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# CASES OF DADE'S CONJECTURE

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We fix a prime $p$. In 1992, Dade published a paper "Counting characters in blocks, I", in which he gave a conjecture on the numbers of irreducible characters with certain heights in a block $B$ and those of irreducible characters with related heights in $p$-local blocks related to $B$. Local blocks are those of the normalizers of chains of $p$-subgroups such that they induce the given block $B$. An important invariant of a character is in fact its defect, i.e., the difference between the exponents of $p$ in the order of the subgroup of which it is an irreducible character and in its degree. For each block, we must count the number of characters of a certain given defect. The conjecture in this paper is called the ordinary form of the conjecture. In his later papers, there appeared several other forms, for example, one concerning the number of characters of twisted group algebras. The ordinary form is the simplest one among others, and the most complicated one is called the inductive form, which implies all the other. Dade claimed that, if the inductive form is true for all finite simple groups, then it is true for all finite groups. So, it is meaningful to see whether the conjecture holds for simple groups and the related groups. Of course, it is conceivable also that looking at many examples is important in order to understand general situation. Anyway, to reach the goal, it seems that we have to take a much closer look at many cases.

In this note, we take a few examples and see what are going on there. As one might guess, most proofs use concrete computations. However, at least in cases of infinite families of groups such as finite algbraic groups, one has to figure out generically which characters can be ignored, which must be really taken into account, and so on. (It happens quite often that we may forget about some characters, because each form is stated in terms of an alternating sum and some characters kill each other in the alternating sum.) Also, in several situations one encounters phenomena which resemble each other. In such cases, it is sometimes possible to apply general assertions to them. It is hoped that some ideas in the proofs in these cases will be used when verifying more general statements or checking more examples.

Parts of my talk at this conference (and thus the contents of this note) and that given at "Darstellungstheorie Endlicher Gruppen", which was held at Oberwolfach, Germany in April 1996 are overlapped. The list of cases where some forms of the conjecture have already been verified may be found in the article of J. An in this proceeding.

## 1. Dade's Conjecture

Let $G$ be a finite group and $p$ a prime. By a $p$-chain we mean a chain

$$C : P_0 < P_1 < \cdots < P_m$$

of $p$-subgroups $P_i$'s of $G$. We denote its normalizer by $N_G(C)$ which is defined by

$$N_G(C) = \bigcap_i N_G(P_i).$$

The above $m$ is called the length of $C$, which is denoted by $|C|$.

The set of $p$-chains with $P_0 = O_p(G)$, where $O_p(G)$ is the maximal normal $p$-subgroup of $G$, is denoted by $\mathcal{P}(G)$ or simply by $\mathcal{P}$. Elements of $G$ act on $\mathcal{P}$ by conjugation, and we let $\mathcal{P}/G$ be a set of representatives of $G$-orbits in $\mathcal{P}$.

It is known that for any $p$-block $b$ of $N_G(C)$, the induced block $b^G$ in the sense of Brauer is defined. For an irreducible character $\chi$ of $G$, we denote by $d(\chi)$ the exponent of the $p$-part of $|G|/\chi(1)$. It is called the defect of $\chi$. For a $p$-block $B$ of $G$, $\max_{\chi \in \mathrm{Irr}(B)} d(\chi)$ is called the defect of $B$.

Given a $p$-chain $C$, a $p$-block $B$ of $G$ and a non-negative integer $d$, let us denote by

$$k(N_G(C), B, d)$$

the number of irreducible characters $\theta$ of $N_G(C)$ such that $\theta$ belong to a $p$-block $b$ of $N_G(C)$ with $b^G = B$ and that $d(\theta) = d$. The following is called the ordinary form of the Dade conjecture.

**Conjecture O.** *Assume that $O_p(G) = \{1\}$ and that $d(B) > 0$. Then is it true that*

$$\sum_{C \in \mathcal{P}/G} (-1)^{|C|} k(N_G(C), B, d) = 0 \ ?$$

Now we explain the other forms briefly. Suppose that $G$ is a normal subgroup of some finite group $E$. Then $E$ acts on $\mathcal{P}$, and for any $C$ in $\mathcal{P}$ its normalizer $N_G(C)$ in $G$ is a normal subgroup of its normalizer $N_E(C)$ in $E$. Thus for an irreducible character $\theta$ of $N_G(C)$, its stabilizer $T(\theta) = T_{N_E(C)}(\theta)$ in $N_E(C)$ is defined. Let $F$ be a subgroup of $E$ containing $G$. Then, for $B$, $d$, and $F$ as above, we denote by

$$k(N_G(C), B, d, F)$$

the number of irreducible characters $\theta$ of $N_G(C)$ such that $\theta$ belong to a $p$-block $b$ of $N_G(C)$ with $b^G = B$, $d(\theta) = d$, and that $T(\theta)G = F$. The following is called the invariant ordinary form of the conjecture.

**Conjecture I-O.** *Assume that $O_p(G) = \{1\}$ and that $d(B) > 0$. Then is it true that*

$$\sum_{C \in \mathcal{P}/G} (-1)^{|C|} k(N_G(C), B, d, F) = 0 \ ?$$

Assume that an irreducible character $\theta$ of $N_G(C)$ satisfies $T(\theta)G = F$. Then we have $F/G \cong T(\theta)/N_G(C)$. Hence, if we take a $\mathbf{C}N_G(C)$-module $V$ which gives $\theta$, the endomorphism algebra $\mathrm{End}(V^{T(\theta)})$ of the induced module $V^{T(\theta)}$ over $\mathbf{C}T(\theta)$ is isomorphic to a twisted group algebra of $F/G$ over $\mathbf{C}$. Recall that isomorphism classes of twised group algebras are determined by elements of the second cohomology group $\mathrm{H}^2(F/G, \mathbf{C}^*)$ of $F/G$ over the multiplicative group $\mathbf{C}^*$. Then, for $B$, $d$, $F$ and $\alpha \in \mathrm{H}^2(F/G, \mathbf{C}^*)$, we denote by

$$k(N_G(C), B, d, F, \alpha)$$

the number of irreducible characters $\theta$ of $N_G(C)$ such that $\theta$ belong to a $p$-block $b$ of $N_G(C)$ with $b^G = B$, $d(\theta) = d$, $T(\theta)G = F$, and that the above endomorphism algebra $\mathrm{End}(V^{T(\theta)})$ gives $\alpha$. The following is called the extended ordinary form of the conjecture.

**Conjecture E-O.** *Assume that $O_p(G) = \{1\}$ and that $d(B) > 0$. Then is it true that*

$$\sum_{C \in \mathcal{P}/G} (-1)^{|C|} k(N_G(C), B, d, F, \alpha) = 0 \quad ?$$

In every form, we can consider characters of twisted group algebras, i.e., projective characters instead of ordinary characters. That is, fixing a twisted group algebra of $G$, we take a $p$-block $B$ of projective irreducible characters and a non-negative integer $d$. Now it is possible to formulate the projective form of the conjecture (**Conjecture P**), whose statement itself is completely the same as that of Conjecture O. For the invariant form, we begin with a twisted group algebra of $E$. The twisted group algebra of $G$ which we must deal with is obtained by taking the subalgebra generated by the bases elements parametrized only by the elements of $G$. We can define $T(\theta)$ for a projective irreducible character $\theta$ of $N_G(C)$, too, and $k(N_G(C), B, d, F)$ makes sense for a subgroup $F$ of $E$ containing $G$. Thus we have the invariant projective form (**Conjecture I-P**). Furthermore, the extended projective form (**Conjecture E-P**) may also be formulated since $\mathrm{End}(V^{T(\theta)})$ is again a twisted group algebra of $F/G$.

So far, we have six forms. However, Dade gives one more, which he calls the inductive form. Here we do not explain its detail, which can be found in [D3]. A rough sketch of the idea is as follows. We must start with a $p$-modular system $(K, R, k)$, where $R$ is a large complete discrete valuation ring, $k$ is its residue class field of characteristic $p$, and $K$ is the field of fractions of $R$. Everything is considered over $R$ at the beginning. That is, we take an $RN_G(C)$-lattice which gives $\theta$ over $K$. Twisted group algebras and endomorphism algebras, which is denoted by $\mathrm{End}(\theta^{T(\theta)})$, are also taken over $R$, and then over $k$, by factoring out by the unique maximal ideal of $R$. Let $B$, $F$, $\alpha$ be as in the extended projective form. Recall that it is necessary to look at only those blocks $b$ that induce the given block $B$ of $G$. Thus $B$ and $b$ are related via the Brauer homomorphism. Moreover, if $\theta$ satisfies $T(\theta)G = F$, then $B$ is $F$-invariant and $b$ is $N_F(C)$-invariant. Hence $B$ and $b$ give twisted group algebras of $F/G$ over the center $Z(B)$ of $B$ and over the center $Z(b)$ of $b$, respectively. Denote the former algebra by $Z(B)^F$. By composing the Brauer homomorphism with the natural embedding of $Z(b)$ to the endomorphism algebra $\mathrm{End}(\theta)$, we have a $k$-algebra homomorphism from $Z(B)$ to $\mathrm{End}(\theta)$. Moreover, we obtain naturally a $k$-algebra homomorphism from $Z(B)^F$ to $\mathrm{End}(\theta^{T(\theta)})$, which is also a twisted group algebra of $F/G$ over $k$. Let us fix a twisted group algebra $k^*(F/G)$ of $F/G$. Thus we fix $\alpha \in \mathrm{H}^2(F/G, \mathbf{C}^*)$. For any given $k$-algebra homomorphism $\iota$ of twisted group algebras of $F/G$ from $Z(B)^F$ to $k^*(F/G)$, we count the number $k(N_G(C), B, d, F, \iota)$ of those $\theta$ with the same conditions concerning $B$, $d$, $F$ as above, and with $\mathrm{End}(\theta^{T(\theta)})$ such that there is an isomorphism of twisted group algebras from $\mathrm{End}(\theta^{T(\theta)})$ to $k^*(F/G)$ with which composing the Brauer homorphism and the natural embedding give $\iota$. The inductive form is of course as follows.

**Conjecture IND.** *Assume that $O_p(G) = \{1\}$ and that $d(B) > 0$. Then is it true that*

$$\sum_{C \in \mathcal{P}/G} (-1)^{|C|} k(N_G(C), B, d, F, \iota) = 0 \quad ?$$

Finally we remark the following.

**Remark.** All the invariants $k(N_G(C), B, d, ***)$ is zero if the final subgroup $P_m$ of $C$ is not contained in any defect group of $B$.

## 2. $p$-chains

Since there are so many $p$-chains, even up to $G$-conjugate, many computations are necessary to check the conjecture. However, it is known that we may concentrate on certain families of $p$-chains, which consisit of fewer $p$-chains.

**Definition.** If a $p$-chain $C : P_0 < P_1 < \cdots < P_m$ satisfies $P_0 = O_p(G)$ and $P_i$ is elementary abelian for all $i$, then it is called an *elementary* chain. If $C$ satisfies $P_0 = O_p(G)$ and

$$P_i = O_p(\cap_{j=0}^{i} N_G(P_j)) \quad \text{for all} \quad i,$$

then it is called a *radical* chain. The sets of elementary chains and radical chains are denoted by $\mathcal{E}$ and $\mathcal{R}$, respectively. The sets $\mathcal{E}$ and $\mathcal{R}$ are $G$-invariant (and also $E$-invariant).

One of the important and useful results is as follows.

**Proposition.** *In any form of the conjecture, the sum can be replaced by the sum taken over either $\mathcal{E}/G$ or $\mathcal{R}/G$ instead of $\mathcal{R}/G$.*

The above is observed by Knörr and Robinbson [KR] and Dade [D1]. In [KR], they consider more general situations and some other families of chains. We can use those families whenever they seem to be suitable for the cases we treat.

If a $p$-subgroup $P$ of $G$ satisfies $P = O_p(N_G(P))$, then it is called a *radical* subgroup of $G$. Thus, a $p$-chain $C : P_0 < P_1 < \cdots < P_m$ is radical if and only if $P_0 = O_p(G)$ and $P_{i+1}/P_i$ is radical in $\cap_{j=0}^{i} N_G(P_j)/P_i$ for all $i \geq 1$. Therefore, to obtain radical chains, we look for radical subgroups at each stage.

**Example.** Let $S(n)$ be the symmetric group on $n$ letters. Radical subgroups of $S(n)$ are determined by Alperin and Fong [AF]. For a non-negative integer $c$, let $A_c$ be the regular elementary abelian group of order $p^c$. For example, if $p = 3$, we have

$$A_2 \cong <(123)(456)(789),(147)(258)(369)> \quad \text{in} \quad S(9).$$

For a sequence $\underline{c} = (c_1, c_2, \cdots)$ of positive integers, let $A_{\underline{c}}$ be the wreath product

$$A_{c_1} \wr A_{c_2} \wr \cdots .$$

Alperin and Fong proved that, if $P$ is a radical subgroup of $S(n)$, then it follows that $P$ is $S(n)$-conjugate to

$$\prod_i (A_{\underline{c}_i})^{m_i}.$$

for some sequences $\underline{c}_i$'s and positive integers $m_i$'s. (Note : The converse is not necessarily true.)

Subgroups of the above type are called basic subgroups. Furthermore, they showed that, if $P$ is as above, then

$$(*) \qquad N_{S(n)}(P)/P \cong \prod_i (GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p)) \wr S(m_i)$$

provided that $\underline{c}_i \neq \underline{c}_j$ if $i \neq j$. Here, we write $\underline{c}_i = (c_{i,1}, c_{i,2}, \cdots, c_{i,n_i})$. Thus, when we look for a radical chain $C$ with $|C| \geq 2$, we must find radical subgroups of $\prod_i (GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p)) \wr S(m_i)$. For this purpose, the following results are of great use.

**Lemma.** (1) *Let $P$ be a radical subgroup of the direct product $G_1 \times G_2$ of two finite groups $G_1$ and $G_2$. Then there are radical subgroups $P_1$ and $P_2$ of $G_1$ and $G_2$, respectively, such that $P = P_1 \times P_2$.*

(2) *Let $G$ be a finite group, and let $P$ be a radical subgroup of the wreath product $G \wr S(n)$ of $G$ with $S(n)$. Then $P$ is $G \wr S(n)$-conjugate to $\tilde{P} \rtimes S$, where $\tilde{P}$ is a radical subgroup of $G^n$ and $S$ is a basic subgroup of $S(n)$.*

By using the above, one shows that our third subgroup $P_2$ in a radical chain satisfies

$$(**) \qquad P_2/P_1 \cong \prod_i Q_i \rtimes R_i,$$

where $Q_i$ is a radical subgroup of $(GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p))^{m_i}$ and $R_i$ is a basic subgroup of $S(m_i)$. Here notice that radical subgroups of finite algebraic groups are known to be the normalizers of parabolic subgroups (when $p$ is the defining characteristic), and that, if $Q_i$ in the above is trivial for all $i$, then $P_2$ is also basic in $S(n)$.

In the rest of this note, we see several examples for which some form of the conjecture is verified.

### 3. Case 1: Symmetric groups

Here we sketch the proof of the ordinary form of the conjecture for $S(n)$ $(n \geq 4)$ for an odd prime $p$, following [OU2].

STEP 1.

Let $C : P_0 < P_1 < \cdots < P_m$ be a radical chain of $S(n)$. Then $P_0 = O_p(S(n)) = \{1\}$, and if $m \geq 1$, then $P_1$ is basic. Moreover, by the results in the previous section, we can define $\ell \geq 1$ so that $P_1$, $P_2$, $\cdots$, $P_\ell$ are basic subgroups of $S(n)$ and $P_{\ell+1}$ is not. Then we have $\cap_{i=0}^{\ell+1} N_{S(n)}(P_i)/P_\ell$ the form

$$\prod_i (GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p)) \wr S(m_i).$$

Also, by the choice of $\ell$, $P_{\ell+1}/P_\ell$ has a "non-trivial" $GL$-part. In case of $|C| \geq \ell+1$, let $s$ with $s \geq \ell$ be such that $P_\ell/P_\ell = \{1\}$, $P_{\ell+1}/P_\ell$, $P_{\ell+2}/P_\ell$, $\cdots$, $P_s/P_\ell$ are contained in the $GL$-part

$$\prod_i (GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p))^{m_i},$$

and $P_{s+1}/P_\ell$ is not. Then $P_{\ell+1}/P_\ell$, $P_{\ell+2}/P_\ell$, $\cdots$, $P_s/P_\ell$ have the form (**) with all $R_i$ trivial, and we may $P_{s+1}/P_\ell \cong \prod_i \tilde{Q}_i \rtimes \tilde{R}_i$, where $\tilde{Q}_i$ is a radical subgroup of $(GL(c_{i,1}, p) \times \cdots \times GL(c_{i,n_i}, p))^{m_i}$ and $\tilde{R}_i$ is a basic subgroup of $S(m_i)$, and moreover, for at least one $i$, $\tilde{R}_i$ is not trivial. Notice also that some $\tilde{Q}_i$ is not trivial by the choice of $\ell$.

Now consider the radical chains $C$ for which the above $s$ satisfies $|C| \geq s+1$. Then define $\tau(C)$ for such a $C$ as follows.

$$\tau(C) : \begin{cases} P_0 < \cdots < P_{s-1} < P_{s+1} < P_{s+2} < \cdots & \text{if } P_s/P_\ell = \prod_i \tilde{Q}_i \\ P_0 < \cdots < P_{s-1} < P_s < Q < P_{s+1} < P_{s+2} < \cdots & \text{if } P_s/P_\ell \neq \prod_i \tilde{Q}_i \end{cases}$$

where $Q$ is the inverse image of $\prod_i \tilde{Q}_i$ in $P_{s+1}$.

Then it is not difficult to show that $\tau(\tau(C)) = C$, $|\tau(C)| = |C| \pm 1$, and that $N_{S(n)}(\tau(C)) = N_{S(n)}(C)$. Thus $\tau$ gives an involutive bijection from the set of the chains $C$ with $|C| \geq s+1$ to itself, which keeps the normalizers of the chains invariant, but changing the parity of their length. Since the number $k(N_G(C), B, d)$ depends only on $N_G(C)$ for given $B$ and $d$, two chains $C$ and $\tau(C)$ contribute nothing to the alternating sum. Therefore, we may consider only those chains $C$ with $m = |C| = s$, that is, $P_\ell/P_\ell = \{1\}$, $P_{\ell+1}/P_\ell$, $\cdots$, $P_m/P_\ell$ are contained in the $GL$-part.

STEP 2.

Recall that in (*) the factor group of the normalizer contains a direct products of $GL$'s, and by Lemma and Step 1, the factor groups $P_\ell/P_\ell = \{1\}$, $P_{\ell+1}/P_\ell$, $\cdots$, $P_m/P_\ell$ are direct products as well. Thus by using an argument similar to the one in Step 1, we may concentrate on only those chains such that every $\underline{c}_i$ has only one part. More precisely, instead of using semi-direct products, we use direct products, and can find similarly cancellations whenever some products have more than one factors.

Here we remark that for those arguments in Steps 1 and 2, the fact that $p \neq 2$ is necessary to conclude that $\tau(C)$ is again a radical chain.

From now on, we assume that every $\underline{c}_i$ has only one part and write $\underline{c}_i = (c_i)$ for all $i$.

STEP 3.

By Step 2, $P_\ell/P_\ell = \{1\}$, $P_{\ell+1}/P_\ell$, $\cdots$, $P_m/P_\ell$ are contained in $\prod_i GL(c_i,p)^{m_i}$. In fact, they give a radical chain of $\prod_i GL(c_i,p) \wr S(m_i)$. In [OU1], it is shown that the ordinary form of the conjecture is true for the general linear groups $GL(n,q)$ and their direct products in the defining characteristic. This also yields that it is true for $\prod_i GL(c_i,p) \wr S(m_i)$ as well. With these results, one can show that the alternating sum vanishes on $\mathcal{R}_1/G$, where $\mathcal{R}_1$ denotes the set of radical chains

$$C : \{1\} < P_1 < P_2 < \cdots < P_m$$

with $|C| = s$ ($s$ is defined as in Step 1), and $P_m/P_\ell$ being contained in the $GL$-part $\prod_i GL(c_i,p)^{m_i}$ such that $c_i \geq 2$ for some $i$. Here we need the condition "$c_i \geq 2$ for some $i$" because of the following reason. We know that the ordinary form is true for $GL(n,q)$ and for the other related groups such as $\prod_i GL(c_i,p) \wr S(m_i)$. However, we must consider also $V \rtimes GL(n,q)$ and $V' \rtimes \prod_i GL(c_i,p) \wr S(m_i)$, for example, where $V$ and $V'$ are vector spaces on which $GL(n,q)$ and $\prod_i GL(c_i,p) \wr S(m_i)$, respectively, act naturally, since we have to deal with $\cap_{i=0}^m N_{S(n)}(P_i)$ and not $\cap_{i=0}^m N_{S(n)}(P_i)/P_\ell$. But, it is known that the alternating sum in the conjecture does not always vanish if $O_p(G) \neq \{1\}$. In fact, the ordinary form is true for $V \rtimes GL(n,q)$ only when $n = \dim V \geq 2$. Since we use these facts, the above condition on $c_i$ is necessary.

Now remaining radical chains are those $C$ such that $|C| = s$ and $P_m/P_\ell$ is contained in the $GL$-part $\prod_i GL(1,p)^{m_i}$. But, since $GL(1,p)$ is a $p'$-group, it follows that $m = |C| = \ell$. Namely, $P_1$, $P_2$, $\cdots$, $P_m$ are all basic subgroups isomorphic to

$$A_1 \times A_1 \cdots \times A_1.$$

STEP 4.

By Step 3, we may consider only the radical chains of the following type.

$$C : \{1\} < (A_1)^{\ell_1} < (A_1)^{\ell_2} < \cdots < (A_1)^{\ell_m},$$

where $\ell_1$, $\ell_2$, $\cdots$, $\ell_m$ are positive integers with $\ell_1 < \ell_2 < \cdots < \ell_m$. For any positive integer $t$, let $\mathcal{R}(t)$ denote the set of radical chains of type

$$C : \{1\} < (A_1)^t < (A_1)^{\ell_2} < \cdots < (A_1)^{\ell_m}.$$

Then the subset of $\mathcal{R}$ consisting of all the remaining chains is precisely the disjoint union $\cup\mathcal{R}(t)$ and the subset consisting only of the trivial chain. On the other hand, for any $p$-block $B$ of $S(n)$, we can define its weight $w(B)$ as follows. Let $D$ be a defect group of $B$. Then $p$ divides $n - f$, where $f$ is the number of the fixed points of $D$. Then $w(B)$ is defined by $w(B) = (n - f)/p$.

Now by induction on $|C|$ and by some local analysis of $p$-blocks of $S(n)$ ([O2]), one can show that, for any $t$ with $t < w(B)$, the alternating sum vanishes on $\mathcal{R}(t)$. On the other hand, again by some local analysis, one can prove that, if $\ell_m > w(B)$, then the number $k(N_{S(n)}(C), B, d)$ is zero. (This is also related to the final remark in §1.) Thus, we may concentrate only on chains with $t \geq w(B)$ and $\ell_m \leq w(B)$. But, since $t \leq \ell_m$, we have $t = \ell_m = w(B)$. This is the chain $C_1 : \{1\} < (A_1)^{w(B)}$. Its normalizer has only the principal block $b_1$ which satisfies $b_1^{S(n)} = B$. Because we have now only the trivial chain and $C_1$, it suffices to check that

$$k(S(n), B, d) = k(N_{S(n)}(C_1), B, d)$$

holds for all $d$. However, the above has already been proved by Olsson in [O1]. Therefore, we can conclude that the ordinary form is true for $S(n)$ when $p$ is odd.

## 4. Case 2 : Cyclic defect group cases

In this section, we consider the case where a defect group $D$ of a given $p$-block $B$ is cyclic. The ordinary form is true in this case ([D1]). Moreover, the invariant form is also true. We give an outline of the proof to it. (See [D4].)

We use $\mathcal{E}$ instead of $\mathcal{R}$. Since $D$ is cyclic, $D$ has the unique cyclic subgroup $P$ of order $p$. By the final remark in §1, it suffices to show that

$$k(G, B, d, F) = k(N_G(P), B, d, F)$$

for all $d$ and $F$. Here, of course, we assume that $G$ is a normal subgroup of some $E$ and that $F$ is a subgroup of $E$ containing $G$. It is known that there is only one block $b$ of $N_G(P)$ such that $d(b) = d(B)$ and $b^G = B$. Furthermore, letting $e$ be the inertia index of $B$, we have

$$k(G, B, d(B)) = k(N_G(P), B, d(b)) = e + \frac{|D| - 1}{e},$$

and $k(G, B, d)$ and $k(N_G(P), B, d)$ for any other $d$ are zero. (See for example [F].) Thus the ordinary form holds.

Let $d = d(B) = d(b)$. Some irreducible characters in $B$ and $b$ are called exceptional characters and the numbers of exceptional characters are $\frac{|D|-1}{e}$ for both $B$ and $b$. The actions of $E$ and $N_E(P)$ on $\mathrm{Irr}(B)$ and on $\mathrm{Irr}(N_G(P))$, respectively, preserve exceptional characters, and moreover, examining their character values, it is not difficult to see how the actions on them are.

On the other hand, $B$ and $b$ both have $e$ irreducible Brauer characters. Let $f$ be the Green correspondence which sends indecomposable $kG$-modules with vertex $D$ to those $kN_G(P)$-modules with the same vertex. (Note : $N_G(D) \leq N_G(P)$.) It is known that irreducible $kG$-modules in $B$ have vertex $D$. Using the fact that

$$D^g \cap D = \{1\} \quad \text{for all} \quad g \in G \setminus N_G(P),$$

one can show that, if $V$ is an irreducible $kG$-module with vertex $D$, then the maximal semisimple submodule of $f(V)$ is irreducible, and that this gives a bijection from the set of irreducible Brauer characters of $G$ belonging to $B$ to that of $N_G(P)$ belonging to $b$. This bijection is compatible with the actions of $E$ and of $N_E(P)$. Recall that $B$ and $b$ have $e$ non-exceptional characters and $e$ irreducible Brauer characters. By looking at the decomposition matrices, it is possible to see the relation between the actions of $E$ and of $N_E(P)$ on non-exceptional characters through those on irreducible Brauer characters. Finally we can conclude that the numbers of non-exceptional characters in $B$ and $b$ whose inertia subgroups give $F$ are the same. This completes the proof of the invariant form in the case where $D$ is cyclic.

## 5. Case 3 : Dihedral defect group cases

In this section, we consider the case where a defect group $D$ of a given $p$-block $B$ is dihedral

$$< x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{-1} >$$

of order $2^n$ with $n \geq 3$. Thus we have $p = 2$. We assume that $B$ is of type (aa) in the sense of Brauer [B]. Namely, we suppose that $x^{2^{n-2}}$, $y$ and $xy$ are conjugate in $G$. Note that, in particular, $< z, y >$ and $< z, xy >$ are not $G$-conjugate. The invariant form is proved in [U], and we give its idea here. Again assume that $G$ is a normal subgroup of $E$ and $F$ is a subgroup of $E$ containing $G$. Ler $z = x^{2^{n-2}}$. Then $< z >$ is the center $Z(D)$ of $D$ and the following are representatives of $G$-orbits in $\mathcal{E}$ whose final subgroups are contained in $D$. (See also the final remark in §1.)

$$\{1\},\quad \{1\} < Z(D),\quad \{1\} < < z, y >,\quad \{1\} < < z, xy >$$

$$\{1\} < Z(D) < < z, y >,\quad \{1\} < Z(D) < < z, xy >$$

Notice that all the local blocks of the normalizers of the above chains have dihedral defect groups. Hence by taking pairs

$$\begin{array}{ccc} G & \text{and} & N_G(Z(D)) \\ N_G(< z, y >) & \text{and} & N_G(< z, y >) \cap N_G(Z(D)) \\ N_G(< z, xy >) & \text{and} & N_G(< z, xy >) \cap N_G(Z(D)), \end{array}$$

it suffices to show that

$$k(G, B, d, F) = k(N_G(Z(D)), B, d, F)$$

holds for any $d$, $F$ and any blocks $B$ with dihedral defect groups. Note that, in fact, we have to prove the above equation for blocks with defect groups of other types (ab) and (bb), because local blocks appearing in the above (even in the case where the given $B$ itself is of type (aa)) may be of those types. However, here we give the idea to show the above equation, only in the case where $B$ is of type (aa). For the other cases, see [U].

It is known by [B] that there is the unique block $b$ of $N_G(Z(D))$ such that $b^G = B$. Moreover, $b$ has also $D$ as a defect group. and

$$k(G,B,d) = k(N_G(Z(D)),B,d) = \begin{cases} 4 & \text{if } d = n \\ 2^{n-2}-1 & \text{if } d = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the ordinary form follows from the above.

For the invariant form, we may assume that $B$ is $E$-invariant. Thus $b$ is also $N_E(Z(D))$-invariant. This yields that $E = N_E(D)G = N_E(Z(D))G$ by the Frattini argument. On the other hand, the actions of $E$ and $N_E(Z(D))$ on characters $\chi$ with $d(\chi) = n-1$ are easy to see, since those characters are mostly determined by their values on $x^j$. Hence, the above observations imply that these actions of $E$ and of $N_E(Z(D))$ on characters of defect $n-1$ are permutation isomorphic. In particular, their inertia groups coincide "in $E$ modulo $G$". This means that

$$k(G,B,n-1,F) = k(N_G(Z(D)),B,n-1,F)$$

for all $F$. However, all the characters with $d = n$ are rational valued, and in general, it is hard to see how those are distributed into $E$-orbits or $N_E(Z(D))$-orbits. But, $E$ can act on the column index set of the generalized decomposition matrix of $B$. The same is true for $N_E(Z(D))$ and $b$. A column index of the generalized decomposition matrix looks like $(g,\eta)$, where $g$ is an element in $D$ and $\eta$ is an irreducible Brauer character of $C_G(g)$ belonging to a block $b_g$ of $C_G(g)$ which induces $B$. The set of column indices consists of representatives of $G$-conjugate classes of such pairs. The row index set is, on the other hand, consists of irreducible ordinary characters in $B$. Brauer proved also in [B] that $B$ has three irreducible Brauer characters, say $\varphi_1$, $\varphi_2$ and $\varphi_3$, and $b$ has only one, say $\varphi$. Let $\chi_1$, $\chi_2$, $\chi_3$, $\chi_4$ be irreducible characters in $B$ whose defects are $n$. Also let $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ be those in $b$. Then we can find the following parts in the generalized decomposition matrices of $B$ and $b$.

$B$

| | $(1,\varphi_1)$ | $(1,\varphi_2)$ | $(1,\varphi_3)$ | $(z,\varphi)$ |
|---|---|---|---|---|
| $\chi_1$ | * | * | * | * |
| $\chi_2$ | * | * | * | * |
| $\chi_3$ | * | * | * | * |
| $\chi_4$ | * | * | * | * |

$b$

| | $(1,\varphi)$ | $(z,\varphi)$ | $(y,\varphi_y)$ | $(xy,\varphi_{xy})$ |
|---|---|---|---|---|
| $\theta_1$ | * | * | * | * |
| $\theta_2$ | * | * | * | * |
| $\theta_3$ | * | * | * | * |
| $\theta_4$ | * | * | * | * |

Here $\varphi_y$ and $\varphi_{xy}$ are the unique irreducible Brauer characters of $C_G(y)$ and $C_G(xy)$, respectively, which belong to blocks inducing $B$. Now, it is clear that $(z,\varphi)$ is $E$-invariant. Also, for $b$, the columns $(1,\varphi)$ and $(z,\varphi)$ are $N_E(Z(D))$-invariant. Moreover, by looking at the decomposition matrix of $B$, it follows that at least one of $\varphi_i$ is $E$-invariant. Say that $\varphi_3$ is $E$-invariant. Since the above parts of the generalized decomposition matrices are invertible and since the actions of $E$ and of $N_E(Z(D))$ on the complements of the above subsets of column and row indices are permutation isomorphic (the actions on those subsets are the same as those on $\{x^j \mid 1 \leq j \leq 2^{n-2} - 1\}$), we can conclude, by using the Brauer's permutation lemma, that

$$k(G, B, n, F) = \begin{cases} 0 & \text{if } T_1 \neq F \neq E \neq T_1 \text{ or } T_1 = E \neq F \\ 2 & \text{if } T_1 = F \neq E \text{ or } T_1 \neq F = E \\ 4 & \text{if } T_1 = F = E, \end{cases}$$

where $T_1$ is the inertia group of $\varphi_1$ in $E$, and that

$$k(N_G(Z(D)), B, n, F) = \begin{cases} 0 & \text{if } T_2 \neq F \neq E \neq T_2 \text{ or } T_2 = E \neq F \\ 2 & \text{if } T_2 = F \neq E \text{ or } T_2 \neq F = E \\ 4 & \text{if } T_2 = F = E, \end{cases}$$

where $T_2 = C_{N_G(Z(D))}(y)G$. Hence, it suffices to prove that $T_1 = T_2$.

Now we look at periodic indecomposable $kG$-modules. There are those modules in $B$ with period three and vertices $< z, y >$ or $< z, xy >$. For some of those modules, their structures are quite well known. (See [E].) In particular, their composition series show that, an element $g$ of $E$ sends those modules with vertices $< z, y >$ to those with $< z, xy >$ by conjugation, if and only if $\varphi_1^g = \varphi_2$, i.e., $\varphi_1$ and $\varphi_2$ are not $g$-invariant. Thus, it follows that $T_1 = N_E(< z, y >)G$. This implies that $T_1 \geq T_2$. On the other hand, the fact that $< z, y >$ and $< z, xy >$ are not $G$-conjugate implies that $T_1 \leq T_2$. This completes the proof.

**Remark**

To check certain forms such as invariant forms, one may have to use some information on indecomposable modules. (See §4 and §5.) It is probably the case that, for proofs of general statements on those forms, information not only on characters but also on block algebras would be necessary. One must study the relations between the structures of algebras (and indecomposable modules over them) of a given block and of the related local blocks.

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## Addendum

In the beginning of February 1997, several e-mails were flying about. They said that there were counterexamples to some forms of the conjecture. However, later, errors were found in the arguments or in the computations, and it turned out that they were not counterexamples at all. So, there seem to be no counterexamples now. As of Monday February 10th, 1997, Dade wrote what had been happening in those days. (In what Dade wrote on 10th, there was still a possibility of a counterexample. But, after a while, at least, an error was found in the argument.) This note was completed on Friday February 14th.