Title	Eigenvalue Asymptotics for the Schrodinger Operator with Asymptotically Flat Magnetic Fields and Decreasing Electric Potential(Spectral and Scattering Theory and Its Related Topics)
Author(s)	SHIRAI, S.; IWATSUKA, A.
Citation	数理解析研究所講究録 (1997), 994: 184-194
Issue Date	1997-05
URL	http://hdl.handle.net/2433/61195
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
994 巻 1997 年 184-194

Eigenvalue Asymptotics for the Schrödinger Operator with Asymptotically Flat Magnetic Fields and Decreasing Electric Potential

SHIRAI, S. and IWATSUKA, A.

Faculty of Sciences, Osaka University
Faculty of Sciences, Kyoto University

1 Introduction

We investigate the asymptotic distribution of eigenvalues of the two dimensional Schrödinger operator with an electromagnetic potential. We consider the operator in $L^2(\boldsymbol{R}^2)$ of the form :

$$H_V = -\frac{\partial^2}{\partial x_1^2} + \left(\frac{1}{i}\frac{\partial}{\partial x_2} - b(x_1)\right)^2 + V(x_1, x_2),$$

where $(0, b(x_1))$ is the (magnetic) vector potential which gives a perturbed constant magnetic field and $V(x_1, x_2)$ is the (electric) scalar potential decaying at infinity.

First, we shall consider the magnetic field $B(x_1)$ obeying :

(B.1) $B(x_1) \in C^2(\boldsymbol{R}; \boldsymbol{R})$, real-valued C^2-functions on $\boldsymbol{R}$. Moreover, $B(x_1)$ is a monotone increasing in x_1 and there exist positive numbers $B_\pm > 0$ such that

$$B_- < B_+ ,$$

$$\lim_{x_1 \to \pm\infty} B(x_1) = B_\pm .$$

Under the assumption (B.1), we define the vector potential $b(x_1)$ as follows.

$$b(x_1) = \int_0^{x_1} B(t)dt\,.$$

In the case where $V(x_1, x_2) \equiv 0$, the spectrum of H_0 has a band structure if (B.1) holds (See, [Iwa]):

$$\sigma(H_0) = \sigma_{ac}(H_0) = \bigcup_{n=1}^{\infty} [\Lambda_n^-, \Lambda_n^+],$$

$$\Lambda_n^{\pm} = (2n-1)B_{\pm}.$$

(B.2)$_\pm$ In addition to (B.1), $B(x_1) \in \mathcal{B}^\infty(\boldsymbol{R})$, moreover, there exists $M > 0$ such that, for each $\alpha \in \boldsymbol{N} \cup \{0\}$,

$$|\,\partial_1^\alpha(B_\pm - B(x_1))| \leq C_{M\alpha}\langle x_1\rangle^{-M} \quad \text{as } x_1 \to \pm\infty$$

holds for some constant $C_{M\alpha}$, where

$$\mathcal{B}^\infty(\boldsymbol{R}) = \{f \in C^\infty(\boldsymbol{R})|\text{ for each } \alpha, \|\partial^\alpha f\|_\infty < \infty\}\ ,$$

and $\|\cdot\|_\infty$ denotes the usual L^∞-norm.

(B.3) In addition to (B.1), assume that $B(x_1)$ fulfills the following conditions:

$$\begin{aligned} B_+ &< 3B_-\ , \\ \|\partial_1 B\|_\infty &\leq B_+ - B_-\ , \\ (B_+ - B_-)(1 &+ \frac{1}{\sqrt{3B_- - B_+}}) < \frac{B_+ + B_-}{6}\ . \end{aligned}$$

(V.1) $V(x_1, x_2) \in C^\infty(\boldsymbol{R}^2; \boldsymbol{R})$, real-valued C^∞-functions on $\boldsymbol{R}^2$, and there exists $m > 0$ such that

$$\left|\partial_1^\alpha \partial_2^\beta V(x_1, x_2)\right| \leq C_{\alpha\beta}\langle x_1; x_2\rangle^{-m-\alpha-\beta}$$

holds for some positive constant $C_{\alpha\beta}$ independent of (x_1, x_2) in $\boldsymbol{R}^2$.

Here ∂_1, ∂_2 denotes $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ respectively and $\langle x_1; x_2\rangle = (1 + x_1^2 + x_2^2)^{\frac{1}{2}}$.

It is well-known that the operator H_V defined on $C_0^\infty(\boldsymbol{R}^2)$ is essentially self-adjoint and V is a relatively compact perturbation with respect to H_0 ([L-S]). Thus one expects that H_V have discrete spectra in the spectral gaps of H_0 and they accumulate at most to the tips of the gap. In the case where $b(x_1)$ is the vector potential which gives a constant magnetic field, the eigenvalue asymptotics around the essential spectrum tips is investigated ([Rai1], [Rai2]).

For $\mu > 0, a_0 \in \boldsymbol{R}$, define

$$\nu_+^{\pm}(\mu; a_0) = \frac{1}{2\pi}\text{vol}\{(x_1, x_2) \in \boldsymbol{R}^2 | x_1 > a_0, \pm V(x_1, x_2) > \mu\}.$$

and

$$\nu_-^{\pm}(\mu; a_0) = \frac{1}{2\pi}\text{vol}\{(x_1, x_2) \in \boldsymbol{R}^2 | -x_1 > a_0, \pm V(x_1, x_2) > \mu\}.$$

For simplicity, we denote $\nu_+^+(\mu; a_0), \nu_-^-(\mu; a_0)$ by $\nu_+(\mu; a_0), \nu_-(\mu; a_0)$ respectively.

For a positive, decreasing function f, we say that f satisfies (T) if

(T) there exist positive numbers γ, γ', μ_0 such that

$$\frac{f(\mu_1)}{f(\mu_2)} \leq \left(\frac{\mu_2}{\mu_1}\right)^{\gamma} \tag{1.1}$$

holds for $\mu_1, \mu_2 \in (0, \mu_0)$ with $\mu_1 < \mu_2$. Moreover

$$f(\mu) \geq \gamma' \mu^{-\frac{2}{m}}$$

holds for $\mu \in (0, \mu_0)$.

Let S be a self-adjoint operator in a Hilbert space, and suppose S has purely discrete spectra in an open interval $(a, b) \subset \boldsymbol{R}$. Then $N((a, b)|S)$ denotes the total multiplicity of the eigenvalues of S lying on (a, b), i.e.,

$$N((a, b)|S) = \dim(\text{Ran}E_S(a, b))$$

where $E_S(a, b)$ denotes the spectral projection of S on (a, b).

We devote ourself to get the asymptotics at some specific gap such that

$$\Lambda_n^+ < \Lambda_{n+1}^-$$

holds where $\Lambda_0^+ = 0$. Thus we shall consider such a gap.

One of the main theorems is:

Theorem 1.1 *Suppose that (V.1), (B.2)$_+$ (resp. (B.2)$_-$) and (B.3) with $m < M$. Moreover suppose that $\nu_+(\mu; a_0)$ (resp. $\nu_-(\mu; a_0)$) satisfies (T) and $\nu_+^-(\mu; a_0)$ (resp. $\nu_-^+(\mu; a_0)$) satysfies (1.1). Then we have*

$$N((\Lambda_n^+ + \mu, M_n)|H_V) = B_+\nu_+(\mu; a_0)(1 + o(1)) \quad as \quad \mu \downarrow 0,$$
$$\Big(resp.\ N((M_n, \Lambda_{n+1}^- - \mu)|H_V) = B_-\nu_-(\mu; a_0)(1 + o(1)) \quad as \quad \mu \downarrow 0,\Big)$$

where we put $M_n = \frac{\Lambda_n^+ + \Lambda_{n+1}^-}{2}$

Remark 1.1 *(i)As $\mu \downarrow 0$, the asymptotic behabior of $\nu_+^{\pm}(\mu; a_0)$ does not depend on a choice of $a_0 > 0$. (A similar assertion holds for $\nu_-^{\pm}(\mu; a_0)$.)*

(ii)In case $V(x_1, x_2)$ is non-negative (resp. non-positive), it follows from the proof that the assumption on $\nu_-(\mu; a_0)$ (resp. $\nu_+(\mu; a_0)$) is not needed.

In the case where the scalar potential $V(x_1, x_2)$ decays slowly, i.e., of order m with $0 < m < 1$, satisfying the assumption (V.2) with the constant m, some of assumptions on $V(x_1, x_2)$ and $B(x_1)$ can be weakend:

(V.2) $V(x_1, x_2) \in C^2(\boldsymbol{R}^2; \boldsymbol{R})$ and there exist $m, m', C > 0$ such that

$$\begin{aligned} 0 < m < 1 \quad &, \quad 2m < m' , \\ |V(x_1, x_2)| \ &\leq \ C\langle x_1; x_2\rangle^{-m} , \\ |\,\partial_1 V(x_1, x_2)| \ + \ &|\,\partial_2 V(x_1, x_2)| \leq C\langle x_1; x_2\rangle^{-m'} . \end{aligned}$$

(B.4)$_\pm$ In addition to (B.1), there exist constants M, M', C such that

$$\begin{aligned} M' \ &> \ 3M \\ |B(x_1) - B_\pm| \ &\leq \ C\langle x_1\rangle^{-M} \quad \text{as } x_1 \to \pm\infty, \\ |\,\partial_1 B(x_1)| \ &\leq \ C\langle x_1\rangle^{-M'} \quad \text{as } x_1 \to \pm\infty . \end{aligned}$$

The other of the main theorems is:

Theorem 1.2 *Suppose that (V.2) and (B.4)$_+$ (resp. (B.4)$_-$) hold with $M > m$. And suppose that $\nu_+(\mu; a_0)$ (resp. $\nu_-(\mu; a_0)$) satisfies (T). Then we have the same eigenvalue asymptotics as in Theorem 1.1.*

We shall give only a proof of Theorem 1.2 in the following sections. Theorem 1.2 can be prove using the min-max principle and estimates of the number of eigenvalues of self-adjoint operators associated with suitable quadratic forms derived from the results in [Col].

2 Direct integral decomposition

[Iwa] proved that H_0 is unitarily equivalent to the self-adjoint operator L acting in $L^2(\boldsymbol{R}_{x_1} \times \boldsymbol{R}_\xi)$ that has the (constant fiber) direct integral decomposition (see, e.g., [R-S4]) :

$$L = \int_{\boldsymbol{R}_\xi}^{\oplus} L(\xi) d\xi , \tag{2.1}$$

using the partial Fourier transformation

$$(\mathcal{F}u)(x_1, \xi) = (2\pi)^{-\frac{1}{2}} \int e^{-ix_2\xi} u(x_1, x_2) dx_2 \tag{2.2}$$

which is a unitary operator from $L(\boldsymbol{R}_{x_1} \times \boldsymbol{R}_{x_2})$ to $L(\boldsymbol{R}_{x_1} \times \boldsymbol{R}_{\xi})$. Here for each ξ in $\boldsymbol{R}$, $L(\xi)$ is a second-order ordinary differential operator in $L^2(\boldsymbol{R}_{x_1})$ of the form :

$$L(\xi) = -\frac{d^2}{dx_1^2} + (\xi - b(x_1))^2 . \tag{2.3}$$

Lemma 2.1 **([Iwa])** *Assume that (B.1) holds. Let ξ be a real number. Then there exists a complete orthonormal system $\{\varphi_n(x_1,\xi)\}_{n=1}^{\infty}$ in $L^2(\boldsymbol{R}_{x_1})$ of eigenfunctions for $L(\xi)$:*

$$L(\xi)\varphi_n(x_1,\xi) = \lambda_n(\xi)\varphi_n(x_1,\xi) , \tag{2.4}$$

$$0 < \lambda_1(\xi) < \lambda_2(\xi) < \lambda_3(\xi) < \cdots \to \infty , \tag{2.5}$$

so that for $n \in \mathbf{N}$

(i) each $\lambda_n(\xi)$ is non-degenerate, and depends analytically on ξ ,

(ii) $\lambda_n(\xi)$ is monotone increasing in ξ, and $\lim_{\xi\to\pm\infty} \lambda_n(\xi) = \Lambda_n^{\pm}$,

(iii) $\varphi_n(\cdot,\xi) \in D(L(0))$ and depends analytically on ξ with respect to the graph norm $\|u\|_{1,0} \equiv (\|u\|^2 + \|L(0)u\|^2)^{\frac{1}{2}}$, where $\|\cdot\|$ stands for the L^2-norm.

(iv) $\varphi_n(x_1,\xi)$ is a real-valued continuous function of x_1 and ξ , and, moreover $\varphi_n(x_1,\xi)$ is infinitely differentiable in x_1 for each ξ and is analytic in ξ for each x_1.

Proof. See lemma 2.3 and a remark at the end of [Iwa]. □

Now we consider the following assumption on the eigenvalues $\{\lambda_n(\xi)\}$:

(Λ.1) There exists a constant $C > 0$ such that for $j,k \in \mathbf{N}$, $j \neq k$, $|\lambda_j(\xi) - \lambda_k(\xi)| \geq C$ holds for all $\xi \in \boldsymbol{R}$.

Although it is not trivial whether the (non-constant) magnetic fields satisfying this condition (Λ.1) in addition to (B.2) exist, but the following lemma gives an answer. We shall give the proof in Sect.12.

Lemma 2.2 *(B.3) implies (Λ.1).*

Hence we get Theorem 1.1 if only we prove the following theorem:

Theorem 2.3 *Under the same assumptions as in Theorem 1.1, except that (B. 3) is replaed by (Λ.1), we have the same eigenvalue asymptotics.*

3 Proof of Theorem 2.3

In the proof of Theorem 2.3, we denote the variables (x_1, x_2) by (x, y) for notational convinience. Coresponding to this, ∂_1, ∂_2 shall be replaced by ∂_x, ∂_y etc. And we shall often denote by C various (positive) constants appeared in estimates. In the case where we want to specify the dependence of some constants, we shall denote them by $C_\varepsilon, C(\eta)$ or $C_{\alpha,\beta}$ etc.

Using the partial Fourier transformation $\mathcal{F}$ defined by (2.2), we consider the operator L_V as follows.

$$L_V = L + \mathcal{F}V\mathcal{F}^{-1} \qquad in\ L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)\ , \tag{3.1}$$

where V stands for the multiplication operator by $V(x, y)$ in $L^2(\boldsymbol{R}_x \times \boldsymbol{R}_y)$ (Generally we shall use the notation f to the multiplication operator $f(x)$ acting in a function space throughout this paper).

In the sequel we denote $\mathcal{F}V\mathcal{F}^{-1}$ by $\widetilde{V}$.

Lemma 3.1 **([Iwa])** *Assume $(B.1)$ holds. For each $n \in \boldsymbol{N}$, let $\mathcal{H}_n$ be the closed subspace of $L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ defined by*

$$\mathcal{H}_n = \{\varphi_n(x,\xi)f(\xi) | f(\xi) \in L^2(\boldsymbol{R}_\xi)\} \tag{3.2}$$

where $\varphi_n(x,\xi)$ is as in Lemma 2.1. Then we have

(i) $L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi) = \sum_n \oplus \mathcal{H}_n$ (the orthogonal sum of Hilbert spaces).

(ii) L is reduced by $\mathcal{H}_n$.

(iii) $L|_{\mathcal{H}_n}$ (restriction of L to $\mathcal{H}_n$) is unitarily equivalent to the operator of multiplication by $\lambda_n(\xi)$ on $L^2(\boldsymbol{R}_\xi)\}$.

where $\varphi_n(x,\xi)$ and $\lambda_n(\xi)$ is as in Lemma 2.1.

Proof. See [Iwa], Lemma 2.5. □

We define the operator

$$T_n : L^2(\boldsymbol{R}_\xi) \longrightarrow \mathcal{H}_n \left(\hookrightarrow L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)\right)$$

by

$$(T_n f)(x,\xi) = \varphi_n(x,\xi)f(\xi) \tag{3.3}$$

for $f(\xi) \in L^2(\boldsymbol{R}_\xi)$ (then we can find

$$T_n^* : \mathcal{H}_n \longrightarrow L^2(\boldsymbol{R}_\xi)$$

by

$$(T_n^* F)(\xi) = \int_{\boldsymbol{R}_x} \varphi_n(x,\xi)F(x,\xi)dx \tag{3.4}$$

for $F(x,\xi) \in \mathcal{H}_n$) , and define $P_n : L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi) \longrightarrow L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ by

$$(P_n u)(x,\xi) = \varphi_n(x,\xi) \int_{\boldsymbol{R}_x} \varphi_n(x,\xi) u(x,\xi) dx \tag{3.5}$$

for $u(x,\xi) \in L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$.

Note that P_n is the orthogonal projection with the range $\mathcal{H}_n$ and T_n is a unitary operator from $L^2(\boldsymbol{R}_\xi)$ to $\mathcal{H}_n$ which gives the equivalence stated in Lemma 2.1(iii). Furthermore $T_n T_n^* P_n = P_n$ on $L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ holds.

Lemma 3.2 *$\widetilde{V} P_n$ is a compact operator on $L^2(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$.*

Proof. We can find that

$$\begin{aligned} \widetilde{V} P_n &= \mathcal{F} V \mathcal{F}^{-1} P_n && (3.6)\\ &= \mathcal{F} V (H_0 - i)^{-1} \mathcal{F}^{-1} \mathcal{F} (H_0 - i) \mathcal{F}^{-1} P_n && (3.7)\\ &= \mathcal{F} V (H_0 - i)^{-1} \mathcal{F}^{-1} (L - i) P_n . && (3.8) \end{aligned}$$

Then $\widetilde{V} P_n$ is compact, since $V(H_0 - i)^{-1}$ is compact(see [A-H-S], Theorem 2.6) and $(L-i)P_n$ is bounded by Lemma 3.1(ii) and the closed graph theorem. □

Set

$$K = -i(P_n \widetilde{V} - \widetilde{V} P_n)$$

and denote by K_+, K_- the positive and negative part of K respectively so that $K = K_+ - K_-, |K| = K_+ + K_-$. Further, for $\varepsilon > 0$ denote by $Y_n^{\pm}(\varepsilon)$ the operator associated with the quadratic form

$$(Y_n^{\pm}(\varepsilon)u, u) = \|i\varepsilon K_+^{\frac{1}{2}} Q_n u \pm \varepsilon K_+^{\frac{1}{2}} P_n u\|^2 + \|i\varepsilon K_-^{\frac{1}{2}} Q_n u \mp \varepsilon K_-^{\frac{1}{2}} P_n u\|^2 \tag{3.9}$$

for $u \in L^2(\boldsymbol{R}^2)$, where $Q_n = I - P_n$. Throughout this paper, $(\cdot,\cdot)$ stands for the standard inner product of L^2. It is easy to see, by the definition, that $Y_n^{\pm}(\varepsilon)$ are compact and nonnegative self-adjoint operators. And direct computations lead us to

$$L_V = P_n L_V P_n + Q_n(L_V \pm \varepsilon^{-2}|K|)Q_n \pm \varepsilon^2 P_n |K| P_n \mp Y_n^{\pm}(\varepsilon) . \tag{3.10}$$

Let us prepare a useful inequality called the Weyl-Ky Fan inequality. We shall frequently make use of it to estimate the upper bound of the number of eigenvalues:

Lemma 3.3 ([Rai1]) *Let A_0, A_1 be bounded self-adjoint operators acting on a Hilbert space. Assume A_1 is compact and set $A = A_0 + A_1$. Then the estimates*

$$\begin{aligned} N((\mu_1,\mu_2)|A_0) \le \quad & N((\mu_1 - \tau_1, \mu_2 + \tau_2)|A) \\ + \quad & N((\tau_1,\infty)| - A_1) + N((\tau_2,\infty)|A_1) \end{aligned}$$

hold for each interval $(\mu_1,\mu_2) \subset \boldsymbol{R}$ and every $\tau_1 > 0, \tau_2 > 0$.

Proof. See [Rai1], Lemma 5.4. □

Applying Lemma 3.3 to the former of (3.10) twice (first, with $A_1 = Y_n^+(\varepsilon)$, $A_0 = L_V$, $\mu_1 = \Lambda_n^+ + \mu$, $\mu_2 = M_n$, $\tau_1 = \frac{\varepsilon}{2}\mu$, $\tau_2 = \frac{\varepsilon}{2}$, and, second, with $A_1 = P_n|K|P_n$, $A = P_nL_VP_n + Q_n(L_V + \varepsilon^{-2}|K|)Q_n$, $\nu_1 = \Lambda_n^+ + (1-\frac{\varepsilon}{2})\mu$, $\mu_2 = M_n + \frac{\varepsilon}{2}$, $\tau_1 = \frac{\varepsilon}{2}\mu$, $\tau_2 = \frac{\varepsilon}{2}$, and using the non-negativity of $Y_n^+(\varepsilon)$ and $\varepsilon^2 P_n|K|P_n$, we find

$$\begin{aligned} & N((\Lambda_n^+ + \mu, M_n)|L_V) \\ \leq \quad & N((\Lambda_n^+ + (1-\varepsilon)\mu, M_n + \varepsilon)|P_nL_VP_n) \\ + \quad & N((\Lambda_n^+ + (1-\varepsilon)\mu, M_n + \varepsilon)|Q_n(L_V + \varepsilon^{-2}|K|)Q_n) \\ + \quad & N((\frac{\mu}{2\varepsilon}, \infty)|P_n|K|P_n) \\ + \quad & N((\frac{\varepsilon}{2}, \infty)|Y_n^+(\varepsilon)) \end{aligned} \tag{3.11}$$

where we also used, at the second inequality, the fact that $P_nL_VP_n + Q_n(L_V + \varepsilon^{-2}|K|)Q_n$ is a direct sum of two operators, for P_n and $Q_n = I - P_n$ are orthogonal projections.

To obtain the converse inequality, apply Lemma 3.3 again to the latter half of (3.10) twice (first with $A_0 = -\varepsilon^2 P_nL_VP_n + Q_n(L_V + |K|)Q_n$, $A_1 = Y_n^-(\varepsilon)$, $\tau_1 = \frac{\varepsilon}{2}\mu$, $\tau_2 = \frac{\varepsilon}{2}$, and second, with $A_1 = -\varepsilon^2 P_n|K|P_n$, $A = L_V$, $\tau_1 = \frac{\varepsilon}{2}\mu$, $\tau_2 = \frac{\varepsilon}{2}$) . Then we get the following estimate as before:

$$\begin{aligned} & N((\Lambda_n^+ + \mu, M_n)|L_V) \\ \geq \quad & N((\Lambda_n^+ + (1+\varepsilon)\mu, M_n - \varepsilon)|P_nL_VP_n) \\ + \quad & N((\Lambda_n^+ + (1+\varepsilon)\mu, M_n - \varepsilon)|Q_n(L_V - \varepsilon^{-2}|K|)Q_n) \\ - \quad & N((\frac{\mu}{2\varepsilon}, \infty)|P_n|K|P_n) \\ - \quad & N((\frac{\varepsilon}{2}, \infty)|Y_n^-(\varepsilon)) \; . \end{aligned} \tag{3.12}$$

In what follows we treat only the asymptyotics of $N((\Lambda_n^+ + \mu, M_n)|L_V)$, since we can prove the case of minus sign of Theorem 2.3 (, as we shall meet later, also in the case of Theorem 1.2) in the same way with obvious modifications.

Lemma 3.4 *For* $\mu > 0$,

$$N((\mu, \infty)|P_n|K|P_n) \leq 2N((\frac{\mu^2}{4}, \infty)|T_n^*\widetilde{V}^2T_n)$$

holds.

Proof. Set

$$E = P_n\widetilde{V}, \quad K' = P_n\widetilde{V} + \widetilde{V}P_n,$$

then we have

$$K^2 \leq K^2 + (K')^2 = 2(E^*E + EE^*) . \tag{3.13}$$

By the variational principle, it is easily seen that $\mu_k(P_n|K|P_n) \leq \mu_k(|K|)$ where $\mu_k(\cdot)$ stands for the k-th eigenvalue, of decreasing order, counting multiplicity and $\langle \psi_1, \ldots, \psi_l \rangle^\perp$ is shorthand for $\{\varphi | (\varphi, \psi_k) = 0, k = 1, \ldots, l\}$. Then it follows that

$$\begin{aligned} N((\mu, \infty)|P_n|K|P_n) &\leq N((\mu, \infty)||K|) \\ &\leq N((\mu^2, \infty)|K^2) \\ &= N((\frac{\mu^2}{2}, \infty)|E^*E + EE^*) \end{aligned} \tag{3.14}$$

where we used (3.13) at the third inequality.

We choose $\varphi_1, \ldots, \varphi_N$(resp. $\varphi_{N+1}, \ldots, \varphi_{N+M}$) to be an orthonormal basis of $\mathrm{Ran}E^{(1)}(\frac{\mu^2}{4}, \infty)$ (resp. $\mathrm{Ran}E^{(2)}(\frac{\mu^2}{4}, \infty)$), where we denote the spectral projection of the self-adjoint operator E^*E (resp. EE^*) by $E^{(1)}(\cdot)$ (resp. $E^{(2)}(\cdot)$) . Now let φ be arbitrary element such that $\varphi \in \langle \varphi, \ldots, \varphi \rangle^\perp$ and $\|\varphi\| = 1$. Then

$$(\varphi, (E^*E + EE^*)\varphi) \leq \frac{\mu^2}{4} + \frac{\mu^2}{4} = \frac{\mu^2}{2}$$

holds. From this inequality and the variational principle, we get

$$\mu_{N+M+1}(E^*E + EE^*) \leq \frac{\mu^2}{2} .$$

Henceforth, it follows that

$$\begin{aligned} &N((\frac{\mu^2}{2}, \infty)|E^*E + EE^*) \\ \leq\ & N + M \\ =\ & N((\frac{\mu^2}{4}, \infty)|E^*E) + N((\frac{\mu^2}{4}, \infty)|EE^*) . \end{aligned}$$

By considering the canonical form of compact operators E^*E and EE^*, we conclude that two terms in the R.H.S. of the above inequality are equal. Finally the statement of the lemma follows from the fact that $P_n\tilde{V}^2P_n|_{\mathcal{H}_n}$ is unitarily equivalent to $T_n^*\tilde{V}^2T_n$. □

Lemma 3.5 *For $\varepsilon > 0$ small enough, there exists $C_1(\varepsilon) > 0$ independent of μ such that*

$$N((\Lambda_n^+ + (1 \pm \varepsilon)\mu, M_n \pm \varepsilon)|Q_n(L_V - \varepsilon^{-2}|K|)Q_n) \leq C_1(\varepsilon)$$

holds.

Proof. Using the fact that L is reduced by $\mathrm{Ran}Q_n$,

$$\begin{aligned} & Q_n(L_V - \varepsilon^{-2}|K|)Q_n(Q_nLQ_n - i)^{-1} \\ = \quad & Q_n(L_V - \varepsilon^{-2}|K|)(L-i)^{-1}(L-i)(Q_nLQ_n - i)^{-1}Q_n \\ & \pm\, \varepsilon^{-2}Q_n|K|(Q_nLQ_n - i)^{-1}Q_n \,. \end{aligned}$$

We observe that $\widetilde{V}(L-i)^{-1}$ is compact, as commented in the proof of Lemma 3.2, and $(L-i)(Q_nLQ_n - i)^{-1}Q_n$ is bounded by the closed graph theorem, and that the last term of the R.H.S. is compact owing to $|K|$. Therefore $Q_n(L_V - \varepsilon^{-2}|K|)Q_n$ is relatively compact with respect to Q_nLQ_n. Finally,

$$\begin{aligned} & \sigma_{ess}(Q_nLQ_n) \cap (\Lambda_n^+ + (1\pm\varepsilon)\mu, M_n \pm \varepsilon) \\ = \quad & \bigcup_{j\neq n}[\Lambda_j^-, \Lambda_j^+] \cap (\Lambda_n^+ + (1\pm\varepsilon)\mu, M_n \pm \varepsilon) \\ = \quad & \emptyset\,, \end{aligned}$$

holds for $\varepsilon > 0$ small enough. Putting together these facts, we come to the conclusion. □

We state a key proposition without proof. This can be proved using the asymptotic estimate of the number of eigenvalues of pseudodifferential operators of negative order ([D-R]) :

Proposition 3.6 *(i) Assume (V.1), (B.2) and (Λ.1) hold. Moreover assume that* $\nu_+^\pm(\mu)$ *satisfy the condition (T). Then we have*

$$N((\Lambda_n^+ + \mu, M_n)|T_n^*L_VT_n) = B_+\nu_+(\mu)(1+o(1)) \quad \textit{as } \mu \downarrow 0\,.$$

(ii) Under the same assumptions as (i), we have

$$\lim_{\varepsilon\downarrow 0}\limsup_{\mu\downarrow 0} N((\frac{\mu^2}{\varepsilon},\infty)|T_n^*\widetilde{V}^2T_n)/\nu_+(\mu) = 0\,.$$

Now let us set about a proof of one of main theorems.

Proof of Theorem 2.3. Since $Y_n^\pm(\varepsilon)$ is compact, for each $\varepsilon > 0$, there exists a constant $C_2(\varepsilon) > 0$ independent of μ, it is derived that

$$N((\frac{\varepsilon}{2},\infty)|Y_n^\pm(\varepsilon)) \le C_2(\varepsilon)\,. \tag{3.15}$$

Putting together (3.11), (3.12), (3.15), lemma 3.4, and lemma 3.5,

$$\begin{aligned} \pm N((\Lambda_n^+ + \mu, M_n)|L_V) \le \ & \pm\ N((\Lambda_n^+ + (1-\varepsilon)\mu, M_n+\varepsilon)|T_n^*L_VT_n) \\ & +\ N((\frac{\mu^2}{8\varepsilon^2},\infty)|T_n^*\widetilde{V}^2T_n) \\ & \pm\ C_1(\varepsilon) + C_2(\varepsilon)\,. \end{aligned}$$

Furthermore, by Proposition 3.6,

$$\pm\lim_{\varepsilon\downarrow 0}\limsup_{\mu\downarrow 0} N((\Lambda_n^+ + \mu, M_n)|L_V)/B_+\nu_+(\mu) \le \pm 1$$

holds where we also used (T). This proves Theorem 2.3. □

References

[A-H-S] Avron, J., Herbst, I. and Simon, B., Schrödinger operators with magnetic fields I. General interactions Duke Math . J., **45**(4) 847-883 (1978).

[Col] Colin de Verdiere, Yves., L'asymptotique de Weyl pour les bouteilles magnétiques, Commun. Math. Phys., **105** 327-335 (1986).

[D-R] Dauge, M. and Robert, D., Weyl's formula for a class of pseudodifferential operators with negative orger on $L^2(\boldsymbol{R}^n)$, Lecture notes in Math. **1256** 91-122 (1987).

[Iwa] Iwatsuka, A., Examples of absolutely continuous Schrödinger operators in magnetic fields, Publ. RIMS, Kyoto Univ., **21** 385-401 (1985).

[L-S] Leinfelder, H. and Simader, C. G., Schrödinger operators with singular magnetic potentials, Math. Z., **176** 1-19 (1981).

[Rai1] Raikov, G. D., Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential I. Behaviour near the essential spectrum tips, Comm. in P.D.E., **15**(3) 407-434 (1990).

[Rai2] ______, Border-line eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, Integral Eq. and Operator Theory, **14** 875-888 (1991).

[R-S4] Reed, M and Simon, B., *Methods of Modern Mathematical Physics*, Vol. IV, Academic Press, New York, 1978.