Title	Singularities of sub-Riemannian exponential mappings, conjugate loci (caustics), wave fronts, cut loci and Carnot-Caratheodory small-balls (Recent results by Agrachev, El-Alaoui, Gauthier, Ge and Kupka)
Author(s)	ISHIKAWA, Go-o
Citation	数理解析研究所講究録 (1997), 1006: 121-137
Issue Date	1997-08
URL	http://hdl.handle.net/2433/61463
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1006巻 1997年 121-137

Singularities of sub-Riemannian exponential mappings, conjugate loci (caustics), wave fronts, cut loci and Carnot-Carathéodory small-balls (Recent results by Agrachev, El-Alaoui, Gauthier, Ge and Kupka).

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31 March 1997

1 Introduction.

After reviewing fundamental notions of sub-Riemannian or nonholonomic or Carnot-Carathéodory (C-C) geometry, we shall explain the recent results [1][2][9], due to Aglachev, El-Alaoui, Gauthier, and Kupka, on singularities appearing in various geometric objects of generic sub-Riemannian or C-C metrics on $\mathbf{R}^3$ with the contact distribution. See also [10][11]. Also we compare these results with the previous results [22] by Vershik and Gershkovich on the left invariant sub-Riemannian metric of the 3-dimensional Heisenberg group.

One of extremely different features of sub-Riemannian geometry from Riemannian geometry appears in the fact that the closure of the conjugate locus as well as the cut locus of a point contains the original point, and, therefore, a C-C small-balls has singularities even if the radius is sufficiently small.

The geodesic flow for a sub-Riemannian metric naturally lives on the cotangent bundle, and it is reasonable to follow the Hamiltonian formalism [18]. In [1][2][9], in particular, using the classical Whitney's theorem on singularities of plane to plane mappings (with estimates), it has been investigated the **diffeomorphism type** of the germ at a point of the closure of the conjugate locus for a generic C-C metric on $\mathbf{R}^3$. However the method used there is limited to the three dimensional case.

To generalize the classification results of [1][2][9], to more higher dimensional cases, for instance to the Engel case on $\mathbf{R}^4$, it is natural, even in the three dimensional case, to apply **Lagrange and Legendre** (L-L) **singularity theory**, namely singularity theory for caustics and wave fronts [5], not the ordinary singularity theory of differentiable mappings, to sub-Riemannian geometry.

However we emphasize that our classification problem is local but **micro-global**; a global version of L-L singularity theory or L-L singularity theory at infinity is not fully investigated yet, as our fortune, (however see [12]), and therefore the application of singularity theory to sub-Riemannian geometry requires more improvement of L-L singularity theory itself.

There are other possibilities of applications of singularity theory to the problem of singularities of end-point mappings and abnormal geodesics can be found in [1][4], and to the problem of singularities of Pfaff systems and rigid curves [25].

The author would like to thank J. Adachi, S. Izumiya, S. Janeczko, T. Morimoto and K. Yamaguchi for helpful information and encouragement. The author very grateful also to the organizer O. Saeki of this symposium and to the editor T. Ohmoto of this volume.

This short survey article is a revival of my talk given at RIMS in 29 January 1997. The subsequent progress on this subject can be seen in [3].

2 Sub-Riemannian geometry

Let M be a connected C^∞-manifold of dimension n, and D a C^∞-subbundle of the tangent bundle TM of M. We call D **non-holonomic** or **bracket generating** if, for each point $P \in M$, any $v \in T_PM$ is represented as a sum of iterated brackets of sections of D. In what follows we assume D is non-holonomic.

A **sub-Riemannian structure** g on (M, D) is a Riemannian metric on the non-holonomic subbundle D of TM; $g : D \oplus D \to \mathbf{R}$, positive definite symmetric bilinear form. We call the triplet (M, D, g) a **sub-Riemannian manifold.**

Example: Let

$$M = \mathbf{R}^3 = G = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \quad x, y, z \in \mathbf{R} \right\},$$

be the 3-dimensional Heisenberg group. In its Lie algebra

$$\mathcal{G} = T_1G = \left\{ \left(\begin{array}{ccc} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) \middle| \quad x, y, z \in \mathbf{R} \right\},$$

we set

$$X = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad Y = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right),$$

and $V = \langle X, Y \rangle_{\mathbf{R}}$. Then

$$[X, Y] = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) (=: Z).$$

Thus V defines a *left-invariant* non-holonomic subbundle D of TG of rank 2. Actually D is a *contact structure* on G defined by $dz - xdz = 0$. Moreover, if we give a metric on V, then we have a left-invariant Riemannian metric on D. We are going to study on generic perturbations of this left-invariant sub-Riemannian structure on $\mathbf{R}^3$.

Rashevsky-Chow's theorem says that, for any two points P, Q of M, there exists a piecewise differentiable path $c : [a, b] \to M$ such that $c(a) = P, c(b) = Q$ and that $\dot{c}(t) \in D_{c(t)}$, for almost every t. Paths satisfying the latter condition are called **admissible** or **horizontal**. The **length** of an admissible path c is defined by

$$L(c) = \int_a^b \|\dot{c}(t)\|_g dt.$$

Then the **Carnot-Carathéodory distance** is defined by

$$d(P,Q) = \text{C-C-}d(P,Q) = \inf\{L(c) \mid c \text{ is an admissible path from } P \text{ to } Q\}.$$

We set, for $x \in M$ and for $\varepsilon > 0$,

$$B_\varepsilon(P) = \{Q \in M \mid d(P,Q) < \varepsilon\}.$$

Fact (1): The metric C-C-d induces on M the original topology (as a manifold). In other words, $\{B_\varepsilon(P)\}_\varepsilon, \varepsilon > 0$, form a system of neighborhoods of P with respect to the manifold topology of M (cf. Ball-Box theorem [7]).

We call D **strongly bracket generating** (SBG) if, for each $P \in M$, and for a section X of D with $X(P) \neq 0$, any $v \in T_P M$ is represented as a sum of a section of D and a single bracket of X and a section of D.

Fact (2): If D is SBG, e.g. contact, then, for a sufficiently small $\varepsilon > 0$, $B_\varepsilon(P)$ is homeomorphic to the Euclidean ball, and the closure

$$\bar{B}_\varepsilon(P) = \{Q \in M \mid d(P,Q) \leq \varepsilon\}$$

is homeomorphic to the Euclidean closed ball. However $\bar{B}_\varepsilon(P)$ $(P \in M, 0 < \varepsilon << 1)$, has always *singularities* with respected to the differentiable structure of M; there exists a point Q on the boundary of $\bar{B}_\varepsilon(P)$ such that the relative germ $(M, \bar{B}_\varepsilon(P), Q)$ at Q is homeomorphic but not diffeomorphic to $(\mathbf{R}^n, \{x_n \geq 0\}, 0)$.

An admissible path $c : [a,b] \to M$ is called a **minimizer** with respect to the C-C distance, if $L(c) = d(c(a), c(b))$. An admissible path $c : [a,b] \to M$ is called a **local**

minimizer if, for any $t_0 \in [a, b]$, there exists a closed interval $[\alpha, \beta]$ containing t_0 as an interior point in $[a, b]$ such that $c|_{[\alpha,\beta]}$ is a minimizer.

It is known that a local minimizer is necessarily an **extremal**: Extremals are divided into **normal extremals** and **abnormal extremals**. The notion of normal extremals, which we will explain below, belongs to sub-Riemannian geometry; while the notion of abnormal extremals is in non-holonomic geometry, that is independent of sub-Riemannian structure g. Abnormal extremals live in $D^{\perp} \subset T^*M$ [18].

Fact (3): If D is SBG, e.g. contact, then there exists no non-constant abnormal extremal. Moreover if $P, Q \in M$ are sufficiently near, then there exists a normal extremal such that $L(c) = d(P, Q)$.

Fix $P \in M$. Take local frame $X_1, \ldots, X_r$ of D over a neighborhood of P. Then a sub-Riemannian structure on (M, D) near P is uniquely determined such that $X_1, \ldots, X_r$ are orthonormal.

Define the **sub-Riemannian Hamiltonian** $h : T^*M \to \mathbf{R}$ by

$$h(\xi) = -\frac{1}{2}(\langle \xi, X_1 \rangle^2 + \ldots + \langle \xi, X_r \rangle^2),$$

for $\xi \in T^*M$. Here $\langle \cdot , \cdot \rangle : T^*M \oplus TM \to \mathbf{R}$ denotes the natural pairing. Then we see that h is critical just along $h^{-1}(0) = D^{\perp} \subset T^*M$. Moreover normal extremals are projections of solutions of the Hamiltonian flow defined by the Hamiltonian h.

To analyze sub-Riemannian structure through the Hamiltonian, we review in the next section on the Hamiltonian formalism.

3 Hamiltonian formalism

Let M be a C^∞ manifold of dimension n, $h : T^*M \to \mathbf{R}$ a C^∞ function. We assume h is homogeneous of degree m with respect to the fiber coordinates of $\pi : T^*M \to M$. (For the sub-Riemannian Hamiltonian in the previous section, we see $m = 2$.)

We denote by $\theta = \theta_M$ the Liouville 1-form on T^*M, and by $\omega = d\theta$ the symplectic 2-form on T^*M. For a local coordinates $q_1, \ldots, q_n$ of M, and for the coresponding fiber coordinates $p_1, \ldots, p_n$, we have $\theta = \sum p_i dq_i$ and $\omega = \sum dp_i \wedge dq_i$. Then the Hamiltonian vector field $\vec{h}$ on T^*M with Hamiltonian h is defined by

$$\vec{h} \rfloor \omega = -dh.$$

Locally

$$\vec{h} = \sum h_{q_i} \frac{\partial}{\partial p_i} - h_{p_i} \frac{\partial}{\partial q_i}.$$

We see that

$$\langle \theta, \vec{h} \rangle = -\sum p_i h_{p_i} = -mh.$$

In other words, $\vec{h} \rfloor \theta = -mh$.

Let $E = \sum p_i \frac{\partial}{\partial p_i}$ denote the *Euler field* over T^*M. Then $Eh = mh$. If $h(P) \neq 0$, then $dh(P) \neq 0$. Therefore the set of critical points of h is contained in $h^{-1}(0)$. In particular, for $c \neq 0$, the level hypersurface $S = h^{-1}(c)$ is non-singular. Also we see that $E \rfloor \omega = \theta$, namely E is a *Liouville field*, and therefore, denoting by L the Lie derivative, we have

$$L_E \omega = E \rfloor d\omega + d(E \rfloor \omega) = d\theta = \omega.$$

Then we see (cf. [14][13]):

Lemma 3.1 *$\theta|_S$ is a contact form on $S = h^{-1}(c)$, $c \neq 0$, and $\vec{h}|_S$ is a contact vector field. In fact more strictly we see $L_{\vec{h}}(\theta|_S) = 0$.*

Proof: We have

$$\theta \wedge (d\theta)^{n-1} = \theta \wedge \omega^{n-1} = (E \rfloor \omega) \wedge \omega^{n-1} = \frac{1}{n} E \rfloor \omega^n \neq 0,$$

on S. Therefore $\theta|_S$ is a contact form. Moreover $\vec{h}$ is tangent to S, and

$$L_{\vec{h}} \theta = \vec{h} \rfloor \omega + d(\vec{h} \rfloor \theta) = -dh - mdh = -(m+1)dh = 0,$$

on S. $\square$

4 Sub-Riemannian wavefronts.

Now we return to the sub-Riemannian geometry.

By Lemma 3.1, $S = h^{-1}(-\frac{1}{2})$ is a contact manifold with the contact form $\theta|_S$. Denote by Φ_t the contact flow on S defined by $\vec{h}$. The constant $c = -\frac{1}{2}$ is chosen so that the time parameter of solution curves (normal extremals), coincide with their C-C arc-lengths. Remark that Φ_t is well-deined for sufficiently small t.

Set $C = S \cap T_P^*M \cong S^{r-1} \times \mathbf{R}^{n-r}$. Then $\theta|_C = 0$ and therefore C is a *Legendre submanifold* of S. Consider the transform $\Phi_t(C) \subset S$ and its projection $W_t = \pi(\Phi_t(C)) \subset M$ by the bundle projection $\pi : T^*M \to M$. We call W_t the **wavefront** from P of time t.

Then, by Fact (3), we observe

Lemma 4.1 *If D is SBG, and $P \in M$, then*

$$\bar{B}_\varepsilon(P) = \{Q \in M \mid d(P,Q) \leq \varepsilon\} = \bigcup_{0 \leq t \leq \varepsilon} W_t.$$

Our fundamental problem is: How singular are W_t, $\bar{B}_\varepsilon$? For the study on singularities of $\bar{B}_\varepsilon(P)$, first we have to investigate the singularities of W_t.

Define the **exponential map** $e : \mathbf{R}_+ \times C \to M$ near $0 \times C$ by $e(t,\xi) = \pi(\Phi_t(\xi))$.

For $\xi \in C$, denote by $\tau(\xi)$ the **escape time**, that is the time that $\pi(\Phi_t(\xi))$ goes out the fixed neighborhood of P. Then set

$$\begin{aligned} t_c(\xi) &= \sup\{t \in \mathbf{R}_+ \mid 0 < t < \tau(\xi);\ 0 < t' < t \Rightarrow \\ &\qquad e_* : T_{(t',\xi)}(\mathbf{R}_+ \times C) \to T_{e(t',\xi)}M \text{ is isomorphic }\}, \end{aligned}$$

the **first conjugate time**.

Lemma 4.2 *$\Phi : \mathbf{R}_+ \times C \to T^*M$, $\Phi(t,\xi) = \Phi_t(\xi)$, is a* **Lagrange immersion**.

Thus the exponential map e is a Lagrange map. The singular locus of e coincides with the trace of singular points of wavefronts.

Proof: It suffices to show that $\vec{h}$ does not tangent to C anywhere. Recall $Eh = -2h$, so, on $T^*M - \{h = 0\}$, $h_{p_i} \neq 0$, for some i. Therefore $\vec{h}$ does not tangent to T_P^*M along $\{h \neq 0\}$. □

Now let $M = \mathbf{R}^3$ and $D \subset TM$ be a contact distribution. Let $P \in \mathbf{R}^3$. Take a local frame X, Y of D. Then recall that

$$h(\xi) = -\frac{1}{2}(\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2).$$

We take the coordinates of $C \cong S^1 \times \mathbf{R}$, cylinder, as follows: Choose the 1-form α satisfying (1) $\ker \alpha = D$, and (2) $d\alpha(X, Y) = 1$. Take the unique vector field ζ on M such that $\zeta \rfloor (\alpha \wedge d\alpha) = d\alpha$. Define a basis $\alpha_1, \alpha_2, \alpha_3$ of T_P^*M by

$$\begin{array}{lll} \langle \alpha_1, X(P)\rangle = 1, & \langle \alpha_1, Y(P)\rangle = 0, & \langle \alpha_1, \zeta(P)\rangle = 0, \\ \langle \alpha_2, X(P)\rangle = 0, & \langle \alpha_2, Y(P)\rangle = 1, & \langle \alpha_2, \zeta(P)\rangle = 0, \\ \langle \alpha_3, X(P)\rangle = 0, & \langle \alpha_3, Y(P)\rangle = 0, & \langle \alpha_3, \zeta(P)\rangle = \langle \alpha, \zeta\rangle(0). \end{array}$$

Then we define the cylindrical coordinates $T_P^*M - \{h = 0\} \cong \mathbf{R}^3 - \{(0,0)\} \times \mathbf{R}$ by

$$\xi = R\cos\varphi\alpha_1 + R\sin\varphi\alpha_2 + r\alpha_3,$$

where $0 \leq R,\ 0 \leq \varphi < 2\pi,\ r \in \mathbf{R}$. Then

$$C = \{\xi \in T_P^*M \mid \langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2 = 1\} = \{\xi \in T_P^*M \mid R = 1\},$$

which is parametrized by φ and r. Thus we have $C \cong S^1 \times \mathbf{R}$.

Then the main result is the following:

Theorem 4.3 ([1][2][9]) *Fix X, Y and $P \in M = \mathbf{R}^3$. Then there exist $a \in \mathbf{R}$ and $b \in \mathbf{R}_+$ such that, setting $\rho = 1/r$,*

$$t_c(\varphi, \rho) = 2\pi\rho + a\rho^3 + O(\rho^4), \quad (\rho > 0).$$

We define $q_c : C \to M$ *by* $q_c(\xi) = e(t_c(\xi), \xi)$. *Then moreover there exists a system of cordinates of* M *near* P *such that*

$$q_c(\varphi, \rho) = \pi\rho^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b\rho^3 \begin{pmatrix} \cos^3\varphi \\ -\sin^3\varphi \\ 0 \end{pmatrix} + O(\rho^4).$$

The image $q_c(C) \subset M$ is called the **first conjugate locus** or the **caustic**. Using the classical Whitney's theorem it is shown in [1][2][9] that the caustic is diffeomorphic to a cone of the *asteroid.*

5 Figures

Figures 1 and **2** are taken from [22]: Figure 1 is a very rough picture of the wavefront for the Heisenberg case. The more detailed one is presented in Figure 2.

Figures 3, 4, 5 show several parts of the Heisenberg wavefront, which are drawn by **Mathematica**.

Figure 6 is from [7], which shows the C-C small balls for the Heisenberg case.

The zoomed-out picture of a generic sub-Riemannian wavefront is presented in **Figure 7**, taken from [2].

Figures 8 and **9** are zoomed-in picture: There exists a curve γ in $M = \mathbf{R}^3$ such that, for $P \in M - \gamma$, each conical point of the Heisenberg wavefront is perturbed into 4 *swallowtails*, while, for $P \in \gamma$, into 6 swallowtails.

Figure 10 and **11** are hand-written pictures: Figure 10 describes the ways of perturbations of conical singularities of the Heisenberg wavefront to a generic one. Figure 11 shows the singuarities of C-C small balls.

Sub-Riemannian caustics in the Heisenberg case and in generic case are given in **Figure 12**: The latter figure is taken from [5].

Figure 13 is from [2], which shows the half part of generic caustic, for $P \in M - \gamma$, and, for $P \in \gamma$, respectively.

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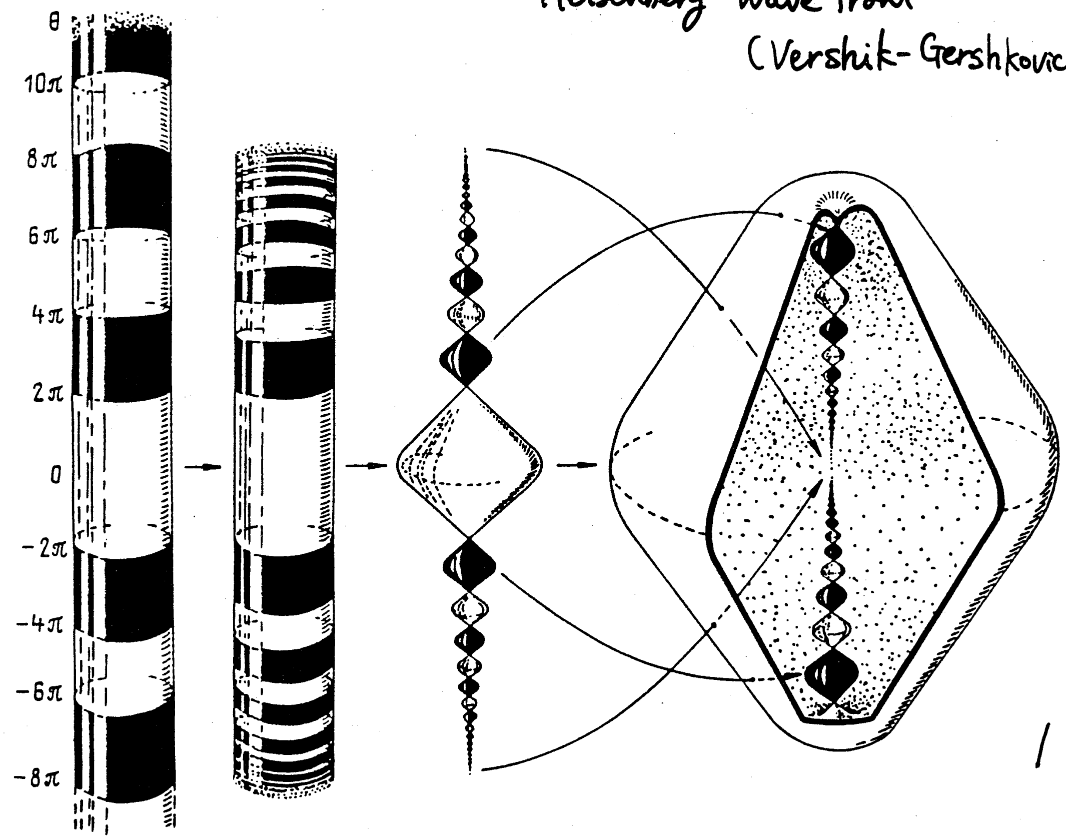
Heisenberg Wave front
(Vershik-Gershkovich)



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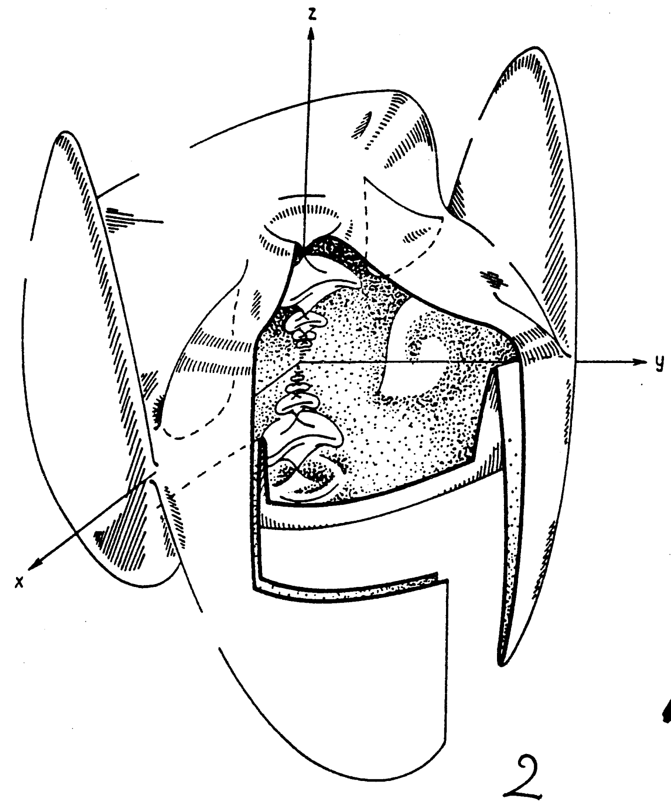

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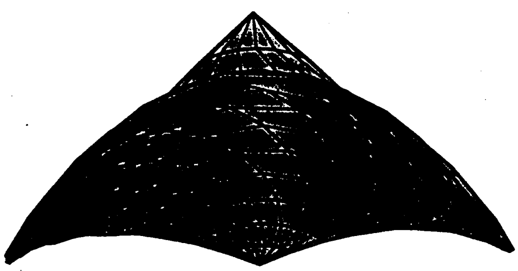

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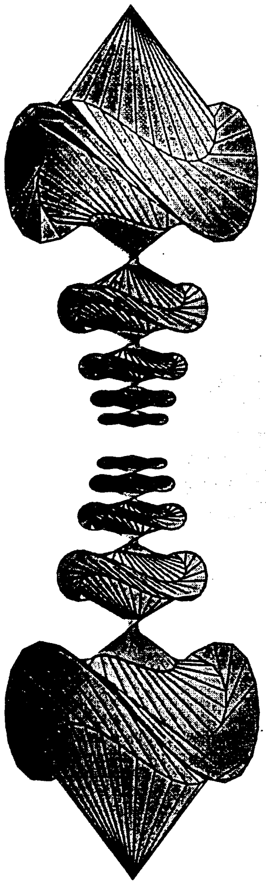

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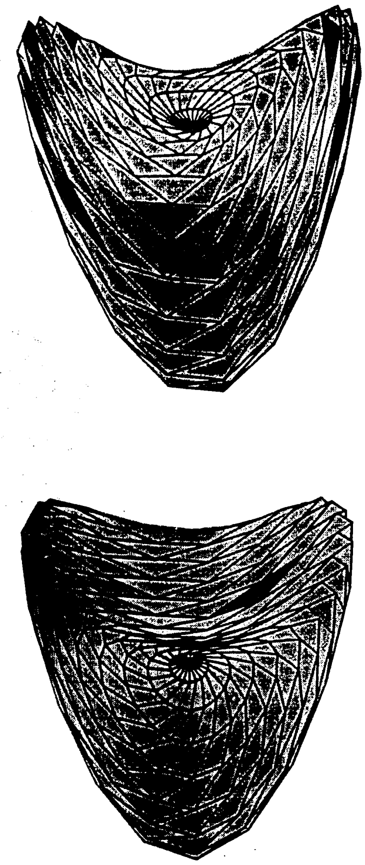

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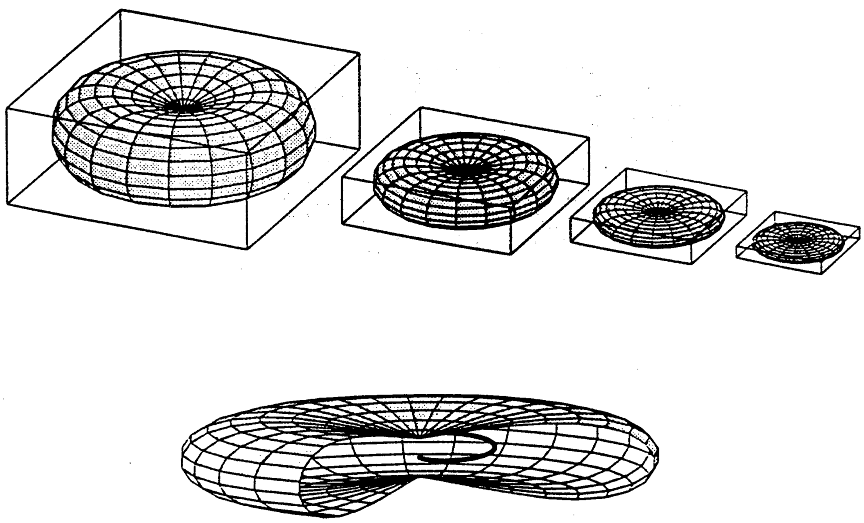
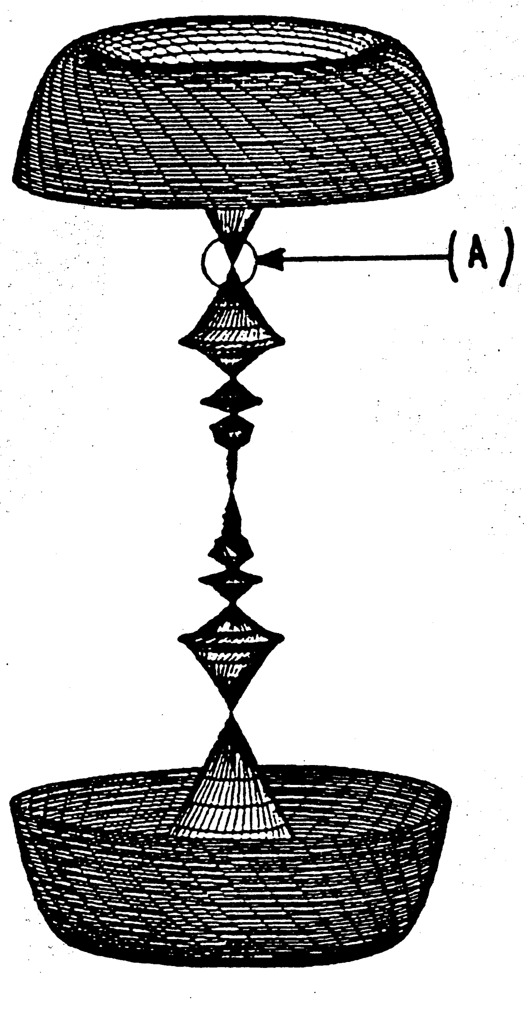


from Bellaïche 6

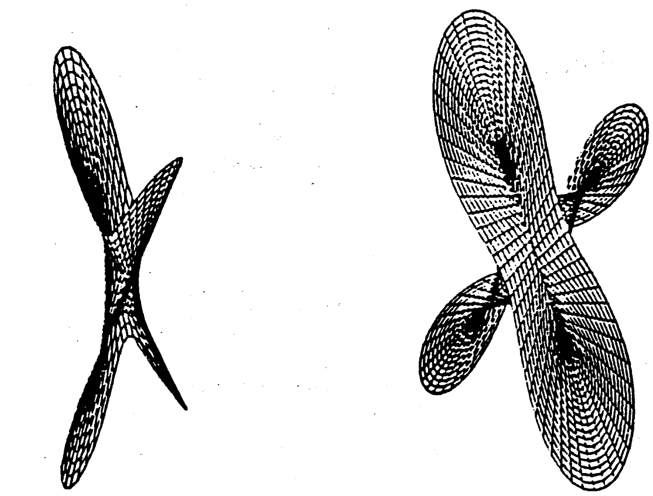
generic Sub-Riemannian wave front



7

from AAGK

generic Sub-Riemannian wave front



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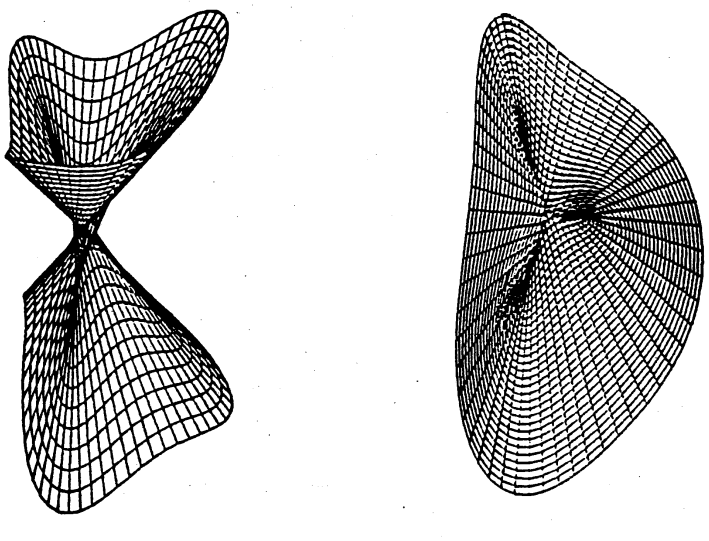
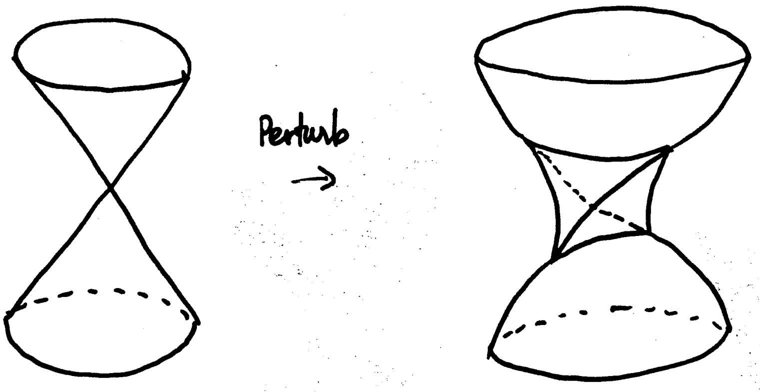
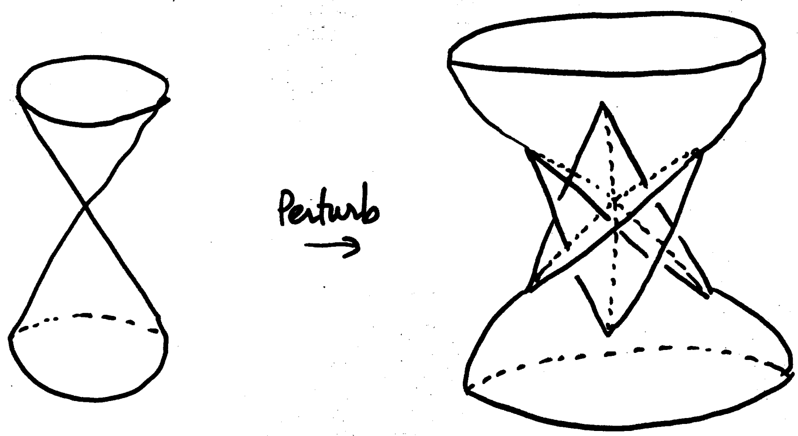


9

from AAGK

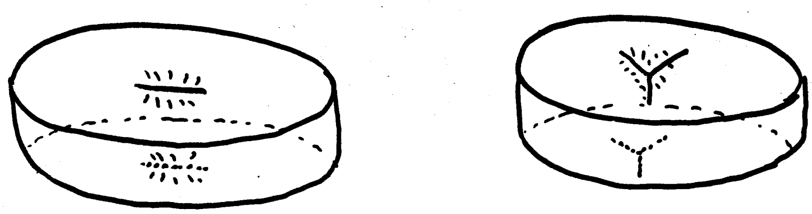
Sub-Riemannian perturbation of a cone (of a wavefront)





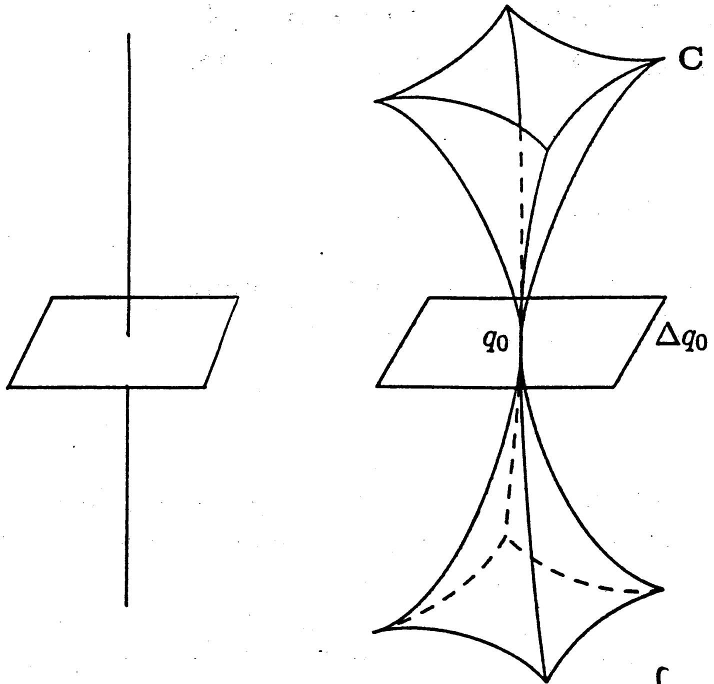
10

Generic C-C small ball



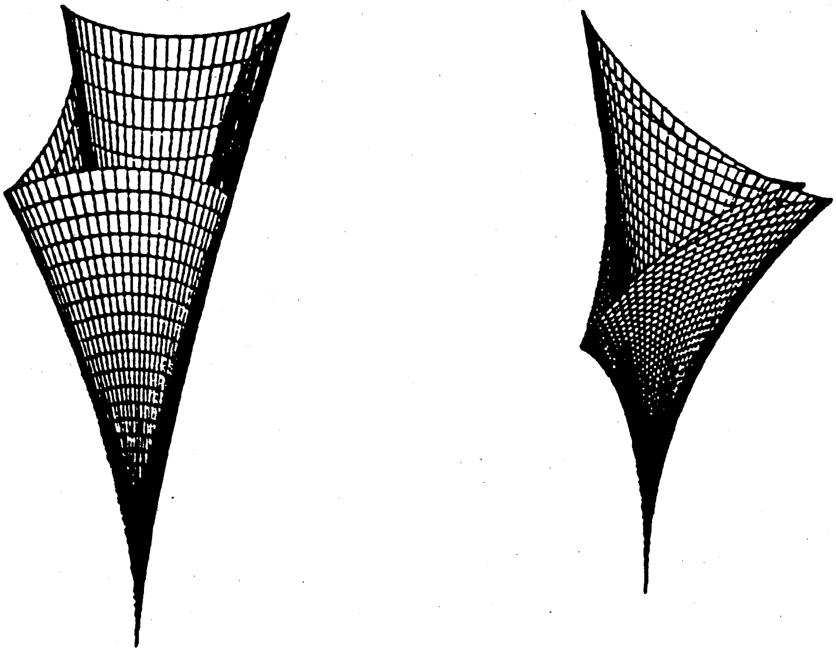
11

Generic sub-Riemannian Caustic.



12

from Agrachev.



from AAGK 13