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BAND GAP OF THE SPECTRUM IN PERIODICALLY-CURVED QUANTUM WAVEGUIDES

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1. INTRODUCTION

In this talk we study the band gap of the spectrum of the Dirichlet Laplacian $-\Delta^D_{\Omega_{d,\gamma}}$ in a strip $\Omega_{d,\gamma}$ in $\mathbf{R}^2$ with constant width d, where the signed curvature γ of the boundary curve is assumed to be periodic with respect to the arc length. Let us recall that a planer curve is uniquely determined by its signed curvature modulo congruent transformations (cf. [4]). Therefore without loss of generality, the boundary curve κ_γ takes the following form:

$$\kappa_\gamma(s) \equiv (a_\gamma(s), b_\gamma(s)), \tag{1.1}$$

$$a_\gamma(s) \equiv \int_0^s \cos\left(-\int_0^{s_1} \gamma(s_2)ds_2\right) ds_1, \tag{1.2}$$

$$b_\gamma(s) \equiv \int_0^s \sin\left(-\int_0^{s_1} \gamma(s_2)ds_2\right) ds_1. \tag{1.3}$$

For $d > 0$, we define

$$\Omega_{d,\gamma} \equiv \left\{ \left(a_\gamma(s) - u\frac{d}{ds}b_\gamma(s),\ b_\gamma(s) + u\frac{d}{ds}a_\gamma(s)\right) \in \mathbf{R}^2\ ;\ \ s \in \mathbf{R},\ 0 < u < d \right\}.$$

Roughly speaking, $\Omega_{d,\gamma}$ is the region obtained by sliding the normal segment of length d along κ_γ. We call κ_γ the reference curve of $\Omega_{d,\gamma}$. Let $-\Delta^D_{\Omega_{d,\gamma}}$ be the Dirichlet Laplacian on $\Omega_{d,\gamma}$. Namely, $-\Delta^D_{\Omega_{d,\gamma}}$ is the Friedrichs extension of the operator

$$-\Delta \quad \text{in} \quad L^2(\Omega_{d,\gamma}) \quad \text{with domain} \quad C_0^\infty(\Omega_{d,\gamma}).$$

$-\Delta^D_{\Omega_{d,\gamma}}$ is the Hamiltonian for an electron confined in a quantum wire on a planer substrate, where the vertical dimension is separated. A typical example of quantum wire is the GaAs-GaAlAs heterostructure. The first mathematical treatment of quantum waveguide (quantum wire) was done by Exner-Šeba (see [4]). Under a suitable decay conditions on $\gamma(s)$ and its derivatives as $s \to \pm\infty$, they proved that $-\Delta^D_{\Omega_{d,\gamma}}$ has at least one bound state for sufficiently small d. Recently, much progress is made by several authors. Bulla et al. (see [2]) and Exner-Vugalter (see [5] and [6]) studied the locally-deformed waveguides obtained by adding some bump to a straight strip or replacing the Dirichlet boundary condition by the Neumann boundary condition on a segment of the boundary of a straight strip. In these cases, they discussed the existence or non-existence of bound states below the essential spectrum.

In this paper, we consider the case that $\gamma(s)$ is periodic. We impose the following assumptions on γ.

$(A.1)$ $\gamma \in C^\infty(\mathbf{R})$.

$(A.2)$ $\gamma(s+2\pi) = \gamma(s)$ *for any* $s \in \mathbf{R}$.

$(A.3)$ *There exists* $d_0 > 0$ *such that*

(i) $-\frac{1}{d_0} < \min_{s\in[0,2\pi]} \gamma(s)$,

(ii) $\Omega_{d_0,\gamma}$ *is not self-intersecting i.e. the map*

$$\mathbf{R} \times (0, d_0) \ni (s,u) \mapsto \left(a_\gamma(s) - u\frac{d}{ds}b_\gamma(s),\ b_\gamma(s) + u\frac{d}{ds}a_\gamma(s)\right) \in \Omega_{d_0,\gamma}$$

is injective.

Simple coordinate transformations and standard elliptic a-priori estimates show that for $d \in (0, d_0]$, $-\Delta^D_{\Omega_{d,\gamma}}$ is unitarily equivalent to the following operator (see §2):

$$H_d \equiv -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) \quad \text{in} \quad L^2(\mathbf{R} \times (0,d)) \tag{1.4}$$

with domain

$$D_d \equiv \{v \in H^2(\mathbf{R} \times (0,d))\,; v(\cdot,0) = v(\cdot,d) = 0 \quad \text{in} \quad L^2(\mathbf{R})\}, \tag{1.5}$$

where

$$V(s,u) \equiv \frac{1}{2}(1+u\gamma(s))^{-3}u\gamma''(s) - \frac{5}{4}(1+u\gamma(s))^{-4}u^2\gamma'(s)^2 - \frac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2. \tag{1.6}$$

Since the coefficients of H_d are periodic with respect to s, one can utilize the Floquet-Bloch reduction scheme in the following way. First, (1.4) is unitarily equivalent to the operator

$$\int_{[0,1]}^{\oplus} H_{\theta,d}\, d\theta,$$

where

$$H_{\theta,d} \equiv -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) \quad \text{in} \quad L^2((0,2\pi) \times (0,d)) \tag{1.7}$$

with domain

$$\begin{aligned} D_{\theta,d} \equiv \{v(s,u) \in H^2((0,2\pi) \times (0,d))\,;\, & v(\cdot,0) = v(\cdot,d) = 0 \text{ in } L^2((0,2\pi)), \\ & v(2\pi,\cdot) = e^{2\pi i\theta}v(0,\cdot) \text{ in } L^2((0,d)), \\ & \tfrac{\partial}{\partial s}v(2\pi,\cdot) = e^{2\pi i\theta}\tfrac{\partial}{\partial s}v(0,\cdot) \text{ in } L^2((0,d))\} \end{aligned} \tag{1.8}$$

for $\theta \in [0,1]$.

We denote by $\mathcal{E}_j(\theta; d)$ the j-th eigenvalue of $H_{\theta,d}$ counted with multiplicity. Then, we have

$$\sigma(-\Delta^D_{\Omega_{d,\gamma}}) = \bigcup_{j=1}^{\infty} \mathcal{E}_j([0,1];d), \quad \text{where} \quad \mathcal{E}_j([0,1];d) = \bigcup_{\theta\in[0,1]} \{\mathcal{E}_j(\theta;d)\}.$$

So, the analysis of $\sigma(-\Delta^D_{\Omega_{d,\gamma}})$ is reduced to that of each $\mathcal{E}_j(\theta;d)$. $\mathcal{E}_j([0,1];d)$ is either a closed interval or a one point set. We call $\mathcal{E}_j([0,1];d)$ the j-th band of $\sigma(-\Delta^D_{\Omega_{d,\gamma}})$.

We consider the asymptotic behavior of $\mathcal{E}_j(\theta;d)$ as d tends to 0. For $\theta \in [0,1]$, let

$$K_\theta \equiv -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2((0,2\pi)) \tag{1.9}$$

with domain

$$F_\theta \equiv \{v \in H^2((0,2\pi))\,; v(2\pi) = e^{2\pi i\theta}v(0),\ v'(2\pi) = e^{2\pi i\theta}v'(0)\}.$$

We call K_θ the reference operator for $H_{\theta,d}$. We denote by $k_j(\theta)$ the j-th eigenvalue of K_θ counted with multiplicity. Then one of our main theorems is the following.

Theorem 1.1. *For $\theta \in [0,1]$ and $j \in \mathbf{N}$, we have*

$$\mathcal{E}_j(\theta; d) = \left(\frac{\pi}{d}\right)^2 + k_j(\theta) + O(d) \ \ (as\ d \to 0),$$

where the error term is uniform with respect to $\theta \in [0,1]$.

It follows from Theorem 1.1 that if there is a band gap of the spectrum for the operator $-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2$ in $L^2(\mathbf{R})$, so is the case for the operator $-\Delta^D_{\Omega_{d,\gamma}}$ for sufficiently small d. In particular, from the classical results about the inverse problem for Hill's equation (cf. [3], [7], and [10]), we have the following.

Corollary 1.2. *If γ is not identically 0, there exists some $j_0 \in \mathbf{N}$ and $C_{j_0} > 0$ such that*

$$\min_{\theta\in[0,1]} \mathcal{E}_{j_0+1}(\theta; d) - \max_{\theta\in[0,1]} \mathcal{E}_{j_0}(\theta; d) = C_{j_0} + O(d) \ \ (d \to 0). \tag{1.10}$$

This corollary says that if γ is not identically 0, at least one band gap appears in the spectrum for sufficiently small d. We prove these results in section 2.

In section 3, we locate the band gap of $\sigma(-\Delta^D_{\Omega_{d,\gamma}})$. Namely, we specify the value of $j_0 \in \mathbf{N}$ such that (1.10) holds. For this purpose, we use the scaling $\gamma \mapsto \epsilon\gamma$, where $\epsilon > 0$ is a small parameter. For $\epsilon > 0$ and $d > 0$, we set $\Omega^\epsilon_d = \Omega_{d,\epsilon\gamma}$. We consider $-\Delta^D_{\Omega^\epsilon_d}$ instead of $-\Delta^D_{\Omega_{d,\gamma}}$. We assume $(A.1)$, $(A.2)$, and the following.

$(A.4)$ *There exist $\epsilon_0 > 0$ and $d_0 > 0$ such that*

(i) $-\dfrac{1}{d_0} < \epsilon_0 \min_{s\in[0,2\pi]} \gamma(s)$,

(ii) *$\Omega^\epsilon_{d_0}$ is not self-intersecting for any $\epsilon \in (0, \epsilon_0]$.*

We substitute $\epsilon\gamma$ for γ in (1.4), (1.7), and (1.9), and denote the resulting operators by $H_{d,\epsilon}$, $H_{\theta,d,\epsilon}$, and $K_{\theta,\epsilon}$ respectively. We denote by $\mathcal{E}_j(\theta; d; \epsilon)$ the j-th eigenvalue of $H_{\theta,d,\epsilon}$ counted with multiplicity. Let $\{v_n\}_{n=-\infty}^{\infty}$ be the Fourier coefficients of $\gamma(s)^2$:

$$\gamma(s)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} v_n e^{ins} \quad \text{in} \quad L^2((0, 2\pi)).$$

Applying the analytic perturbation theory (cf. [8]) to the reference operator $K_{\theta,\epsilon}$, we get the following.

Theorem 1.3. *Let γ be not identically 0, and $n \in \mathbf{N}$ be such that $v_n \neq 0$. Then, there exists $\widetilde{\epsilon} \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \widetilde{\epsilon}\,]$, there exists $C_\epsilon > 0$ for which*

$$\min_{\theta\in[0,1]} \mathcal{E}_{n+1}(\theta; d; \epsilon) - \max_{\theta\in[0,1]} \mathcal{E}_n(\theta; d; \epsilon) = C_\epsilon + O(d) \ \ (d \to 0).$$

In the end of section 3, we give a simple example of $\gamma(s)$ satisfying $(A.3)$ or $(A.4)$.

2. Asymptotic Expansion of Band Functions and Existence of Band Gaps

Our main purpose in this section is to prove Theorem 1.1 and Corollary 1.2. We assume $(A.1)$, $(A.2)$, and $(A.3)$ throughout this section. As in [4], we first transform $-\Delta^D_{\Omega_{d,\gamma}}$ into the operator (1.4). We begin with the following remark.

For $d > 0$, we denote by Φ_d the map

$$\mathbf{R} \times (0,d) \ni (s,u) \mapsto \left(a_\gamma(s) - u\frac{d}{ds}b_\gamma(s), b_\gamma(s) + u\frac{d}{ds}a_\gamma(s)\right) \in \Omega_{d,\gamma}.$$

We denote by $J\Phi_d$ the Jacobian matrix of Φ_d. We have by a direct computation

$$\det(J\Phi_d)(s,u) = 1 + u\gamma(s) \quad \text{for} \quad (s,u) \in \mathbf{R} \times (0,d).$$

Then, **(i)** of $(A.3)$ implies that for $d \in (0, d_0]$,

$$\det(J\Phi_d)(s,u) \geq 1 + d_0\gamma_- > 0 \quad \text{for} \quad (s,u) \in \mathbf{R} \times (0,d),$$

where

$$\gamma_- = \min_{s\in[0,2\pi]} \min\{\gamma(s), 0\} \; (> -\tfrac{1}{d_0}). \tag{2.1}$$

So, Φ_d is a local diffeomorphism for $d \in (0, d_0]$. This and **(ii)** of $(A.3)$ imply that Φ_d is a global diffeomorphism for $d \in (0, d_0]$. We assume $d \in (0, d_0]$ throughout this section. For $f \in L^2(\Omega_{d,\gamma})$, we define

$$(U_d f)(s,u) \equiv (1 + u\gamma(s))^{1/2} f(\Phi_d(s,u)).$$

Then, U_d is a unitary operator from $L^2(\Omega_{d,\gamma})$ to $L^2(\mathbf{R} \times (0,d))$, and U_d maps $C_0^\infty(\Omega_{d,\gamma})$ into $C_0^\infty(\mathbf{R} \times (0,d))$ bijectively.

We are going to show that H_d in (1.4) and $H_{\theta,d}$ in (1.7) are self-adjoint with respective domains in (1.5) and (1.8), and the direct integral representation

$$H_d \cong \int_{[0,1]}^{\oplus} H_{\theta,d}\, d\theta.$$

We recall that $-\Delta^D_{\Omega_{d,\gamma}}$ is the Friedrichs extension of the operator

$$-\Delta \quad \text{in} \quad L^2(\Omega_{d,\gamma}) \quad \text{with domain} \quad C_0^\infty(\Omega_{d,\gamma}).$$

Let H_d° be the Friedrichs extension of the operator

$$-\frac{\partial}{\partial s}(1 + u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) \quad \text{in} \quad L^2(\mathbf{R} \times (0,d))$$

with domain $C_0^\infty(\mathbf{R} \times (0,d))$, where $V(s,u)$ is defined by (1.6). We recall (1.5):

$$D_d \equiv \{v \in H^2(\mathbf{R} \times (0,d)) \; ; \; v(\cdot,0) = v(\cdot,d) = 0 \quad \text{in} \quad L^2(\mathbf{R})\}.$$

Proposition 2.1. *We have*

$$U_d(-\Delta^D_{\Omega_{d,\gamma}})U_d^{-1} = H_d^\circ, \tag{2.2}$$

$$D_d \subset \mathcal{D}(H_d^\circ), \tag{2.3}$$

and

$$H_d^\circ v = -\frac{\partial}{\partial s}(1 + u\gamma(s))^{-2}\frac{\partial}{\partial s}v - \frac{\partial^2}{\partial u^2}v + V(s,u)v \quad \textit{for} \quad v \in D_d. \tag{2.4}$$

Proof. One can prove (2.2) by a direct computation and the first representation theorem. (2.3) and (2.4) follow from the first representation theorem and the following fact.

$C_0^\infty(\mathbf{R} \times (0,d))$ is dense in D_d with respect to the norm $\|\cdot\|_{H^1(\mathbf{R}\times(0,d))}$. □

Next we introduce the translational operators which reduce our problem to that of differential operators on a torus. We set

$$L \equiv 2\pi\mathbf{Z},\ \Lambda_d \equiv \mathbf{R} \times (0,d), \quad \text{and} \quad \Sigma_d \equiv (0,2\pi) \times (0,d).$$

For $l \in L$ and $v = v(s,u) \in L^2_{loc}(\Lambda_d)$, we define $T_l v \in L^2_{loc}(\Lambda_d)$ by

$$(T_l v)(s,u) \equiv v(s-l,u), \ \ (s,u) \in \Lambda_d.$$

$\{T_l\}_{l\in L}$ is an abelian group and each T_l commutes with H_d°. We define

$$\mathcal{B}_d \equiv \{v \in D_d\ ;\ \exists R > 0 \quad \text{s.t.} \quad \operatorname{supp} v \subset [-R,R] \times [0,d]\}.$$

One can easily see that $\mathcal{B}_d$ is dense in D_d with respect to the norm $\|\cdot\|_{H^2(\Lambda_d)}$. For $v \in \mathcal{B}_d$ and $\theta \in [0,1]$, we define

$$\begin{aligned}(\mathcal{U}v)(s,u,\theta) &\equiv \sum_{l\in L} e^{il\theta}(T_l v)(s,u)\\ &= \sum_{l\in L} e^{il\theta} v(s-l,u), \ \ (s,u) \in \Lambda_d.\end{aligned}$$

We easily see that for $l \in L$ and $\theta \in [0,1]$,

$$(\mathcal{U}v)(s+l,u,\theta) = e^{il\theta}(\mathcal{U}v)(s,u,\theta) \quad \text{in} \quad \Lambda_d.$$

Using the Parseval's identity, we have the following.

Proposition 2.2. *$\mathcal{U}$ is uniquely extended to a unitary operator from $L^2(\Lambda_d)$ to* $\mathcal{H} \equiv \int_{[0,1]}^{\oplus} L^2(\Sigma_d)\, d\theta.$

Now let us recall the operator $H_{\theta,d}$ defined by (1.7) with domain $D_{\theta,d}$ from (1.8). We prove the self-adjointness of $H_{\theta,d}$, which is not only important itself but also needed later to determine $\mathcal{D}(H_d^\circ)$ (equivalently $\mathcal{D}(-\Delta^D_{\Omega_{d,\gamma}})$) explicitly.

Proposition 2.3. *$H_{\theta,d}$ is self-adjoint.*

Proof. Using Green's formula, one can show that $H_{\theta,d}$ is symmetric.

We choose $k > 0$ such that

$$\inf_{(s,u)\in\Sigma_d} V(s,u) > -k.$$

Let us show the following.

(2.5) $H_{\theta,d} + k$ is 1 to 1 and onto. Namely, for any $f \in L^2(\Sigma_d)$, there exists unique $w \in D_{\theta,d}$ such that $(H_{\theta,d} + k)w = f$.

For convenience, we enlarge $\Sigma_d = (0,2\pi) \times (0,d)$. We choose $\epsilon \in (0,\pi)$. We set

$$\Sigma'_d \equiv (-\epsilon, 2\pi+\epsilon) \times (0,d),$$

and

$$\begin{aligned}Q_\theta \equiv \{v \in H^1(\Sigma'_d)\ ;\ & v(\cdot,0) = v(\cdot,d) = 0 \quad \text{in} \quad L^2((-\epsilon,2\pi+\epsilon)),\\ & v(s+2\pi,u) = e^{2\pi i\theta} v(s,u) \quad \text{a.e. in} \quad (-\epsilon,\epsilon)\times(0,d)\},\end{aligned}$$

equipped with the inner product

$$(v,w)_{Q_\theta} \equiv (v,w)_{H^1(\Sigma_d)}.$$

Then Q_θ is a Hilbert space. For $p \in (-\epsilon,\epsilon)$, we set $\Sigma_d^p \equiv (p, p+2\pi) \times (0,d)$. We define a quadratic form $q_\theta(\cdot,\cdot)$ on Q_θ by

$$q_\theta(v,w) \equiv \int_{\Sigma_d^p} \left\{ (1+u\gamma(s))^{-2} \frac{\partial}{\partial s} v \overline{\frac{\partial}{\partial s} w} + \frac{\partial}{\partial u} v \overline{\frac{\partial}{\partial u} w} + V(s,u) v\overline{w} + k v\overline{w} \right\} dsdu, \tag{2.6}$$

for $v, w \in Q_\theta$. We note that the right-hand side of (2.6) is independent of the choice of $p \in (-\epsilon,\epsilon)$. We easily see that

$$|q_\theta(v,w)| \leq C_1 \|v\|_{Q_\theta} \|w\|_{Q_\theta} \quad \text{for any} \quad v,w \in Q_\theta, \tag{2.7}$$

$$q_\theta(v,v) \geq C_2 \|v\|_{Q_\theta}^2 \quad \text{for any} \quad v \in Q_\theta, \tag{2.8}$$

where C_1 and $C_2 > 0$ are constants independent of $v, w \in Q_\theta$ and $v \in Q_\theta$ respectively. Let $f \in L^2(\Sigma_d)$. We extend f to the function in Σ_d' by

$$f(s,u) = \begin{cases} e^{2\pi i\theta} f(s-2\pi, u) & \text{for} \quad (s,u) \in (2\pi, 2\pi+\epsilon) \times (0,d), \\ e^{-2\pi i\theta} f(s+2\pi, u) & \text{for} \quad (s,u) \in (-\epsilon, 0) \times (0,d). \end{cases}$$

Because $(\cdot, f)_{L^2(\Sigma_d)}$ is a bounded linear functional on Q_θ, and q_θ satisfies (2.7) and (2.8), the Lax-Milgram theorem implies the following.

There exists unique $w \in Q_\theta$ such that

$$q_\theta(v,w) = (v,f)_{L^2(\Sigma_d)} \quad \text{for any} \quad v \in Q_\theta. \tag{2.9}$$

Next we show $w \in D_{\theta,d}$. For $y \in \mathbf{R}^2$ and $r > 0$, we set $B(y,r) \equiv \{x \in \mathbf{R}^2 \; ; \; |x-y| < r\}$. For $y \in \Sigma_d'$, there exists $r \in (0,\pi)$ such that $B(y,r) \subset\subset \Sigma_d'$. We choose $p \in (-\epsilon,\epsilon)$ such that $B(y,r) \subset \Sigma_d^p$. Let $l = (2\pi, 0)$. Then $B(y \pm l, r) \cap \Sigma_d^p = \emptyset$. For $v \in C_0^\infty(B(y,r))$, we extend v to the function in Q_θ and denote it by $\tilde{v}$. Then $\operatorname{supp} \tilde{v} \cap \Sigma_d^p \subset B(y,r)$. So, (2.6) and (2.9) imply that

$$\begin{aligned} &\left(\left\{ -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) + k \right\} v, w \right)_{L^2(B(y,r))} \\ =&(v,f)_{L^2(B(y,r))}, \end{aligned}$$

for any $v \in C_0^\infty(B(y,r))$. Therefore, the local regularity estimate for elliptic differential equations (cf. [1] Theorem 6.3.) implies

$$w \in H_{loc}^2(B(y,r)).$$

So, we get

$$w \in H_{loc}^2(\Sigma_d'). \tag{2.10}$$

For $y = (y_1, 0)$ $(y_1 \in (0,2\pi))$ and $r \in (0,\epsilon)$, we set $B_h(y,r) \equiv B(y,r) \cap \Sigma_d'$. Using the above method, we have

$$\begin{aligned} &\left(\left\{ -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) + k \right\} v, w \right)_{L^2(B_h(y,\epsilon))} \\ =&(v,f)_{L^2(B_h(y,\epsilon))}, \end{aligned}$$

for any $v \in C_0^\infty(B_h(y,\epsilon))$. Moreover $w \in Q_\theta$ implies

$$w(\cdot,0)=0 \quad \text{in} \quad L^2((-\epsilon+y_1, y_1+\epsilon)).$$

Hence, the regularity estimate up to the boundary for elliptic differential equations (cf. [1] Theorem 9.5.) implies

$$w \in H^2(B_h(y,\tfrac{\epsilon}{2})) \quad \text{for any} \quad y \in (0,2\pi)\times\{0\}. \tag{2.11}$$

Similarly, we have

$$w \in H^2(B_h(y,\tfrac{\epsilon}{2})) \quad \text{for any} \quad y \in (0,2\pi)\times\{d\}, \tag{2.12}$$

where $B_h(y,\frac{\epsilon}{2}) \equiv B(y,\frac{\epsilon}{2}) \cap \Sigma_d'$. So, (2.10), (2.11), and (2.12) imply that there exists $r \in (0,\epsilon)$ such that

$$w \in H^2((-r, 2\pi+r)\times(0,d)).$$

Combining this with $w \in Q_\theta$, we have

$$w(2\pi,\cdot) = e^{2\pi i\theta} w(0,\cdot) \quad \text{in} \quad L^2((0,d)),$$

and

$$\frac{\partial}{\partial s} w(2\pi,\cdot) = e^{2\pi i\theta} \frac{\partial}{\partial s} w(0,\cdot) \quad \text{in} \quad L^2((0,d)).$$

So, we get

$$w \in D_{\theta,d}.$$

Therefore, we can integrate (2.9) by parts, and get

$$(v, (H_{\theta,d}+k)w)_{L^2(\Sigma_d)} = (v,f)_{L^2(\Sigma_d)} \quad \text{for any} \quad v \in Q_\theta.$$

Hence, we have

$$(v, (H_{\theta,d}+k)w)_{L^2(\Sigma_d)} = (v,f)_{L^2(\Sigma_d)} \quad \text{for any} \quad v \in C_0^\infty(\Sigma_d).$$

Therefore,

$$(H_{\theta,d}+k)w = f, \quad w \in D_{\theta,d}.$$

On the other hand, we have

$$((H_{\theta,d}+k)v, v)_{L^2(\Sigma_d)} \geq \mu \|v\|^2_{L^2(\Sigma_d)} \quad \text{for any} \quad v \in D_{\theta,d}, \tag{2.13}$$

where $\mu = \inf_{(s,u)\in\Sigma_d} V(s,u) + k \ (>0)$. So we have shown (2.5).

Using (2.5) and (2.13), one can easily show that $H_{\theta,d}$ is closed. Thus, $H_{\theta,d}$ is self-adjoint. □

We recall the following operator defined by (1.4):

$$H_d \equiv -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u) \quad \text{in} \quad L^2(\Lambda_d)$$

with domain

$$D_d \equiv \{v \in H^2(\Lambda_d)\ ;\ v(\cdot,0)=v(\cdot,d)=0 \quad \text{in} \quad L^2(\mathbf{R})\}.$$

Recall H_d° defined in the head of this section. Then, we have the following.

Proposition 2.4. *We have*

$$H_d = \mathcal{U}^{-1}\left(\int_{[0,1]}^{\oplus} H_{\theta,d}\, d\theta\right)\mathcal{U}, \tag{2.14}$$

and

$$H_d^\circ = H_d. \tag{2.15}$$

Proof. Because one can easily show (2.14) by using a standard density argument, we show (2.15) only. Because $H_{\theta,d}$ is self-adjoint for $\theta \in [0,1]$ and $\mathcal{U}$ is unitary, (2.14) implies that H_d is self-adjoint. On the other hand, H_d° is a self-adjoint extension of H_d by Proposition 2.1. Therefore, we have $H_d = H_d^\circ$. This completes the proof of Proposition 2.4. □

Combining the above proposition with Proposition 2.1, $-\Delta^D_{\Omega_{d,\gamma}}$ is unitarily equivalent to $\int^{\oplus}_{[0,1]} H_{\theta,d}\, d\theta$. So, the analysis of $\sigma(-\Delta^D_{\Omega_{d,\gamma}})$ is precisely reduced to that of each $\sigma(H_{\theta,d})$.

As a final preliminary, we describe the band structure of $\sigma(-\Delta^D_{\Omega_{d,\gamma}})$. Because $H_{\theta,d}$ has a compact resolvent and is bounded from below, $\sigma(H_{\theta,d})$ is discrete. As we have defined in section 1, for $j \in \mathbf{N}$, $\mathcal{E}_j(\theta;d)$ denotes the j-th eigenvalue of $H_{\theta,d}$ counted with multiplicity:

$$\mathcal{E}_1(\theta;d) \le \mathcal{E}_2(\theta;d) \le \cdots \le \mathcal{E}_j(\theta;d) \le \cdots \to \infty.$$

One can easily show that $\mathcal{E}_j(\cdot;d)$ is Lipschitz continuous. Therefore,

$$\mathcal{E}_j([0,1];d) = \bigcup_{\theta\in[0,1]} \{\mathcal{E}_j(\theta;d)\}$$

is either a closed interval or a one-point set for $j \in \mathbf{N}$. We have also that

$$\sigma(-\Delta^D_{\Omega_{d,\gamma}}) = \bigcup_{j=1}^{\infty} \mathcal{E}_j([0,1];d).$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. In this proof, we mainly use the min-max principle. As in [4], we first introduce an approximate operator for $H_{\theta,d}$. We recall (2.1):

$$\gamma_- \equiv \min_{s\in[0,2\pi]} \min\{\gamma(s),0\} \ \left(> -\tfrac{1}{d_0}\right).$$

Let

$$\gamma_+ \equiv \max_{s\in[0,2\pi]} \max\{\gamma(s),0\}.$$

Then, we have for any $d \in (0,d_0)$,

$$0 < (1+d\gamma_+)^{-1} \le (1+u\gamma(s))^{-1} \le (1+d\gamma_-)^{-1} \quad \text{on} \quad \Sigma_d. \tag{2.16}$$

We define

$$V_+(s) \equiv \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma''_+ - \frac{1}{4}(1+d\gamma_+)^{-2}\gamma(s)^2,$$

and

$$V_-(s) \equiv -\frac{1}{2}(1+d\gamma_-)^{-3} d\gamma''_+ - \frac{5}{4}(1+d\gamma_-)^{-4} d^2(\gamma'_+)^2 - \frac{1}{4}(1+d\gamma_-)^{-2}\gamma(s)^2,$$

where

$$\gamma''_+ \equiv \max_{s\in[0,2\pi]} |\gamma''(s)| \ (<\infty), \text{ and } \gamma'_+ \equiv \max_{s\in[0,2\pi]} |\gamma'(s)| \ (<\infty).$$

Then, $V_+(s)$ and $V_-(s)$ satisfy the following.

$$V_-(s) \le V(s,u) \le V_+(s) \quad \text{on} \quad \Sigma_d. \tag{2.17}$$

We define the following approximate operators similar to those of [4]. For $\theta \in [0,1]$, we define

$$H^{\pm}_{\theta,d} \equiv -(1+d\gamma_{\mp})^{-2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + V_{\pm}(s) \quad \text{in} \quad L^2(\Sigma_d) \quad \text{with domain} \quad D_{\theta,d}.$$

We note that both $H^+_{\theta,d}$ and $H^-_{\theta,d}$ are self-adjoint and have compact resolvents. According to (2.16) and (2.17), we have

$$H^-_{\theta,d} \le H_{\theta,d} \le H^+_{\theta,d}. \tag{2.18}$$

We estimate the eigenvalues of $H^+_{\theta,d}$ and $H^-_{\theta,d}$. For this purpose we introduce the following operators. For $\theta \in [0,1]$, we define

$$T^{\pm}_{\theta,d} \equiv -(1+d\gamma_{\mp})^{-2}\frac{d^2}{ds^2} + V_{\pm}(s) \quad \text{in} \quad L^2((0,2\pi)) \quad \text{with domain} \quad F_\theta,$$

where

$$F_\theta \equiv \{v \in H^2((0,2\pi))\ ;\ v(2\pi) = e^{2\pi i\theta}v(0),\ v'(2\pi) = e^{2\pi i\theta}v'(0)\}.$$

Both $T^+_{\theta,d}$ and $T^-_{\theta,d}$ are self-adjoint and have compact resolvents. For $j \in \mathbf{N}$, we denote dy $\mathcal{E}^{\pm}_j(\theta;d)$ the j-th eigenvalue of $T^{\pm}_{\theta,d}$ counted with multiplicity respectively. Let $\{\phi^{\pm}_j\}_{j=1}^{\infty}$ be the complete orthonormal system of $L^2((0,2\pi))$, where $\phi^{\pm}_j(\theta,d,\cdot)$ is the eigenfunction of $T^{\pm}_{\theta,d}$ associated with the eigenvalue $\mathcal{E}^{\pm}_j(\theta;d)$. We have $\phi^{\pm}_j(\theta,d,\cdot) \in F_\theta \cap C^\infty([0,2\pi])$. We further introduce the following operator

$$-\frac{d^2}{du^2} \quad \text{in} \quad L^2((0,d)) \quad \text{with domain} \quad \{v \in H^2((0,d))\ ;\ v(0)=v(d)=0\}. \tag{2.19}$$

For $k \in \mathbf{N}$, the k-th eigenvalue of (2.19) is $(\frac{\pi k}{d})^2$. The associated eigenfunction is $\sqrt{\frac{2}{d}}\sin(\frac{\pi k}{d}u)$. We have also that $\left\{\sqrt{\frac{2}{d}}\sin(\frac{\pi k}{d}u)\right\}_{k=1}^{\infty}$ is a complete orthonormal system of $L^2((0,d))$. We set

$$\psi^{\pm}_{j,k}(\theta,d,s,u) \equiv \phi^{\pm}_j(\theta,d,s)\sqrt{\frac{2}{d}}\sin\left(\frac{\pi k}{d}u\right)$$

for $(s,u) \in \Sigma_d$ and $j,k \in \mathbf{N}$. Then we have for any $j,k \in \mathbf{N}$,

$$\psi^{\pm}_{j,k}(\theta,d,\cdot,\cdot) \in D_{\theta,d},$$

and

$$H^{\pm}_{\theta,d}\psi^{\pm}_{j,k}(\theta,d,\cdot,\cdot) = \mu^{\pm}(j;k;\theta;d)\psi^{\pm}_{j,k}(\theta,d,\cdot,\cdot),$$

where

$$\mu^{\pm}(j;k;\theta;d) \equiv \left(\frac{\pi k}{d}\right)^2 + \mathcal{E}^{\pm}_j(\theta;d). \tag{2.20}$$

Moreover, $\{\psi^{\pm}_{j,k}(\theta,d,\cdot,\cdot)\}_{j,k\in\mathbf{N}}$ is a complete orthonormal system of $L^2(\Sigma_d)$. Therefore, $\{\mu^{\pm}(j;k;\theta;d)\}_{j,k\in\mathbf{N}}$ is the set of all eigenvalues of $H^{\pm}_{\theta,d}$ counted with multiplicity.

We estimate $\mu^{\pm}(j;k;\theta;d)$. We recall (1.9): for $\theta \in [0,1]$,

$$K_\theta \equiv -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2((0,2\pi)) \quad \text{with domain} \quad F_\theta,$$

and that $k_j(\theta)$ denotes the j-th eigenvalue of K_θ counted with multiplicity for $j \in \mathbf{N}$.

We first show the following.

For any $j \in \mathbf{N}$, *we have*

$$\mathcal{E}_j^+(\theta; d) = k_j(\theta) + O_j(d) \ \ (d \to 0), \tag{2.21}$$

and

$$\mathcal{E}_j^-(\theta; d) = k_j(\theta) + O_j(d) \ \ (d \to 0), \tag{2.22}$$

where each error term is uniform with respect to $\theta \in [0,1]$.

We rewrite $T^+_{\theta,d}$ and $T^-_{\theta,d}$ as follows.

$$\begin{aligned} T^+_{\theta,d} =& (1+d\gamma_-)^{-2}\left\{-\frac{d^2}{ds^2} - \frac{1}{4}(1+d\gamma_-)^2(1+d\gamma_+)^{-2}\gamma(s)^2\right\} \\ &+ \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+''. \end{aligned} \tag{2.23}$$

$$\begin{aligned} T^-_{\theta,d} =& (1+d\gamma_+)^{-2}\left\{-\frac{d^2}{ds^2} - \frac{1}{4}(1+d\gamma_+)^2(1+d\gamma_-)^{-2}\gamma(s)^2\right\} \\ &- \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+'' - \frac{5}{4}(1+d\gamma_-)^{-4} d^2(\gamma_+')^2. \end{aligned} \tag{2.24}$$

A straightfoward calculation follows that

$$(1+d\gamma_-)^2(1+d\gamma_+)^{-2} - 1 = \frac{d(\gamma_- - \gamma_+)\{2 + d(\gamma_+ + \gamma_-)\}}{(1+d\gamma_+)^2} \le 0, \tag{2.25}$$

and

$$(1+d\gamma_+)^2(1+d\gamma_-)^{-2} - 1 = \frac{d(\gamma_+ - \gamma_-)\{2 + d(\gamma_+ + \gamma_-)\}}{(1+d\gamma_-)^2} \ge 0. \tag{2.26}$$

We set

$$\gamma_1 \equiv \max_{s\in[0,2\pi]} |\gamma(s)|.$$

Then, (2.23) $\sim$ (2.26) implies

$$\begin{aligned} &(1+d\gamma_-)^{-2}\left(-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2\right) + \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+'' \\ \le& T^+_{\theta,d} \\ \le& (1+d\gamma_-)^{-2}\left(-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2\right) + \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+'' \\ &+ \frac{1}{4}\cdot\frac{d(\gamma_+ - \gamma_-)\{2 + d(\gamma_+ + \gamma_-)\}}{(1+d\gamma_+)^2(1+d\gamma_-)^2}\gamma_1^2 \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} &(1+d\gamma_+)^{-2}\left(-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2\right) - \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+'' \\ &- \frac{1}{4}\cdot\frac{d(\gamma_+ - \gamma_-)\{2 + d(\gamma_+ + \gamma_-)\}}{(1+d\gamma_+)^2(1+d\gamma_-)^2}\gamma_1^2 - \frac{5}{4}(1+d\gamma_-)^{-4} d^2(\gamma_+')^2 \\ \le& T^-_{\theta,d} \\ \le& (1+d\gamma_+)^{-2}\left(-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2\right) - \frac{1}{2}(1+d\gamma_-)^{-3} d\gamma_+'' \\ &- \frac{5}{4}(1+d\gamma_-)^{-4} d^2(\gamma_+')^2. \end{aligned} \tag{2.28}$$

Applying the min-max principle to (2.27) and (2.28), we get

$$\mathcal{E}_j^+(\theta; d) = (1 + d\gamma_-)^{-2} k_j(\theta) + O(d) \quad (d \to 0),$$

and

$$\mathcal{E}_j^-(\theta; d) = (1 + d\gamma_+)^{-2} k_j(\theta) + O(d) \quad (d \to 0),$$

where each error term is uniform with respect to $\theta \in [0,1]$. Because $k_j(\cdot)$ is continuous on $[0,1]$, we get (2.21) and (2.22).

Next we show the following.

For any $j_0 \in \mathbf{N}$, there exists $\tilde{d} = \tilde{d}(j_0)$ such that for any $d < \tilde{d}$,

$$\mu^{\pm}(j; k; \theta; d) \geq \mu^{\pm}(j_0; 1; \theta; d) + 1 \tag{2.29}$$

for any $k \geq 2$, $j \geq 1$, and $\theta \in [0,1]$.

We fix any $j_0 \in \mathbf{N}$. Using (2.20), we have for any $k \geq 2$, $j \in \mathbf{N}$, and $\theta \in [0,1]$,

$$\begin{aligned}
&\mu^{\pm}(j; k; \theta; d) - \mu^{\pm}(j_0; 1; \theta; d) \\
=&\frac{\pi^2(k^2-1)}{d^2} + \mathcal{E}_j^{\pm}(\theta; d) - \mathcal{E}_{j_0}^{\pm}(\theta; d) \\
\geq&\frac{3\pi^2}{d^2} + \mathcal{E}_1^{\pm}(\theta; d) - \mathcal{E}_{j_0}^{\pm}(\theta; d) \\
\geq&\frac{3\pi^2}{d^2} + \min_{\theta \in [0,1]} k_1(\theta) + O(d) - \max_{\theta \in [0,1]} k_{j_0}(\theta) + O_{j_0}(d),
\end{aligned}$$

where we used (2.21) and (2.22) in the fourth line. Therefore, we have (2.29).

(2.29) implies the following. For any $j_0 \in \mathbf{N}$, there exists $\tilde{d} = \tilde{d}(j_0)$ such that for any $d < \tilde{d}$, the j-th eigenvalue of $H_{\theta,d}^{\pm}$ is $\mu^{\pm}(j; 1; \theta; d)$ for any $j \leq j_0$ and $\theta \in [0,1]$. Hence, (2.18) and the min-max principle imply the following. For any $d < \tilde{d}(j_0)$, we have

$$\mu^-(j; 1; \theta; d) \leq \mathcal{E}_j(\theta; d) \leq \mu^+(j; 1; \theta; d)$$

for any $j \leq j_0$ and $\theta \in [0,1]$. Using (2.20), (2.21), and (2.22), we have

$$\mathcal{E}_{j_0}(\theta; d) = \left(\frac{\pi}{d}\right)^2 + k_{j_0}(\theta) + O(d) \quad (d \to 0),$$

where the error term is uniform with respect to $\theta \in [0,1]$. This completes the proof of Theorem 1.1 □

Next we prove the existence of the band gap of $\sigma(-\Delta_{\Omega_{d,\gamma}}^D)$. Recall (1.9). We define

$$K \equiv -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2(\mathbf{R}) \quad \text{with domain} \quad H^2(\mathbf{R}).$$

The following facts are well-known (see [9] Chapter XIII16).

i) Each $k_j(\cdot)$ is continuous on $[0,1]$, and

$$k_j(1-\theta) = k_j(\theta) \quad \text{for any} \quad \theta \in [0,1].$$

ii) For j odd (even), $k_j(\theta)$ is monotone increasing (decreasing) as θ increases from 0 to $\frac{1}{2}$. Especially, we have

$$k_1(0) < k_1(\tfrac{1}{2}) \le k_2(\tfrac{1}{2}) < k_2(0) \le \cdots \le k_{2j-1}(0) < k_{2j-1}(\tfrac{1}{2}) \le k_{2j}(\tfrac{1}{2}) < k_{2j}(0) \le \cdots.$$

iii) We set

$$B_j \equiv \begin{cases} [k_j(0), k_j(\frac{1}{2})] & \text{(for } j \text{ odd)}, \\ [k_j(\frac{1}{2}), k_j(0)] & \text{(for } j \text{ even)}, \end{cases}$$

and

$$G_j \equiv \begin{cases} (k_j(\frac{1}{2}), k_{j+1}(\frac{1}{2})) & \text{(for } j \text{ odd such that } k_j(\frac{1}{2}) \ne k_{j+1}(\frac{1}{2})), \\ (k_j(0), k_{j+1}(0)) & \text{(for } j \text{ even such that } k_j(0) \ne k_{j+1}(0)), \\ \emptyset & \text{(otherwise)}. \end{cases}$$

Then, we have

$$\sigma(K) = \bigcup_{j=1}^{\infty} B_j.$$

We call B_j the j-th band of $\sigma(K)$, and G_j the gap of $\sigma(K)$ if $G_j \ne \emptyset$.

Combining these with Theorem 1.1, we have the following.

Corollary 2.5. *For each $l \in \mathbf{N}$, we have*

$$\min_{\theta\in[0,1]} \mathcal{E}_{l+1}(\theta; d) - \max_{\theta\in[0,1]} \mathcal{E}_l(\theta; d) = |G_l| + O(d) \quad (d \to 0), \tag{2.30}$$

and

$$|\mathcal{E}_l([0,1]; d)| = |B_l| + O(d) \quad (d \to 0),$$

where $|\cdot|$ is the Lebesgue measure.

Here we recall the following classical result about the inverse problem for Hill's equation, which was first proved by Borg ([3]). For alternative proofs, see [7] and [10].

Theorem (Borg). *Suppose that W is a real-valued, piecewise continuous function on $[0, 2\pi]$. Let $\lambda_j^{\pm}$ be the j-th eigenvalue of the following operator counted with multiplicity respectively*

$$-\frac{d^2}{ds^2} + W(s) \quad in \quad L^2((0, 2\pi))$$

with domain

$$\{v \in H^2((0, 2\pi))\ ;\ v(2\pi) = \pm v(0),\ v'(2\pi) = \pm v'(0)\}.$$

We suppose that

$$\lambda_j^+ = \lambda_{j+1}^+ \quad for\ all\ even \quad j,$$

and

$$\lambda_j^- = \lambda_{j+1}^- \quad for\ all\ odd \quad j.$$

Then, W is constant on $[0, 2\pi]$.

We are now in a position to prove Corollary 1.2.

Proof of Colollary 1.2. In virtue of (2.30), it suffices to show the following:

There exists some $j_0 \in \mathbf{N}$ such that $G_{j_0} \ne \emptyset$.

We show this by contradiction. We suppose that $G_j = \emptyset$ for all $j \in \mathbf{N}$. Then, the above Borg's theorem and $(A.2)$ impliy that $\gamma(s)^2$ is constant in $\mathbf{R}$. Because γ is continuous, $\gamma(s)$ is constant in $\mathbf{R}$. On the other hand, γ is not identically 0 by assumption. So, there exists a constant $k \ne 0$ such that $\gamma(s) = k$ in $\mathbf{R}$. Then, the reference curve κ_γ must be a circumference of radius $\frac{1}{|k|}$. This violates (**ii**) of $(A.3)$. □

3. Location of Band Gaps

The purpose in this section is to prove Theorem 1.3. First we recall (2.30):

$$\min_{\theta\in[0,1]} \mathcal{E}_{l+1}(\theta;d) - \max_{\theta\in[0,1]} \mathcal{E}_l(\theta;d) = |G_l| + O(d) \ \ (d \to 0).$$

So, we will specify the value of $l \in \mathbf{N}$ such that $|G_l| > 0$. For this purpose, we use the scaling $\gamma \mapsto \epsilon\gamma$, where $\epsilon > 0$ is a small parameter. As we have introduced in section 1, we set $\Omega_d^\epsilon = \Omega_{d,\epsilon\gamma}$. Now we consider $-\Delta^D_{\Omega_d^\epsilon}$ instead of $-\Delta^D_{\Omega_{d,\gamma}}$. We assume $(A.1)$, $(A.2)$, and $(A.4)$. Besides, we assume $d \in (0, d_0]$ and $\epsilon \in (0, \epsilon_0]$. We recall that $H_{d,\epsilon}$, $H_{\theta,d,\epsilon}$, and $K_{\theta,\epsilon}$ denote the operators obtained by substituting $\epsilon\gamma$ for γ in (1.4), (1.7), and (1.9) respectively, and $\mathcal{E}_j(\theta;d;\epsilon)$ denotes the j-th eigenvalue of $H_{\theta,d,\epsilon}$ counted with multiplicity. Especially, $K_{\theta,\epsilon}$ has the following expression.

$$K_{\theta,\epsilon} \equiv -\frac{d^2}{ds^2} - \frac{1}{4}\epsilon^2\gamma(s)^2 \quad \text{in} \quad L^2((0,2\pi)) \quad \text{with domain} \quad F_\theta \tag{3.1}$$

for $\theta \in [0,1]$. Let $k_j(\theta;\epsilon)$ be the j-th eigenvalue of $K_{\theta,\epsilon}$ counted with multiplicity. We set

$$G_j(\epsilon) = \begin{cases} (k_j(\frac{1}{2};\epsilon), k_{j+1}(\frac{1}{2};\epsilon)) & (\text{for } j \text{ odd such that } k_j(\frac{1}{2};\epsilon) \neq k_{j+1}(\frac{1}{2};\epsilon)), \\ (k_j(0;\epsilon), k_{j+1}(0;\epsilon)) & (\text{for } j \text{ even such that } k_j(0;\epsilon) \neq k_{j+1}(0;\epsilon)), \\ \emptyset & (\text{otherwise}). \end{cases} \tag{3.2}$$

Then, (2.30) implies that for each $l \in \mathbf{N}$ and $\epsilon \in (0, \epsilon_0]$, we have

$$\min_{\theta\in[0,1]} \mathcal{E}_{l+1}(\theta;d;\epsilon) - \max_{\theta\in[0,1]} \mathcal{E}_l(\theta;d;\epsilon) = |G_l(\epsilon)| + O_\epsilon(d) \ \ (d \to 0). \tag{3.3}$$

We consider the asymptotic behavior of $|G_l(\epsilon)|$ as ϵ tends to 0, and specify the value of $l \in \mathbf{N}$ such that $|G_l(\epsilon)| > 0$ for sufficiently small ϵ. For this purpose, we use the analytic perturbation theory (see [8]).

To treat this problem in a more general situation, we introduce the new notations. Let

$(A.6)$ $\quad W \in L^2([0,2\pi]\,;\,\mathbf{R})$.

We define

$$L_0^+ \equiv -\frac{d^2}{ds^2} \quad \text{in} \quad L^2((0,2\pi))$$

with domain

$$F_0 \equiv \{v \in H^2((0,2\pi))\,;\, v(2\pi) = v(0),\ v'(2\pi) = v'(0)\},$$

and

$$L_0^- \equiv -\frac{d^2}{ds^2} \quad \text{in} \quad L^2((0,2\pi))$$

with domain

$$F_{\frac{1}{2}} \equiv \{v \in H^2((0,2\pi))\,;\, v(2\pi) = -v(0),\ v'(2\pi) = -v'(0)\}.$$

For $\beta \in \mathbf{C}$, we define

$$\begin{aligned} L^+(\beta) &\equiv L_0^+ + \beta W \\ &= -\frac{d^2}{ds^2} + \beta W(s) \quad \text{in} \quad L^2((0,2\pi)) \quad \text{with domain} \quad F_0, \end{aligned}$$

and

$$L^-(\beta) \equiv L_0^- + \beta W$$
$$= -\frac{d^2}{ds^2} + \beta W(s) \quad \text{in} \quad L^2((0,2\pi)) \quad \text{with domain} \quad F_{\frac{1}{2}}.$$

We regard $L_0^\pm$ as the unperturbed operators and βW as a perturbation. For $\beta \in \mathbf{R}$, we denote by $l_j^\pm(\beta)$ the j-th eigenvalue of $L^\pm(\beta)$ counted with multiplicity. Let us write down the eigenvalues and eigenfunctions of unperturbed operator $L_0^\pm$. For $n \in \mathbf{N}$, we set

$$\psi_0 = \frac{1}{\sqrt{2\pi}}, \ \psi_{n,1} = \frac{1}{\sqrt{2\pi}}e^{ins}, \ \psi_{n,2} = \frac{1}{\sqrt{2\pi}}e^{-ins}. \tag{3.4}$$

Then, $\{\psi_0, \ \psi_{n,1}, \ \psi_{n,2}\}_{n\in\mathbf{N}}$ is a complete orthonormal system of $L^2((0,2\pi))$, and we have

$$L_0^+\psi_0 = 0, \ L_0^+\psi_{n,1} = n^2\psi_{n,1}, \ L_0^+\psi_{n,2} = n^2\psi_{n,2} \tag{3.5}$$

for $n \in \mathbf{N}$. Therefore, we have

$$l_1^+ = 0, \ l_{2n}^+(0) = l_{2n+1}^+(0) = n^2 \quad \text{for} \quad n \in \mathbf{N}. \tag{3.6}$$

Next, for $n \in \mathbf{N}$, we set

$$\varphi_{n,1} = \frac{1}{\sqrt{2\pi}}e^{i(n-\frac{1}{2})s}, \ \varphi_{n,2} = \frac{1}{\sqrt{2\pi}}e^{-i(n-\frac{1}{2})s}. \tag{3.7}$$

Then, $\{\varphi_{n,1}, \ \varphi_{n,2}\}_{n\in\mathbf{N}}$ is a complete orthonormal system of $L^2((0,2\pi))$, and we have

$$L_0^-\varphi_{n,1} = \left(n - \frac{1}{2}\right)^2 \varphi_{n,1}, \ L_0^-\varphi_{n,2} = \left(n - \frac{1}{2}\right)^2 \varphi_{n,2} \tag{3.8}$$

for $n \in \mathbf{N}$. Therefore, we have

$$l_{2n-1}^-(0) = l_{2n}^-(0) = \left(n - \frac{1}{2}\right)^2 \quad \text{for} \quad n \in \mathbf{N}. \tag{3.9}$$

For $\beta \in \mathbf{R}$ and $n \in \mathbf{N}$, we set

$$\delta_n^+(\beta) \equiv l_{2n+1}^+(\beta) - l_{2n}^+(\beta), \tag{3.10}$$

and

$$\delta_n^-(\beta) \equiv l_{2n}^-(\beta) - l_{2n-1}^-(\beta). \tag{3.11}$$

For $\beta \in \mathbf{R}$ and $n \in \mathbf{N}$, we define

$$\delta_{2n-1}(\beta) \equiv \delta_n^-(\beta), \ \delta_{2n}(\beta) \equiv \delta_n^+(\beta).$$

We consider the asymptotic expansion of $\delta_n(\beta)$ as $\beta \to 0$. Now we recall the analytic perturbation theorem due to Kato and Rellich, which is rewritten suitably in our situation (see [8] Chapter VII and Theorem 2.6 in Chapter VIII).

Theorem (Kato and Rellich). *Let H_0 be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Suppose that E_0 is an eigenvalue of H_0 with multiplicity m, and there exists $\epsilon > 0$ such that $\sigma(H_0) \cap (E_0 - \epsilon, E_0 + \epsilon) = \{E_0\}$. Let $\{\Omega_1, \cdots, \Omega_m\}$ be an orthonormal system of eigenvectors of H_0 associated with the eigenvalue E_0:*

$$\Omega_j \in \mathcal{D}(H_0),\ H_0\Omega_j = E_0\Omega_j \quad for \quad 1 \leq j \leq m,$$

$$(\Omega_i, \Omega_j)_{\mathcal{H}} = \delta_{ij} \quad for \quad 1 \leq i,j \leq m.$$

Let V be a symmetric, H_0-bounded operator. For $\beta \in \mathbf{C}$, we define

$$H(\beta) \equiv H_0 + \beta V \quad in \quad \mathcal{H} \quad with\ domain \quad \mathcal{D}(H_0).$$

Let $\mu_1, \cdots, \mu_m$ be the all eigenvalues of the matrix $((V\Omega_i, \Omega_j)_{\mathcal{H}})_{1\leq i,j\leq m}$ counted with multiplicity. Then, we have the following.

There exist m (single-valued) analytic functions $u_1(\beta), \cdots, u_m(\beta)$ in $\beta \in \mathbf{C}$ near 0 such that $u_1(\beta), \cdots, u_m(\beta)$ are the all eigenvalues of $H(\beta)$ counted with multiplicity in $\{E \in \mathbf{C}\ ;\ |E - E_0| < \frac{\epsilon}{2}\}$ for $\beta \in \mathbf{C}$ near 0, and

$$u_j(\beta) = E_0 + \mu_j\beta + O(|\beta|^2) \ \ (\beta \to 0) \tag{3.12}$$

for $1 \leq j \leq m$.

As a simple consequence of this theorem, we can see the splitting of a doubly degenerate eigenvalue when a purturbation is turned on:

Corollary 3.1. *Let $m = 2$ in the statement of above theorem. Then, we have*

$$\begin{aligned}&(u_1(\beta) - u_2(\beta))^2\\ =&[\{(V\Omega_1, \Omega_1)_{\mathcal{H}} - (V\Omega_2, \Omega_2)_{\mathcal{H}}\}^2 + 4|(V\Omega_1, \Omega_2)_{\mathcal{H}}|^2]\beta^2 + O(|\beta|^3)\\ &(\beta \to 0).\end{aligned} \tag{3.13}$$

Using this corollary, we compute the asymptotic expansion of $\delta_n(\beta)$ as $\beta \to 0$. Let $\{w_n\}_{n=-\infty}^{\infty}$ be the Fourier coefficients of $W(s)$:

$$W(s) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} w_n e^{ins} \quad \text{in} \quad L^2((0, 2\pi)). \tag{3.14}$$

Because W is real-valued, we have

$$w_n = \overline{w_{-n}} \quad \text{for} \quad n \in \mathbf{Z}. \tag{3.15}$$

Then, we have the following.

Theorem 3.2. *For each $n \in \mathbf{N}$, we have*

$$\delta_n(\beta) = \sqrt{\frac{2}{\pi}}\, |w_n| \cdot |\beta| + O(|\beta|^2) \ \ (\beta \to 0, \beta \in \mathbf{R}). \tag{3.16}$$

Proof. We show (3.16) only for even n because odd case is similar. We recall (3.4), (3.5), (3.6), and (3.10). We apply the preceding Kato and Rellich's theorem and Corollary 3.1 by setting

$$\mathcal{H} \equiv L^2((0, 2\pi)),\ H_0 \equiv L_0^+,\ V \equiv W,\ m = 2, \tag{3.17}$$

$$E_0 \equiv l_{2n}^+(0) = l_{2n+1}^+(0) = n^2,$$

$$\Omega_1 \equiv \psi_{n,1} = \frac{1}{\sqrt{2\pi}} e^{ins}, \ \Omega_2 \equiv \psi_{n,2} = \frac{1}{\sqrt{2\pi}} e^{-ins} \quad \text{for} \quad n \in \mathbf{N}.$$

Let $u_1(\beta)$ and $u_2(\beta)$ be as in the preceding theorem under the situation (3.17). Then, we have

$$\begin{aligned}
&(\delta_n^+(\beta))^2 \\
=&(l_{2n+1}^+(\beta) - l_{2n}^+(\beta))^2 \\
=&(u_1(\beta) - u_2(\beta))^2 \\
=&[\{(W\psi_{n,1}, \psi_{n,1})_{\mathcal{H}} - (W\psi_{n,2}, \psi_{n,2})_{\mathcal{H}}\}^2 + 4|(W\psi_{n,1}, \psi_{n,2})_{\mathcal{H}}|^2]\beta^2 + O(|\beta|^3) \\
&(\beta \to 0, \ \beta \in \mathbf{R}).
\end{aligned}$$

We compute

$$\begin{aligned}
&\{(W\psi_{n,1}, \psi_{n,1})_{\mathcal{H}} - (W\psi_{n,2}, \psi_{n,2})_{\mathcal{H}}\}^2 + 4|(W\psi_{n,1}, \psi_{n,2})_{\mathcal{H}}|^2 \\
=&\left(\int_0^{2\pi} W(s)ds - \int_0^{2\pi} W(s)ds\right)^2 + 4\left|\frac{1}{2\pi}\int_0^{2\pi} W(s)e^{2ins}ds\right|^2 \\
=&4\left|\frac{1}{\sqrt{2\pi}} w_{-2n}\right|^2 \\
=&\frac{2}{\pi}|w_{2n}|^2,
\end{aligned}$$

where we used (3.14) in the third line and (3.15) in the fourth line. Thus we have

$$\begin{aligned}
(\delta_{2n}(\beta))^2 =&(\delta_n^+(\beta))^2 \\
=&\frac{2}{\pi}|w_{2n}|^2\beta^2 + O(|\beta|^3) \ \ (\beta \to 0, \ \beta \in \mathbf{R}).
\end{aligned} \tag{3.18}$$

We note that $u_1(\beta) - u_2(\beta)$ is analytic in $\beta \in \mathbf{C}$ near 0, and

$$\delta_n^+(\beta) = |u_1(\beta) - u_2(\beta)| \quad \text{for} \quad \beta \in \mathbf{R} \quad \text{near} \quad 0.$$

Then, (3.18) implies

$$\begin{aligned}
\delta_{2n}(\beta) =&\delta_n^+(\beta) \\
=&\sqrt{\frac{2}{\pi}}\,|w_{2n}| \cdot |\beta| + O(|\beta|^2) \ \ (\beta \to 0, \ \beta \in \mathbf{R}).
\end{aligned}$$

Thus we showed (3.16) for even n. □

Now we turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. We recall (3.1), (3.2), and (3.3). As we have introduced in section 1, $\{v_n\}_{n=-\infty}^{\infty}$ denote the Fourier coefficients of $\gamma(s)^2$:

$$\gamma(s)^2 = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} v_n e^{ins} \quad \text{in} \quad L^2((0, 2\pi)).$$

By assumption, $\gamma^2 \neq 0$ in $L^2((0, 2\pi))$. Let $n \in \mathbf{N}$ be such that $v_n \neq 0$. Then, Theorem 3.2 implies that

$$|G_n(\epsilon)| = \frac{1}{4}\sqrt{\frac{2}{\pi}}\epsilon^2|v_n| + O(\epsilon^4) \ \ (\epsilon \to 0).$$

Therefore, $|G_n(\epsilon)| > 0$ for sufficiently small ϵ. Combining this with (3.3), we get the conclusion. □

Next we give an example of $\gamma(s)$ satisfying $(A.3)$ or $(A.4)$. Now we suppose $(A.1)$ and $(A.2)$. For $\epsilon \in (0,1]$, we define

$$\kappa_\gamma^\epsilon \ : \ \mathbf{R} \ni s \mapsto (a_\gamma^\epsilon(s), b_\gamma^\epsilon(s)) \in \mathbf{R}^2,$$

where

$$a_\gamma^\epsilon(s) \equiv \int_0^s \cos(-\epsilon h(s_1))\, ds_1,$$

$$b_\gamma^\epsilon(s) \equiv \int_0^s \sin(-\epsilon h(s_1))\, ds_1,$$

$$h(s) \equiv \int_0^s \gamma(s_2)\, ds_2.$$

Then, κ_γ^ϵ is a C^∞ curve whose curvature at $\kappa_\gamma^\epsilon(s)$ is $\epsilon\gamma(s)$. We define a map Φ^ϵ by

$$\Phi^\epsilon \ : \ \mathbf{R}^2 \ni (s,u) \mapsto \left(a_\gamma^\epsilon(s) - u\frac{d}{ds}b_\gamma^\epsilon(s),\ b_\gamma^\epsilon(s) + u\frac{d}{ds}a_\gamma^\epsilon(s) \right) \in \mathbf{R}^2.$$

Then, we have the following.

Proposition 3.5. *Suppose*

$$h(2\pi) = 0,$$

and

$$\max_{s\in[0,2\pi]} |h(s)| < \frac{\pi}{2}.$$

Then, there exists some $d_0 > 0$ such that for any $\epsilon \in (0,1]$, $\Phi^\epsilon|_{\mathbf{R}\times(0,d_0)}$ is injective.

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