}$ maximize firm B's payoff. This proves that randomizing over $y^* = \frac{1}{6}$ and $y^* = \frac{5}{6}$ is an optimal response of firm B to firm A's strategy.

The second part of the proposition follows by symmetry. Q.E.D.

Proposition 4 : In the above game there is a pair of location vectors $(\xi, v) = ((x_1, ..., x_n), (y_1, ..., y_{n-1}))$ and a pair of probability vectors $(\alpha, \beta) = ((\alpha_1, ..., \alpha_n), (\beta_1, ..., \beta_{n-1}))$ such that firm A choosing x_i with probability $\alpha_i > 0$ and firm B choosing y_i with probability β_i is an equilibrium for any number $n \geq 2$. Moreover, $x_i < y_i < x_{i+1}$ for all $i = 1, ..., n-1$.

Proof: Define $z \in \mathbf{R}^{2n-1}$ by $Z \equiv \{\xi, v | 0 \leq x_i \leq y_i \leq x_{i+1} \leq 1 \text{ for all } i = 1, ..., n-1\}$. Obviously, Z is convex and compact. Let

$$\varphi_A(x, v, \beta) \equiv \sum_j \pi_A(x, y_i)\beta_j, \tag{3.5}$$

$$\varphi_B(y, \xi, \alpha) \equiv \sum_j \pi_B(x_j, y)\alpha_j. \tag{3.6}$$

Note that $\dfrac{\partial^2 \pi_A(x,y)}{\partial^2 x} = -\dfrac{3}{2}t < 0$ for $x > y$ and $x < y$. Therefore $\varphi_A(\bullet, v, \beta)$ is a strictly concave function of x for all $x \in (y_{i-1}, y_i)$, where $y_0 \equiv 0$ and $y_n \equiv 1$. This , together with the maximum theorem , implies that

$$f_{Ai}(v, \beta) \equiv arg \ \ max_{x \in [y_{i-1}, y_i]} \varphi_A(x, v, \beta) \tag{3.7}$$

is a continuous function of (v, β). Similarly,

$$f_{Bi}(\xi, \alpha) \equiv arg \ \ max_{y \in [x_i, x_{i+1}]} \varphi_B(y, \xi, \alpha) \tag{3.8}$$

is a continuous function of (ξ, α).
Define $f_A(\bullet) \equiv [f_{A1}(\bullet), ..., f_{An}(\bullet)]$ and $f_B(\bullet) \equiv [f_{B1}(\bullet), ..., f_{Bn-1}(\bullet)]$.
Define $S_A \equiv \{\alpha \in R^n | \sum_i \alpha_i = 1\}$ and $S_B \equiv \{\beta \in R^{n-1} | \sum_i \beta_i = 1\}$. Then

$$g_A(\xi, v, \beta) \equiv arg \ \ min_{\alpha \in S_A} \sum_i \alpha_i \sum_j \pi_A(x_i, y_j)\beta_j, \tag{3.9}$$

$$g_B(\xi, v, \alpha) \equiv arg \ \ min_{\beta \in S_B} \sum_i \beta_i \sum_j \pi_B(x_j, y_i)\alpha_j \tag{3.10}$$

are convex valued, upperhemicontinuous correspondences. As a result, the correspondences $h(\xi, v, \alpha, \beta) \equiv f_A(v, \beta) \times f_B(\xi, \alpha) \times g_A(\xi, v, \beta) \times g_B(\xi, v, \alpha)$ maps $Z \times S_A \times S_B$ into itself. Also, it is convex valued and upperhemicontinuous so that by Kakutani's theorem it has a fixed point $(\xi^*, v^*, \alpha^*, \beta^*)$. We will prove that the point $(\xi^*, v^*, \alpha^*, \beta^*)$ satisfies the conditions of Proposition 4.

First we show that $\varphi_A(x_i^*, v^*, \beta^*) = \varphi_A(x_{i+1}^*, v^*, \beta^*)$ for all $i = 1, ..., n-1$. Suppose $\varphi_A(x_i^*, v^*, \beta^*) \neq \varphi_A(x_{i+1}^*, v^*, \beta^*)$. Note that by definition of $g_A(\bullet)$ one has $\alpha_i^* = 0$ for all i such that $\varphi_A(x_i^*, v^*, \beta^*) > min_j \varphi_A(x_j^*, v^*, \beta^*)$. Suppose there is a $k > 1$ such that $\varphi_A(x_i^*, v^*, \beta^*) > min_j \varphi_A(x_j^*, v^*, \beta^*)$ for all $i < k$ and $\varphi_A(x_k^*, v^*, \beta^*) = min_j \varphi_A(x_j^*, v^*, \beta^*)$. Then $\varphi_B(y, \xi^*, \alpha^*)$ is strictly decreasing over $[x_1^*, x_k^*]$ because $\alpha_i^* = 0$ for all $i < k$. Accordingly, by definition of $f_{Bi}(\bullet)$ one has $y_i^* = x_i^*$ for all $i < k$. Therefore, by definition of $f_{Ak}(\bullet)$, x_k^* must maximize $\varphi_A(x, v^*, \beta^*)$ subject to $x_{k-1}^* \leq x \leq y_k^*$. As $\varphi_A(x, v^*, \beta^*)$ is strictly concave over $[x_{k-1}^*, y_k^*]$ this yields a contradiction $\varphi_A(x_{k-1}^*, v^*, \beta^*) > \varphi_A(x_k^*, v^*, \beta^*)$. The same argument shows that there cannot be a $k < n$ such that $\varphi_A(x_i^*, v^*, \beta^*) > min_j \varphi_A(x_j^*, v^*, \beta^*)$ for all $i < k$.

Suppose there is a k and an l such that $k < l-1$ and $\varphi_A(x_i^*, v^*, \beta^*) > min_j \varphi_A(x_j^*, v^*, \beta^*)$ for all $k < i < l$, and $\varphi_A(x_k^*, v^*, \beta^*) = \varphi_A(x_l^*, v^*, \beta^*) = min_j \varphi_A(x_j^*, v^*, \beta^*)$. Then $\varphi_B(y, \xi^*, \alpha^*)$ is strictly concave over $[x_k^*, x_l^*]$ and so one has $y_k^* = x_{k+1}^*$ and /or $y_{l-1}^* = x_{l-1}^*$. In the first case, x_k^* must maximize $\varphi_A(x, v^*, \beta^*)$ subject to $y_{k-1}^* \leq x \leq x_{k+1}^*$. But then $\varphi_A(x_{k+1}^*, v^*, \beta^*) > \varphi_A(x_k^*, v^*, \beta^*)$ leads to a contradiction because $\varphi_A(\bullet, v^*, \beta^*)$ is strictly concave over $[y_{k-1}^*, x_{k+1}^*]$. In the second case, a similar argument yields a contradiction. This proves $\varphi_A(x_i^*, v^*, \beta^*) = \varphi_A(x_{i+1}^*, v^*, \beta^*)$ for all $i = 1, ..., n-1$. The same argument as above can be used to show that $\varphi_B(y_i^*, \xi^*, \alpha^*) = \varphi_B(y_{i+1}^*, \xi^*, \alpha^*)$ for all $i = 1, ..., n-1$.

Next we will show that $x_i^* < y_i^* < x_{i+1}^*$ for all $i = 1, ..., n-1$. Clearly, one cannot have $x_1^* = x_n^*$ because , otherwise, decreasing x_1 or increasing $x_n$would increase firm A's payoff $\varphi_A(x, v^*, \beta^*)$. Suppose there is a k such that $x_k^* = y_k^* < x_{k+1}^*$. Then x_{k+1}^* must maximize $\varphi_A(x, v^*, \beta^*)$ subject to $x_k^* \leq x \leq y_{k+1}^*$. As $\varphi_A(x, v^*, \beta^*)$ is strictly concave over $[x_k^*, y_{k+1}^*]$ this leads to a contradiction to $\varphi_A(x_k^*, v^*, \beta^*) = \varphi_A(x_{k+1}^*, v^*, \beta^*)$. By the same argument one can rule out that $x_k^* < y_k^* = x_{k+1}^*$ for some k . Finally, $y_{k-1}^* = x_k^* < y_k^*$ or $y_k^* < x_{k+1}^* = y_{k+1}^*$ would contradict that y_k^* must maximize $\varphi_B(y, \xi^*, \alpha^*)$ subject to $x_k^* \leq y \leq x_{k+1}^*$ because $\varphi_B(y, \xi^*, \alpha^*)$ is strictly concave over $[x_k^*, x_{k+1}^*]$ and $\varphi_B(y_i^*, \xi^*, \alpha^*) = \varphi_B(y_{i+1}^*, \xi^*, \alpha^*)$ for all $i = 1, ..., n-1$.

Last we will show that $\alpha_i^* > 0$ and $\beta_i^* > 0$ for all i . Suppose $\alpha_i^* = 0$. Then $\varphi_B(y, \xi^*, \alpha^*)$ is strictly decreasing over $[x_1^*, x_2^*]$ and so $y_1^* = x_1^*]$, this leads to a contradiction to our above result that $x_i^* < y_i^*]$ for all $i = 1, ..., n-1$. Similarly, $\alpha_i^* = 0$ is impossible. Suppose there is a k and an l such that $k < l-1$ and $\alpha_k^* > 0$, $\alpha_l^* > 0$, and $\alpha_i^* = 0$ for all $k < i < l$. Then $\varphi_B(y, \xi^*, \alpha^*)$ is strictly concave over $[x_k^*, x_l^*]$ and so $y_k^* = x_{k+1}^*$ and/or $y_{l-1}^* = x_{l-1}^*$. This again contradicts our above result. The same argument proves that $\beta_i^* > 0$ for all $i = 1, ..., n-1$. Q.E.D.

By symmetry of payoffs we can show that there also is an equilibrium in which firm A randomizes over $n-1$ and firm B over n locations. Moreover, the same arguments as in the proof Proposition 4 can be used to prove the next Proposition which both firms randomly select one of n locations.

Proposition 5 : In the above game there is a pair of location vectors $(\xi, v) = ((x_1, ..., x_n), (y_1, ..., y_n))$ and a pair of probability vectors $(\alpha, \beta) = ((\alpha_1, ..., \alpha_n), (\beta_1, ..., \beta_n))$ such that firm A choosing x_i with probability $\alpha_i > 0$ and firm B choosing y_i with probability β_i is an equilibrium for any number $n \geq 2$. Moreover, $x_i < y_i < x_{i+1} < y_{i+1}$ for all $i = 1, ..., n-1$.

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