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STATIONARY SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS AND THE BOUSSINESQ EQUATIONS UNDER GENERAL OUTFLOW CONDITION

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Let D be a bounded domain in $\mathbf{R}^n$ (n=2 or 3), ∂D its smooth boundary. We consider the existence of solutions to the stationary Navier-Stokes equations

$$(1)\quad \begin{cases} -\nu\Delta \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p &= \boldsymbol{f} \quad \text{in } D, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } D, \end{cases}$$

under the boundary condition

$$(2)\quad \boldsymbol{u}=\boldsymbol{\beta} \qquad \text{on} \quad \partial D$$

where $\boldsymbol{u}$ is the velosity vector, p the pressure, $\boldsymbol{f}$ the external force, ν kinematic viscosity, $\boldsymbol{\beta}$ the velosity vector given on the boundary.

We consider also the similar problem for the stationary Boussinesq equations

$$(3)\quad \begin{cases} -\nu\Delta \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p+\alpha \boldsymbol{g}T &= \boldsymbol{f} \quad \text{in } D, \\ -\chi\Delta T+(\boldsymbol{u}\cdot\nabla)T &= 0 \quad \text{in } D, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } D, \end{cases}$$

under the boundary condition

$$(4)\quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} \quad \text{on } \partial D \\ T = \theta \quad \text{on } \partial D \end{cases}$$

where T is the temperature, α coefficient of volume expansion, χ thermal diffusivity, $\boldsymbol{\beta}$ and θ prescribed velosity and temperature on the boundary, respectively.

Firstly, we mention some known results for these problem.

The existence of the stationary solutions to the Navier-Stokes equations (1), (2) and the Boussinesq equations (3), (4) is known in general context if, for any $\varepsilon>0$, there exists an extension $\boldsymbol{b}_\varepsilon$ of the boundary value $\boldsymbol{\beta}$ to the domain D such that $\operatorname{div}\boldsymbol{b}_\varepsilon=0$ in D and the inequality

$$(L)\quad |((\boldsymbol{u}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{u})|\le\varepsilon\|\nabla\boldsymbol{u}\|^2, \qquad \forall\boldsymbol{u}\in \boldsymbol{C}_{0,\sigma}^\infty(D)$$

holds, where

$$(\boldsymbol{u},\boldsymbol{v})=\sum_i\int_D u_i(x)v_i(x)dx$$

$$||\boldsymbol{u}|| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$$

$$C^{\infty}_{0,\sigma}(D) = \{\boldsymbol{u} \in C^{\infty}_{0}(D) \; ; \; \mathrm{div}\boldsymbol{u} = 0 \text{ in D}\}.$$

Suppose that the boundary ∂D of D is multiply connected,

(5) $\quad \partial D = \cup_{i=1}^{k} \Gamma_i \; (k \geq 2) \; (\Gamma_i :$ connected component of $\partial D)$

and D is inside of Γ_k.

Since div$\boldsymbol{u} = 0$, the integral $\int_{\partial D} \boldsymbol{u} \cdot \boldsymbol{n} d\sigma$ must vanish, where $\boldsymbol{n}$ denotes the outward normal vector to the boundary. Let us call this condition general outflow condition (GOC).

$$(GOC) \quad \int_{\partial D} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = \sum_{i=1}^{k} \int_{\Gamma_i} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = 0$$

Theorem 1. *(Leray[8], Hopf[6], Fujita[3], Ladyzhenskaya[7])*
Suppose the following condition is satisfied.

$$(OC) \quad \int_{\Gamma_i} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = 0 \; (1 \leq i \leq k)$$

Then the inequality (L) holds true .

Remark 1. *If the boundary is multiply connected, the condition (OC) is stronger than the condition (GOC). On the other hand, if ∂D is connected, then (GOC) and (OC) are equivalent and*

$$\int_{\partial D} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = 0 \implies (L) \text{ holds true.}$$

When (OC) does not holds, we know the following fact due to the work of Takeshita.

Theorem 2. *(Takeshita[14]) Let D be a bounded domain in $\mathbf{R}^2$ the boundary of which consists of 2 connected components $\partial D = \Gamma_1 \cup \Gamma_2$. Suppose that we can insert a circle between Γ_1 and Γ_2. If the boundary integral*

$$\int_{\Gamma_1} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = - \int_{\Gamma_2} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma \neq 0,$$

then (L) does not hold true.

Therefore we can not use the method of Theorem 1 to show the existence of stationary solutions to the Navier-Stokes equations (1), (2). Nevertherless this does not mean the non-existence of solutions. In fact, Amick showed the existence of solution under the assumption of " symmetry" for 2-D case.

Theorem 3. *(Amick[1]) Let D be a bounded domain in $\mathbf{R}^2$. If D, $\boldsymbol{f}$, $\boldsymbol{\beta}$ are symmetric with respect to a line ℓ, and every Γ_i intersects with ℓ, then a solution exists.*

Motivated the work of Takeshita, we found the following exact solution for 2-D annular domain

$$D = \{\boldsymbol{x} \in \mathbf{R}^2; R_1 < |\boldsymbol{x}| < R_2\},\ \partial D = \Gamma_1 \cup \Gamma_2,\ \Gamma_i = \{|\boldsymbol{x}| = R_i\}(i = 1, 2).$$

Example 1. *(Morimoto[9], see also [10]) Suppose $\boldsymbol{f} = \mathbf{0}$ and the boundary value:*

$$\boldsymbol{\beta} = \frac{\mu}{R_i}\boldsymbol{e}_r + \omega_i R_i \boldsymbol{e}_\theta \quad on \quad \Gamma_i\ (i = 1, 2),$$

where μ, ω_1, ω_2 are given constants. Then the boundary value problem (1) (2) has the following solution. The velocity $\boldsymbol{u}_0$ is given by

$$\boldsymbol{u}_0 = \boldsymbol{u}_0(\mu) = \frac{\mu}{r}\boldsymbol{e}_r + b(\mu, r)\boldsymbol{e}_\theta.$$

(i) If $\mu \neq -2\nu$, $b(\mu, r) = \frac{1}{r}(c_1 + c_2 r^{2+\frac{\mu}{\nu}})$,

(ii) If $\mu = -2\nu$, $b(\mu, r) = \frac{1}{r}(c_1 + c_2 \log r)$,

where c_1, c_2 are appropriate constants. The pressure $p_0 = p_0(\mu)$ can be obtained from the equation.

As for the perturbation of the above solution, we have

Theorem 4. *(Morimoto-Ukai[13])*
Let $D = \{\boldsymbol{x} \in \mathbf{R}^2; R_1 < |\boldsymbol{x}| < R_2\}$, $\boldsymbol{f} = \mathbf{0}$ and the boundary value:

$$\boldsymbol{\beta} = \{\frac{\mu}{R_i} + \varphi_i(\theta)\}\boldsymbol{e}_r + \{\omega_i R_i + \psi_i(\theta)\}\boldsymbol{e}_\theta \quad on \quad \Gamma_i\ (i = 1, 2),$$

where μ, ω_1, ω_2 are given constants and $\varphi_i(\theta)$, $\psi_i(\theta)$ are 2π-periodic functions, the integral of which over the interval $[0, 2\pi]$ vanishes. Suppose the inequality

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1}\right)^2 < 2\nu$$

hold. Then there exists at most discrete countable set $\mathcal{M}$ such that for each $\mu \in \mathbf{R}\backslash\mathcal{M}$ the boundary value problem $(1), (2)$ has a solution for sufficiently small $\varphi_i(\theta)$, $\psi_i(\theta)$ $(i = 1, 2)$.

Remark 2. *$\omega_i (i = 1, 2)$ can be large but the diffenrence $|\omega_1 - \omega_2|$ should be small.*

For the general domain D in $\mathbf{R}^2$ or $\mathbf{R}^3$, the boundary of which is multiply connected, we have

Theorem 5. *(Fujita-Morimoto[4])*
Suppose that $\boldsymbol{f} \in V'$ is a potential force, that $\boldsymbol{\beta} = \mu\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1$, where μ is a constant, $\boldsymbol{\beta}_0$ is the boundary value of gradient of a harmonic function $\varphi \in H^2(D)$, and that $\boldsymbol{\beta}_1$ is in $H^{1/2}(\partial D)$ with

$$\int_{\partial D} \boldsymbol{\beta}_1 \cdot \boldsymbol{n} d\sigma = 0.$$

Then, there exists a discrete countable set $\mathcal{M} \subset \mathbf{R}$ such that for each $\mu \in \mathbf{R} \setminus \mathcal{M}$, there exists a weak solution to $(1), (2)$ if $\boldsymbol{\beta}_1$ satisfies the inequality $||\boldsymbol{\beta}_1||_{H^{1/2}(\partial D)} < C^*$ for some positive constant $C^* = C^*(\nu, \mu, D. \boldsymbol{\beta}_0)$.

Remark 3. *The boundary value* $\boldsymbol{\beta}_0$ *may not satisfy the vanishing outflow condition. A non-trivial example of such* $\boldsymbol{\beta}_0$ *in 3-dimensinal case is*

$$\sum_{i=1}^{k-1} \nabla \left(\frac{q_i}{4\pi|x-a_i|} \right)$$

where q_i*'s are constants and* a_i*'s are points outside* D*, each* a_i *being enclosed by* Γ_i*.*

In the following case, the set $\mathcal{M}$ is void, that is, for every μ, solutions exist for sufficiently small $\boldsymbol{\beta}_1$.

Theorem 6. *(Fujita-Morimoto-Okamoto[5], Morimoto[12])*
In case of 2-D annular domain and

$$\boldsymbol{\beta}_0 = \nabla \log r \Big|_{\partial D}$$

the set of exceptional values $\mathcal{M}$ *in Theorem 5 is void.*

As for the Boussinesq equations, we obtain the following results.

Theorem 7. *(Morimoto[11]) Suppose that* $\boldsymbol{f} \in V'$ *is a potential force, that* $\boldsymbol{\beta} = \mu\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1$*, where* μ *is a constant,* $\boldsymbol{\beta}_0$ *is the boundary value of gradient of a harmonic function* $\varphi \in H^2(D)$*, and that* $\boldsymbol{\beta}_1$ *is in* $H^{1/2}(\partial D)$ *with*

$$\int_{\partial D} \boldsymbol{\beta}_1 \cdot \boldsymbol{n} d\sigma = 0.$$

Suppose that θ_0 *is in* $H^{1/2}(\partial D)$*. Then, there exists a discrete countable set* $\mathcal{M} \subset \mathbf{R}$ *such that for each* $\mu \in \mathbf{R} \setminus \mathcal{M}$*, there exists a solution to (3) (4), if* α*,* $||\boldsymbol{\beta}_1||_{H^{1/2}(\partial D)}$*,* $||\theta_0||_{H^{1/2}(\partial D)} < C^*$ *holds for some positive constant* $C^* = C^*(\nu, \chi, \mu, D, \boldsymbol{\beta}_0)$*.*

Remark 4. *The set of exceptional value* $\mathcal{M}$ *is the same as in the Navier-Stokes equations case.*

Theorem 8. *(Morimoto[11]) In case of 2-D annular domain and*

$$\boldsymbol{\beta}_0 = \nabla \log r \Big|_{\partial D}$$

the set of exceptional values $\mathcal{M}$ *in Theorem 7 is void.*

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