Title	CONSTRUCTING LOW-DISCREPANCY SEQUENCES BY USING $ \beta$-ADIC TRANSFORMATIONS
Author(s)	NINOMIYA, SYOITI
Citation	数理解析研究所講究録 (1997), 1011: 64-76
Issue Date	1997-08
URL	http://hdl.handle.net/2433/61526
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1011 巻 1997 年 64-76

CONSTRUCTING LOW-DISCREPANCY SEQUENCES BY USING β-ADIC TRANSFORMATIONS

SYOITI NINOMIYA

IBM Research, Tokyo Research Laboratory

Abstract. A new class of low-discrepancy sequences is constructed by the use of β-adic transformations. Here, β is a real number greater than 1. When β is an integer greater than 2, this sequence becomes the generalized van der Corput sequence in base β. It is also shown that for some special β, the discrepancy of this sequence decreases in the fastest order.

0. Introduction and background

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points ([Niederreiter 1]). A sequence $x_1, x_2, \dots$ in the s-dimensional unit cube $I^s = \prod_{i=1}^{s}[0,1)$ is said to be uniformly distributed in I^s when

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_J(x_n) = \lambda_s(J)$$

holds for all subintervals $J \in I^s$, where c_J is the characteristic function of J, and λ_s is the s-dimensional Lebesgue measure. If $x_1, x_2, \dots \in I^s$ is a uniformly distributed sequence, the formula

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{I^s} f(x)\, dx \tag{0.1}$$

holds for any Riemann integrable function on I^s. The discrepancy of the point set $P = \{x_1, x_2, \dots, x_N\}$ in I^s is defined as follows:

$$D_N(\mathcal{B}; P) = \sup_{B\in\mathcal{B}} \left| \frac{A(B;P)}{N} - \lambda_s(B) \right| \tag{0.2}$$

where $\mathcal{B} \subset \wp(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B;P)$ is the counting function that indicates the number of n, where $1 \le n \le N$, for which $x_n \in B$. When $\mathcal{J}^* = \{\prod_{i=1}^{s}[0,u_i), 0 \le u_i < 1\}$,the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(\mathcal{J}^*; P)$. When $S = \{x_1, x_2, \dots\}$ is a sequence in I^s, we

Key words and phrases. β-adic transformation, discrepancy, ergodic theory, numerical integration, van der Corput sequence.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX define $D_N^*(S)$ as $D_N^*(S_N)$, where S_N is the point set $\{x_1, x_2, \ldots, x_N\}$. Let S be a sequence in I^s. It is known that the following two conditions are equivalent:

(a) S is uniformly distributed in I^s;

(b) $\lim_{N\to\infty} D_N^*(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy.

Theorem 0.1 (Koksma-Hlawka)[1]. *If f has bounded variation $V(f)$ on $\bar{I}^s$ in the sense of Hardy and Krause, then for any $x_1, x_2, \ldots, x_N \in I^s$, we have*

$$\left| \frac{1}{N}\sum_{n=1}^{M} f(x_n) - \int_{I^s} f(x)\,dx \right| \leq V(f) D_N^*(x_1, \ldots, x_N).$$

Schmidt [4] showed that, when $s = 1, 2$, there exists a positive constant C that depends only on s, and the following inequality holds for an arbitrary point set P consisting of N elements:

$$D_N^*(P) \geq C\frac{(\log N)^{s-1}}{N}. \tag{0.3}$$

If (0.3) holds, then there exists a positive constant C that depends only on s, and any sequence $S \subset I^s$ satisfies

$$D_N^*(S) \geq C\frac{(\log N)^{s}}{N} \tag{0.4}$$

for infinitely many N. Taking account of (0.3) and (0.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

Definition 0.1. Let S be an one-dimensional sequence in $[0, 1)$. If S satisfies

$$\overline{\lim_{N\to\infty}} \frac{N D_N^*(S)}{\log N} = C \text{ (const)},$$

then S is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [1].

Definition 0.2. Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n) p^j, \qquad a_j(n) \in \{0, 1, \ldots, p-1\} \text{ for all } j \geq 0,$$

in base p. Then, the radical-inverse function ϕ_p is defined by

$$\phi_p(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n)) p^{-j-1} \qquad \text{for all integers} \qquad n \geq 0,$$

where τ_j is a permutation of $\{0, 1, \ldots, p-1\}$. The van der Corput sequence in base p is the sequence $V_p = \{\phi_p(n)\}_{n=0}^{\infty} \subset [0, 1)$.

Theorem 0.2 [1]. *For an arbitrary integer $p \geq 2$, V_p is a low-discrepancy sequence.*

In the following part of this paper, the author defines a class of sequences by the use of β-adic transformation ([Rény 3], [Parry 2]) and shows that any member of this class is a low-discrepancy sequence when $\beta = (L + \sqrt{L^2 + 4K})/2$, where L and K are integers greater than 1 and satisfy $K \leq L$. When β is an integer greater than 2, the sequence becomes V_β.

1. β-adic transformation

In this section we define the fibred system and the β-adic transformation, following [Schweiger 5] and [Takahashi 6].

R, **Z**, and **N** are the sets of all real numbers ,all integers, and all natural numbers, respectively. For $x \in \mathbf{R}$, $[x]$ denotes the integer part of x.

Definition 1.1. Let B be a set and $T : B \to B$ be a map. The pair (B, T) is called a fibred system if the following conditions are satisfied:

(a) There is a finite countable set A.

(b) There is a map $k : B \to A$, and the sets

$$B(i) = k^{-1}(\{i\}) = \{x \in B : k(x) = i\}$$

form a partition of B.

(c) For an arbitrary $i \in A$, $T|_{B(i)}$ is injective.

Definition 1.2. Let $\Omega = A^{\mathbf{N}}$ and $\sigma : \Omega \to \Omega$ be the one-sided shift operator. Let $k_j(x) = k(T^{j-1}x)$. We derive a canonical map $\varphi : B \to \Omega$ from

$$\varphi(x) = (k_j(x))_{n=1}^{\infty}.$$

φ is called the representation map.

We have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{T} & B \\ \varphi \downarrow & & \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

Definition 1.3. If a representation map φ is injective, φ is called a valid representation.

Definition 1.4. Let $\omega \in \Omega$. If $\omega \in \text{Im}(\varphi)$, ω is called an admissible sequence.

Definition 1.5. The cylinder of rank n defined by $a_1, a_2, \ldots, a_n \in A$ is the set

$$B(a_1, a_2, \ldots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \cdots \cap T^{-n+1}B(a_n).$$

We define B to be a cylinder of rank 0.

Definition 1.6. Let $\beta > 1$ and $\beta \in \mathbf{R}$. Let $f_\beta : [0,1) \to [0,1)$ be a function defined by

$$f_\beta(x) = \beta x - [\beta x].$$

Let $A = \mathbf{Z} \cap [0, \beta)$. Then, we have the following fibred system $([0,1), f_\beta)$:

$$\begin{array}{ccc} [0,1) & \xrightarrow{f_\beta} & [0,1) \\ \varphi \downarrow & & \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array} \tag{1.1}$$

The representation map φ of this fibred system is defined by

$$x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^{n+1}} \iff \varphi(x) = (a_0, a_1, \ldots, a_n, \ldots) \in \Omega.$$

This fibred system $([0,1), f_\beta)$ is called a β-adic transformation. In this situation, we define $\zeta_\beta \in \Omega$ by

$$\zeta_\beta = \lim_{x \nearrow 1} \varphi(x). \tag{1.2}$$

We also define $X_\beta \subset \Omega$ to be the set of all admissible sequences.

For a sequence $a \in \Omega$, we write the i-th element of a as $a(i)$, that is, $a = (a(1), a(2), \ldots)$. We remark that φ is not a valid representation at this point, because $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ and $(a_1, a_2, \ldots, a_n - 1, \zeta_\beta(1), \zeta_\beta(2), \ldots)$ are two different representations of the same $x = \sum_{i=1}^{n} a_i \beta^{-i}$. In this paper we adopt the former representation and make φ valid. We derive the following propositions directly from this definition.

Proposition 1.1.

$$X_\beta = \{\omega \in \Omega \mid \forall n \in \mathbf{Z}_{\geq 0} \ \ \sigma^n \omega \prec \zeta_\beta\},$$

where $\omega \prec \psi$ means that ω precedes ψ in lexicographical order.

Proposition 1.2. *For an arbitrary $i \in A$,*

$$B(i) = \begin{cases} [\frac{i}{\beta}, \frac{i+1}{\beta}), & 0 \leq i < [\beta] \\ [\frac{[\beta]}{\beta}, 1), & \textit{otherwise} \end{cases}$$

holds.

Let $\rho_\beta(x) = \sum_{n=0}^{\infty} a_n \beta^{-n-1}$; then, we have

$$\rho_\beta(X_\beta) = [0,1]$$

and the following commutative diagram:

$$\begin{array}{ccc} [0,1) & \xrightarrow{f_\beta} & [0,1) \\ \varphi \downarrow \ \ \uparrow \rho_\beta & & \rho_\beta \uparrow \ \ \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array} \tag{1.3}$$

2. Constructing the sequence

In this section, a sequence $N_\beta \subset [0,1)$ is defined by the use of β-adic transformation. Let $\beta \in \mathbf{R}_{>1}$ and let $([0,1), f_\beta)$ be a fibred system (1.3). Let $B = [0,1)$, and $A, \Omega, (X_\beta, \sigma), \rho_\beta, \varphi, \zeta_\beta, B(a_1, \ldots, a_n)$ be the same as in the previous section.

Definition 2.1. For an arbitrary $n \in \mathbf{Z}_{\geq 0}$, $X_\beta(n), Y_\beta(n) \subset X_\beta$,$F_\beta(n) \in \mathbf{Z}$, and $G_\beta(n) \in \mathbf{Z}$ are defined as follows:

$$X_\beta(n) = \begin{cases} \{(0,0,\ldots)\}, & n = 0 \\ \{\omega \in X_\beta \mid \sigma^{n-1}\omega \neq (0,0,\ldots) \text{ and } \sigma^n\omega = (0,0,\ldots)\}, & n \neq 0 \end{cases}$$

$$Y_\beta(n) = \cup_{i=0}^{n} X_\beta(i)$$

$$F_\beta(n) = \sharp X_\beta(n)$$

$$G_\beta(n) = \sum_{i=0}^{n} F_\beta(i) = \sharp Y_\beta(n)$$

It is apparent that

$$F_\beta(n) \leq ([\beta]+1)^{n-1}.$$

Definition 2.2. For an arbitrary $n \in \mathbf{N}$, define $l_n \in \mathbf{N}$ to satisfy $G_\beta(l_n) < n \leq G_\beta(l_n + 1)$. Define $\tau_n : X_\beta(n) \to \oplus_{i=1}^{n} A$ by $\tau_n((k_1, \ldots, k_n)) = (k_n, \ldots, k_1)$. Induce the right-to-left lexicographical or reverse right-to-left lexicographical order to $X_\beta(l_n+1) = \{\omega_1, \omega_2, \ldots, \omega_{F_\beta(l_n+1)}\}$; that is to say, for all $i < j$, $\tau_n(\omega_i) \prec \tau_n(\omega_j)$ or $\tau_n(\omega_j) \prec \tau_n(\omega_i)$ holds, respectively. In this situation, the sequence N_β is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_{n-l_n})\}_{n=1}^{\infty}$$

In this paper, we assume that the elements of $X_\beta(l_n+1)$ are arranged in right-to-left lexicographical order.

From this definition, we immediately have the following proposition:

Proposition 2.1. *If $\beta \in \mathbf{Z}_{\geq 2}$ then N_β is V_β.*

From this proposition, we see that, if $\beta \in \mathbf{Z}_{\geq 2}$, N_β is a low-discrepancy sequence. We also have the following theorem:

Theorem 2.1. *Let $L, K \in \mathbf{N}$ and $K \leq L$. If $\beta = (L + \sqrt{L^2+4K})/2$, then N_β is a low-discrepancy sequence.*

To prove this theorem, we provide several lemmas, propositions, and definitions. We use the following notation for periodic sequences:

$$\begin{aligned}&(a_1, a_2, \ldots, \dot{a}_n, \ldots \dot{a}_{n+m}) \\ &= (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}, a_n, a_{n+1}, \ldots, a_{n+m}, \ldots)\end{aligned}$$

Let $\beta \in \mathbf{R}_{>1}$.

Lemma 2.1. *If* $\zeta_\beta = (\dot{a}_1, a_2, \ldots, (\dot{a}_m - 1))$, *then* $\{F_\beta(n)\}_{n=1}^\infty$ *and* $\{G_\beta(n)\}_{n=1}^\infty$ *satisfy the following linear recurrent equations:*

$$\text{(2.1.F)}\qquad \begin{aligned} &F_\beta(n+m) - \sum_{i=1}^{m} a_i F_\beta(n+m-i) = 0 \quad \textit{for all } n \geq 1-m,\ n \neq 0 \\ &F_\beta(m) - \sum_{i=1}^{m} a_i F_\beta(m-i) + 1 = 0 \end{aligned}$$

$$\text{(2.1.G)}\qquad G_\beta(n+m) - \sum_{i=1}^{m} a_i G_\beta(n+m-i) = 0 \quad \textit{for all } n > 0.$$

Here we extend the definition of $F_\beta(n)$ to $F_\beta(-n) = 0 \quad (n > 0)$.

Proof. It is apparent from the definition of β-adic transformation that

$$\text{(2.2.a)}\qquad a_1 = \begin{cases} [\beta], & \beta \notin \mathbf{Z} \\ \beta - 1, & \beta \in \mathbf{Z} \end{cases}$$

and

$$\text{(2.2.b)}\qquad a_1 \geq \begin{cases} a_j, & j = 1, \ldots, m-1 \\ a_m - 1, & \end{cases}$$

hold. From Proposition 1.1, we have

$$\begin{aligned} X_\beta(n+m) = {} & \{(x, \omega_1) \mid x \in \{0, \ldots, a_1 - 1\},\quad \omega_1 \in X_\beta(n+m-1)\} \\ & \cup \{(a_1, x, \omega_2) \mid x \in \{0, \ldots, a_2 - 1\},\quad \omega_2 \in X_\beta(n+m-2)\} \\ & \vdots \\ & \cup \{(a_1, \ldots, a_{m-1}, x, \omega_m) \mid x \in \{0, \ldots, a_m - 1\},\quad \omega_m \in X_\beta(n)\} \end{aligned}$$

for all $n \geq 1$, and

$$\begin{aligned} X_\beta(0) &= \{(\dot{0})\} \\ X_\beta(1) &= \{(x, \dot{0}) \mid x \in \{1, \ldots, a_1\}\} \\ X_\beta(2) &= \{(x, \omega_1) \mid x \in \{0, \ldots, a_1 - 1\},\quad \omega_1 \in X_\beta(1)\} \\ &\quad \cup \{(a_1, x, \dot{0}) \mid x \in \{1, \ldots, a_2\}\} \\ &\vdots \\ X_\beta(m-1) &= \{(x, \omega_{m-2}) \mid x \in \{0, \ldots, a_1 - 1\},\quad \omega_{m-2} \in X_\beta(m-2)\} \\ &\quad \cup \{(a_1, x, \omega_{m-3}) \mid x \in \{0, \ldots, a_2 - 1\},\quad \omega_{m-3} \in X_\beta(m-3)\} \\ &\quad \vdots \\ &\quad \cup \{(a_1, \ldots, a_{m-2}, x, \dot{0}) \mid x \in \{1, \ldots, a_{m-1}\}\} \\ X_\beta(m) &= \{(x, \omega_{m-1}) \mid x \in \{0, \ldots, a_1 - 1\},\quad \omega_{m-1} \in X_\beta(m-1)\} \\ &\quad \cup \{(a_1, x, \omega_{m-2}) \mid x \in \{0, \ldots, a_2 - 1\},\quad \omega_{m-2} \in X_\beta(m-2)\} \\ &\quad \vdots \\ &\quad \cup \{(a_1, \ldots, a_{m-1}, x, \dot{0}) \mid x \in \{1, \ldots, a_m - 1\}\}. \end{aligned}$$

In the above expressions, we set $\{0,\dots,a_i-1\}=\emptyset$ when $a_i=0$. Remark $a_1,\ a_m\geq 1$. Then (2.1.F) holds. From Definition 2.1, (2.1.F), and

$$F_\beta(m)+F_\beta(0)=\sum_{i=1}^{m}a_iF_\beta(m-i),$$

we have

$$\begin{aligned}
G_\beta(n+m)=&F_\beta(n+m)+F_\beta(n+m-1)+\cdots+F_\beta(0)\\
=&\ a_1F_\beta(n+m-1)+a_2F_\beta(n+m-2)+\cdots+a_mF_\beta(n)\\
&+a_1F_\beta(n+m-2)+a_2F_\beta(n+m-3)+\cdots+a_mF_\beta(n-1)\\
&+\vdots\\
&+a_1F_\beta(m)+a_2F_\beta(m-1)+\cdots+a_mF_\beta(1)\\
&+a_1F_\beta(m-1)+a_2F_\beta(m-2)+\cdots+a_{m-1}F_\beta(0)\\
&+a_1F_\beta(m-2)+a_2F_\beta(m-3)+\cdots+a_{m-2}F_\beta(0)\\
&+\vdots\\
&+a_1F_\beta(0)\\
=&\ a_1G_\beta(n+m-1)+a_2G_\beta(n+m-2)+\cdots+a_mG_\beta(n).
\end{aligned}$$

Thus (2.1.G) holds.

Definition 2.3. For $(k_1,k_2,\dots,k_n)\in X_\beta(n)$, define

$$d(k_1,k_2,\dots,k_n)=\min\{\max\{0,n-m\}\leq d\leq n \mid 1\in\overline{B(\sigma^d(k_1,\dots,k_n))}\}.$$

Lemma 2.2. *Let $(k_1,\dots,k_n)\in Y_\beta(n)$. When $(k_1,\dots,k_n)\in X_\beta(l)$ and $l<n$, we set $k_{l+1}=\cdots=k_n=0$. If $\zeta_\beta=(\dot{a}_1,a_2,\dots,\dot{(a_m-1)})$, then*

$$\lambda(B(k_1,\dots,k_n))=\begin{cases}\dfrac{1}{\beta^d}\displaystyle\sum_{i=n-d+1}^{m}\frac{a_i}{\beta^i}, & \textit{when}\quad d>n-m\\[2ex] \dfrac{1}{\beta^n}, & \textit{when}\quad d=n-m\end{cases}$$

where $d=d(k_1,\dots,k_n)$ and λ is a one-dimensional Lebesgue measure.

Proof. From $\zeta_\beta=(\dot{a}_1,a_2,\dots,\dot{(a_m-1)})$ we have

$$1-\sum_{i=1}^{m}\frac{a_i}{\beta^i}=0 \tag{2.3.a}$$

$$1-\sum_{i=1}^{lm}\frac{\zeta_\beta(i)}{\beta^i}=\frac{1}{\beta^{ml}}, \tag{2.3.b}$$

where l is an arbitrary positive integer. If $\beta \in \mathbf{N}_{\geq 2}$, this lemma is trivial. We assume that $\beta \neq \mathbf{N}$. We prove the lemma by induction on n. Consider the case in which $n = 1$. From the definition of f_β, (2.2), and (2.3.a), we have

$$\lambda(B(0)) = \lambda(B(1)) = \cdots = \lambda(B(a_1 - 1)) = \frac{1}{\beta}$$

and

$$\lambda(B(a_1)) = \sum_{i=2}^{m} \frac{a_i}{\beta^i}.$$

This means that the lemma's statement holds when $n = 1$. We show that this statement holds for $(k_1, \ldots, k_n, k_{n+1}) \in \cup_{i=1}^{n+1} X_\beta(i)$ under the induction hypothesis. For any $n \geq 1$ and $J \subset [0, 1)$,

$$f_\beta(f_\beta^{-n}(J)) = f_\beta^{-n+1}(J) \tag{2.4}$$

holds from f_β's surjectivity. Consider the case in which $k_1 = 0, 1, \ldots, a_1 - 1$, that is to say, the case in which $d = d(k_1, \ldots, k_{n+1}) \geq 1$ and $d(k_2, \ldots, k_{n+1}) = d - 1$. In this case, $f_\beta(B(k_1)) = [0, 1)$ holds; therefore, considering (2.4), we have

$$f_\beta(B(k_1, \ldots, k_{n+1})) = B(k_2, \ldots, k_{n+1}) \tag{2.5}$$

and

$$\lambda(f_\beta(J)) = \beta\lambda(J) \tag{2.6}$$

for an arbitrary $J \subset B(k_1)$. From the induction hypothesis,

$$\lambda(B(k_2, \ldots, k_{n+1})) = \begin{cases} \dfrac{1}{\beta^{d-1}} \displaystyle\sum_{i=n-d}^{m} \frac{a_i}{\beta^i}, & \text{when} \quad d - 1 > n - m \\ \dfrac{1}{\beta^n}, & \text{when} \quad d - 1 = n - m \end{cases}$$

holds. Therefore, from (2.5) and (2.6), this lemma's statement holds. When $d = 0$, the statement follows from (2.3.a) and (2.3.b).

For a sequence S, $S[N]$ denotes the point set consisting of the first N elements of S, and $S[N; M] = S[N + M] \setminus S[N]$.

Lemma 2.3. *For an arbitrary* $(k_1, \ldots, k_n) \in Y_\beta(n)$, *we have*

$$A(B(k_1, \ldots, k_n); N_\beta[G_\beta(m + d + l)])$$
$$= \begin{cases} \displaystyle\sum_{i=1}^{m-n+d} a_{n-d+i} G_\beta(m + d + l - n - i) & \textit{when} \quad d > n - m \\ G_\beta(l) & \textit{when} \quad d = n - m \end{cases}$$

where $d = d(k_1, \ldots, k_n)$ and $l \in \mathbf{Z}_{\geq 0}$.

Proof. When $d = n - m$ holds, it is trivial. Assume that $d > n - m$. Let $K = (k_1, \ldots, k_n)$. From Proposition 1.1,

$$
\begin{aligned}
&\{\omega \in \cup_{i=0}^{m+d+l} X_\beta(i) \mid \rho_\beta(\omega) \in B(k_1, \ldots, k_n)\} \\
&= \{(K, x, \omega_1) \mid x \in \{0, \ldots, a_{n-d+1} - 1\},\ \omega_1 \in Y_\beta(m + d + l - n - 1)\} \\
&\cup \{(K, a_{n-d+1}, x, \omega_2) \mid x \in \{0, \ldots, a_{n-d+2} - 1\},\ \omega_2 \in Y_\beta(m + d + l - n - 2)\} \\
&\vdots \\
&\cup \{(K, a_{n-d+1}, \ldots, a_{m-1}, x, \omega_{m-n+d}) \\
&\qquad \mid x \in \{0, \ldots, a_m - 1\},\ \omega_{m-n+d} \in Y_\beta(l)\},
\end{aligned}
$$

holds. In the above expressions, we set $\{0, \ldots, a_i - 1\} = \emptyset$ when $a_i = 0$. Therefore, we have

$$
\begin{aligned}
&A(B(k_1, \ldots, k_n); N_\beta[G_\beta(n + l)]) \\
&= \sum_{i=1}^{m-n+d-1} a_{n-d+i} G_\beta(m + d + l - i) + a_m G_\beta(n + l) \\
&= \sum_{i=1}^{m-n+d} a_{n-d+i} G_\beta(m + d + l - i).
\end{aligned}
$$

Proof of **Theorem 2.1.** From the conditions of the theorem,

$$\zeta_\beta = (\dot{L}, (K \dot{-} 1)) \tag{2.7}$$

holds. Let $\alpha = (L - \sqrt{L^2 + 4K})/2$. Then we have

$$F_\beta(n) = \begin{cases} 1, & n = 0 \\ \dfrac{1}{\beta - \alpha}(\beta^{n-1}(\beta^2 - 1) - \alpha^{n-1}(\alpha^2 - 1)), & n \geq 1 \end{cases} \tag{2.8.F}$$

$$G_\beta(n) = \begin{cases} 1, & n = 0 \\ \dfrac{1}{\beta - \alpha}(\beta^n(\beta + 1) - \alpha^n(\alpha + 1)), & n \geq 1 \end{cases} \tag{2.8.G}$$

from (2.7) and Lemma 2.1. Define $Z_\beta(n)$ and $H_\beta(n)$ as follows:

$$
\begin{aligned}
Z_\beta(n) &= \{\omega \in Y_\beta(n) \mid \omega(n) \neq L\} \\
H_\beta(n) &= \sharp Z_\beta(n)
\end{aligned}
$$

The following partitionings of $Y_\beta(n)$ and $Z_\beta(n)$ hold.

$$
\begin{aligned}
Y_\beta(n + 1) = &\{(\omega, x) \mid x \in \{0, 1, \ldots, K - 1\},\ \omega \in Y_\beta(n)\} \\
&\cup \{(\omega, x) \mid x \in \{K, K + 1, \ldots, L\},\ \omega \in Z_\beta(n)\}
\end{aligned} \tag{2.9.Y}
$$

$$(2.9.Z)\qquad \begin{aligned} Z_\beta(n+1) =&\{(\omega,x) \mid x\in\{0,1,\ldots,K-1\},\ \omega\in Y_\beta(n)\}\\ &\cup\{(\omega,x)\mid x\in\{K,K+1,\ldots,L-1\},\ \omega\in Z_\beta(n)\}\end{aligned}$$

Then we have

$$(2.10)\qquad \begin{aligned} H_\beta(n+1) &= KG_\beta(n)+(L-K)H_\beta(n)\\ G_\beta(n+1) &= KG_\beta(n)+(L-K-1)H_\beta(n).\end{aligned}$$

From (2.10) and Lemma 2.1, we have

$$H_\beta(n+2)-LH_\beta(n+1)-KH_\beta(n)=0,\quad n\geq 1.$$

From the same discussion as in the proof of Lemma 2.3,

$$(2.11)\qquad A(B(k_1,\ldots,k_n);\rho_\beta(Z_\beta(2+d+l))) = \begin{cases} H_\beta(l), & d=n-2\\ KH_\beta(l), & d=n-1\\ H_\beta(l+2), & d=n\end{cases}$$

$$d=d(k_1,\ldots,k_n)$$

holds for an arbitrary $(k_1,\ldots,k_n)\in Y_\beta(n)$. Define

$$\Delta(B;P)=A(B;P)-M\lambda(B),$$

where B is an interval in $[0,1)$ and $P=\{x_1,x_2,\ldots,x_M\}\subset[0,1)$. For any set of points P,S in $[0,1)$, and any interval $B\subset[0,1)$,

$$\Delta(B;P\cup S)=\Delta(B;P)+\Delta(B;S)$$

holds. Considering the order of N_β that we gave in Definition 2.2, we have

$$(2.12)\qquad N_\beta[H_\beta(n)]=\rho_\beta(Z_\beta(n)).$$

From Lemma 2.2, Lemma 2.3, (2.8.G), (2.11) and (2.12), we have

$$(2.13)\qquad \begin{aligned}&\Delta(B(k_1,\ldots,k_n);N_\beta[G_\beta(2+d+l)])\\ &= \begin{cases} \dfrac{\alpha+1}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^n-1\right)\alpha^l, & d=n-2\\ \dfrac{K(\alpha+1)}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^{n+1}-1\right)\alpha^l, & d=n-1\\ \dfrac{\alpha+1}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^n-1\right)\alpha^{l+2}, & d=n\end{cases}\end{aligned}$$

and

$$(2.14)\qquad \begin{aligned}&\Delta(B(k_1,\ldots,k_n);N_\beta[H_\beta(2+d+l)])\\ &= \begin{cases} \dfrac{1}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^n-1\right)\alpha^{l+1}, & d=n-2\\ \dfrac{K}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^{n+1}-1\right)\alpha^{l+1}, & d=n-1\\ \dfrac{1}{\beta-\alpha}\left(\left(\dfrac{\alpha}{\beta}\right)^n-1\right)\alpha^{l+3}, & d=n\end{cases}\end{aligned}$$

where $(k_1,\dots,k_n) \in Y_\beta(n)$, $l \in \mathbf{Z}$ and $d = d(k_1,\dots,k_n)$. Define the truncating operator $r_k : X_\beta \to Y_\beta(k)$ as follows:

$$r_k(\omega) = \begin{cases} \omega, & \text{when } \omega \in X_\beta(j),\ j \le k \\ (\omega(1),\dots,\omega(k)) & \text{otherwise} \end{cases}$$

For any $i, j \in \mathbf{Z}$ and any cylinder B of rank less than k,

$$A(B; N_\beta[i;j]) = A(B; r_k(N_\beta[i;j])) \tag{2.15}$$

holds. Let $(k_1,\dots,k_n) \in Y_\beta(n)$, let $d = d(k_1,\dots,k_n)$, and let M be an arbitrary integer greater than $G_\beta(2+d)$. Let l be an integer satisfying

$$G_\beta(2+d+l) \le M < G_\beta(2+d+l+1).$$

Applying partitioning (2.9.Y) and (2.9.Z) recursively for $Y_\beta(2+d+l+1)$, we obtain the following partitioning of $N_\beta[G_\beta(2+d+l+1)]$:

$$\begin{aligned}
&N_\beta[G_\beta(2+d+l+1)] \\
&=N_\beta[G_\beta(2+d+l)] \\
&\quad \cup N_\beta[G_\beta(2+d+l); G_\beta(2+d+l)] \\
&\quad \vdots \\
&\quad \cup N_\beta[(K-1)G_\beta(2+d+l); G_\beta(2+d+l)] \\
&\quad \cup N_\beta[KG_\beta(2+d+l); H_\beta(2+d+l)] \\
&\quad \vdots \\
&\quad \cup N_\beta[KG_\beta(2+d+l)+(L-K-1)H_\beta(2+d+l); H_\beta(2+d+l)] \\
&\quad \cup N_\beta[KG_\beta(2+d+l)+(L-K)H_\beta(2+d+l); G_\beta(2+d+l-1)] \\
&\quad \vdots \\
&\quad \cup N_\beta[KG_\beta(2+d+l)+(L-K)H_\beta(2+d+l)+KG_\beta(2+d+l-1) \\
&\qquad ; H_\beta(2+d+l-1)] \\
&\quad \cup \\
&\quad \vdots
\end{aligned} \tag{2.16}$$

Partition $N_\beta[M]$ in the same way as (2.16); then, from (2.15), the additivity of Δ, (2.9.Y), (2.9.Z), and the order we induced to N_β, we have

$$
\begin{aligned}
&\Delta(B; N_\beta[M]) \\
&\le K\,|\Delta(B; N_\beta[G_\beta(2+d+l)])| + (L-K)\,|\Delta(B; N_\beta[H_\beta(2+d+l)])| \\
&+ K\,|\Delta(B; N_\beta[G_\beta(1+d+l)])| + (L-K-1)\,|\Delta(B; N_\beta[H_\beta(1+d+l)])| \\
&+ K\,|\Delta(B; N_\beta[G_\beta(d+l)])| + (L-K-1)\,|\Delta(B; N_\beta[H_\beta(d+l)])| \\
&\vdots \\
&+ K\,|\Delta(B; N_\beta[G_\beta(2+d+1)])| + (L-K-1)\,|\Delta(B; N_\beta[H_\beta(2+d+1)])| \\
&+ K\,|\Delta(B; N_\beta[G_\beta(2+d)])| + (L-K)\,|\Delta(B; N_\beta[H_\beta(2+d)])| \\
&\le K \sum_{i=0}^{l} |\Delta(B; N_\beta[G_\beta(2+d+i)])| \\
&\quad + (L-K) \sum_{i=0}^{l} |\Delta(B; N_\beta[H_\beta(2+d+i)])|
\end{aligned}
\tag{2.17}
$$

where $B = B(k_1, \ldots, k_n)$. From (2.13), (2.14), (2.17) and the fact that $|\alpha| < 1 < |\beta|$, there exists a constant C_1 that satisfies the following inequality (2.18) for any cylinder $B(k_1, \ldots, k_n)$ of any rank n and any integer $M > G_\beta(2+d)$.

$$
|\Delta(B(k_1, \ldots, k_n); N_\beta[M])| < C_1 \tag{2.18}
$$

Choose an arbitrary $u \in [0, 1)$. Let $M \in \mathbf{N}$ and l be an integer that satisfies

$$
G_\beta(l) \le M < G_\beta(l+1).
$$

Let $B(u_1, \ldots, u_l)$ be a cylinder of rank l that satisfies $u \in B(u_1, \ldots, u_l)$. Then we have

$$
\begin{gathered}
[0, u) = B_{t_1} \sqcup B_{t_2} \sqcup \cdots \sqcup B_{t_k} \sqcup R \\
0 \le t_1 < t_2 < \cdots < t_k = l
\end{gathered}
\tag{2.19}
$$

where B_{t_i} is a cylinder of rank t_i and $\lambda(R) < \beta^{-l}$. From (2.8.G), there exist constants C_2 and C_3 that satisfy $l < C_2 \log M$ and $M\beta^{-l} < C_3$. Then, from (2.18) and (2.19), we have

$$
|\Delta([0, u); N_\beta[M])| < C_1 C_2 \log M + C_3.
$$

The theorem follows from this.

References

1. Harald Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 1992.
2. W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar **11** (1960), 401–416.
3. A. Rény, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar **8** (1957), 477–493.
4. W. M. Schmidt, *Irregularities of distribution* VII, Acta Arith. **21** (1972), 45–50.
5. Fritz Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford science publications, 1995.
6. Y. Takahashi, *Kukanrikigakukei no Kaosu to Shuhkiten*, (in Japanese), Tokyo Metropolitan University Seminar Report.

IBM Research, Tokyo Research Laboratory 1623-14, Shimotsuruma, Yamato-shi, Kanagawa-ken 242, Japan

E-mail address: ninomiya@trl.ibm.co.jp