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SYMMETRIC DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH PSEUDO-INVEXITY*

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1. INTRODUCTION

Dantzig, Eisenberg and Cottle [1] first formulated a pair of symmetric dual nonlinear programs in which the dual of dual equals the prime and established the weak and strong duality for these problems concerning convex and concave functions. Mond and Hanson [5] extended the symmetric duality results to variational problems, giving continuous analogues of the results of the above authors. Since the invexity conditions on functions were first defined by Hanson [2] as a generalization of convexity ones, many authors ([4],[8],[9]) have extended the concepts of invexity to continuous functions. Smart and Mond [9] extended the symmetric duality results to variational problems by using the continuous version of invexity.

Recently, Kim and Lee [3] presented a pair of symmetric dual variational problems in the spirit of Mond and Weir [6] different from the one formulated by Smart and Mond [9], using the continuous version of pseudo-invexity which is a generalization of that of invexity.

On the other hand, Mond and Weir [6] gave a different pair of symmetric dual nonlinear programming problems in which the convexity and concavity assumptions were

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reduced to the pseudo-convexity and pseudo-concavity ones, and obtained the weak and strong duality of these problems.

In this paper, we formulate a pair of multiobjective symmetric variational problems. Weak, strong and converse duality theorems are established under pseudo-invexity assumptions for these problems by using the concept of efficiency. Self-dual problems and static symmetric dual programs are included as special cases. Also, Kim and Lee's results [3] are obtained as special cases.

2. NOTATIONS AND STATEMENT OF THE PROBLEMS

The following conventions for vectors in R^n will be used :

$$x < y \Leftrightarrow x_i < y_i, \quad i = 1, 2, \cdots, n;$$

$$x \leqq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n;$$

$$x \leq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n \quad \text{but} \quad x \neq y;$$

$$x \nleq y \quad \text{is the negation of} \quad x \leq y.$$

Let $[a, b]$ be a real interval and $f : [a, b] \times R^n \times R^n \times R^m \times R^m \to R^p$. Consider the vector valued function $f(t, x, x', y, y')$, where $t \in [a, b]$, x and y are functions of t with $x(t) \in R^n$ and $y(t) \in R^m$, and x' and y' denote the derivatives of x and y, respectively, with respect to t. Assume that f has continuous fourth-order partial derivatives with respect to x, x', y and y'. f_x and $f_{x'}$ denote the $p \times n$ matrices of first partial derivatives with respect to x and x'. i.e.,

$$f_x^i = \left(\frac{\partial f^i}{\partial x_1}, \cdots, \frac{\partial f^i}{\partial x_n} \right) \quad \text{and} \quad f_{x'}^i = \left(\frac{\partial f^i}{\partial x_1'}, \cdots, \frac{\partial f^i}{\partial x_n'} \right), \quad i = 1, 2, \cdots, p.$$

Similarly, f_y and $f_{y'}$ denote the $p \times m$ matrices of first partial derivatives with respect to y and y'. We consider the problem of finding functions $x : [a, b] \to R^n$ and

$y : [a, b] \to R^m$, with $(x'(t), y'(t))$ piecewise smooth on $[a, b]$, to solve the following pair of multiobjective variational problems.

(MSP)

$$
\begin{aligned}
\text{Minimize} \quad & \int_a^b f(t, x, x', y, y')dt \\
& = \Big(\int_a^b f^1(t, x, x', y, y')dt, \cdots, \int_a^b f^p(t, x, x', y, y')dt \Big) \\
\text{subject to} \quad & x(a) = x_0, \;\; x(b) = x_1, \;\; y(a) = y_0, \;\; y(b) = y_1, \\
& \lambda^T f_y(t, x, x, y, y') - \frac{d}{dt}\lambda^T f_{y'}(t, x, x', y, y') \leqq 0, \qquad (1) \\
& y^T \Big[\lambda^T f_y(t, x, x', y, y') - \frac{d}{dt}\lambda^T f_{y'}(t, x, x', y, y') \Big] \geqq 0, \qquad (2) \\
& \lambda > 0, \;\; \lambda^T e = 1,
\end{aligned}
$$

(MSD)

$$
\begin{aligned}
\text{Maximize} \quad & \int_a^b f(t, u, u', v, v')dt \\
& = \Big(\int_a^b f^1(t, u, u', v, v')dt, \cdots, \int_a^b f^p(t, u, u', v, v')dt \Big) \\
\text{subject to} \quad & u(a) = x_0, \;\; u(b) = x_1, \;\; v(a) = y_0, \;\; v(b) = y_1, \\
& \lambda^T f_x(t, u, u, v, v') - \frac{d}{dt}\lambda^T f_{x'}(t, u, u', v, v') \geqq 0, \qquad (3) \\
& u^T \Big[\lambda^T f_x(t, u, u', v, v') - \frac{d}{dt}\lambda^T f_{x'}(t, u, u', v, v') \Big] \leqq 0, \qquad (4) \\
& \lambda > 0, \;\; \lambda^T e = 1,
\end{aligned}
$$

where $\lambda \in R^p$ and $e = (1, \cdots, 1)^T \in R^p$.

Remark 2.1. Observe that if $p = 1$ in (MSP) and (MSD), then (MSP) and (MSD) become (SP) and (SD) given by Kim and Lee [3].

3. SYMMETRIC DUALITY

We consider the following multiobjective variational problem.

$$(MP) \qquad \text{Minimize} \quad \int_a^b f(t,x,x')dt = \left(\int_a^b f^1 dt, \cdots, \int_a^b f^p dt\right)$$
$$\text{subject to} \quad x(a)=\alpha, \;\; x(b)=\beta,$$
$$g(t,x,x') \leqq 0, \;\; t \in [a,b],$$

where $f : [a,b] \times R^n \times R^n \to R^p$, $g : [a,b] \times R^n \times R^n \to R^m$. Let $K = \Big\{x \in C([a,b],R^n) \mid x(a) = \alpha, x(b) = \beta, \; g(t,x(t),x'(t))) \leqq 0, \; t \in [a,b]\Big\}$ be the set of feasible solutions for (MP).

Definition 3.1. A point x^* in K is an efficient solution (Pareto optimum) of (MP) if for all x in K,

$$\int_a^b f(t,x,x')dt \nleq \int_a^b f(t,x^*,x^{*\prime})dt.$$

(i.e., there exists no other $x \in K$ such that $\int_a^b f(t,x,x')dt \leq \int_a^b f(t,x^*,x^{*\prime})dt$)

Now we defined the pseudo-invexity as follows;

Definition 3.2. The functional $\int_a^b \lambda^T f$ is pseudo-invex in x and x' if for each $y : [a,b] \to R^m$, with y' piecewise smooth, there exists a function $\eta : [a,b] \times R^n \times R^n \times R^n \times R^n \to R^n$ such that for all $x : [a,b] \to R^n, u : [a,b] \to R^n$, with $(x'(t), u'(t))$ piecewise smooth on $[a,b]$,

$$\int_a^b \eta(t,x,x',u,u')^T \left[\lambda^T f_x(t,u,u',y,y') - \frac{d}{dt}\lambda^T f_{x'}(t,u,u',y,y')\right] dt \geqq 0 \text{ implies}$$
$$\int_a^b \lambda^T f(t,x,x',y,y')dt - \int_a^b \lambda^T f(t,u,u',y,y')dt \geqq 0.$$

Definition 3.3. The functional $-\int_a^b \lambda^T f$ is pseudo-invex in y and y' if for each $x : [a,b] \to R^n$, with x' piecewise smooth, there exists a function $\xi : [a,b] \times R^m \times R^m \times$

$R^m \times R^m \to R^m$ such that for all $v : [a,b] \to R^m, y : [a,b] \to R^m$, with $(v'(t), y'(t))$ piecewise smooth on $[a,b]$,

$$-\int_a^b \xi(t,v,v',y,y')^T \left[\lambda^T f_y(t,x,x',y,y') - \frac{d}{dt}\lambda^T f_{y'}(t,x,x',y,y')\right] dt \geqq 0 \text{ implies}$$
$$-\int_a^b \lambda^T f(t,x,x',v,v')dt + \int_a^b \lambda^T f(t,x,x',y,y')dt \geqq 0.$$

In the sequel, we will write $\eta(x,u)$ for $\eta(t,x,x',u,u')$ and $\xi(v,y)$ for $\xi(t,v,v',y,y')$.

Remark 3.1. If f is independent of t, the definitions (3.2-3.3) reduce to the definitions of pseudo-invexity of the static case in [3].

Theorem 3.1. (Weak duality) *Let (x,y,λ) be feasible for (MSP) and (u,v,λ) be feasible for (MSD). Assume that $\int_a^b \lambda^T f$ is pseudo-invex in x and x', and $-\int_a^b \lambda^T f$ is pseudo-invex in y and y', with $\eta(x,u) + u(t) \geqq 0$ and $\xi(v,y) + y(t) \geqq 0$ for all $t \in [a,b]$ (except perhaps at corners of $(x'(t), y'(t))$ or $(u'(t), v'(t))$).*
Then $\int_a^b f(t,x,x',y,y')dt \nleq \int_a^b f(t,u,u',v,v')dt$.

Proof: Assume the contrary that $\int_a^b f(t,x,x',y,y')dt \leq \int_a^b f(t,u,u',v,v')dt$. Then, since $\lambda > 0$,

$$\int_a^b \lambda^T f(t,x,x',y,y')dt < \int_a^b \lambda^T f(t,u,u',v,v')dt. \tag{5}$$

From (3) and (4),

$$\int_a^b \eta(x,u)^T \left[\lambda^T f_x(t,u,u',v,v') - \frac{d}{dt}\lambda^T f_{x'}(t,u,u',v,v')\right] dt$$
$$\geqq \int_a^b [\eta(x,u) + u(t)]^T \left[\lambda^T f_x(t,u,u',v,v') - \frac{d}{dt}\lambda^T f_{x'}(t,u,u',v,v')\right] dt \geqq 0.$$

Since $\int_a^b \lambda^T f$ is pseudo-invex in x and x', we have

$$\int_a^b \lambda^T f(t,x,x',v,v')dt - \int_a^b \lambda^T f(t,u,u',v,v')dt \geqq 0. \tag{6}$$

From (1) and (2),

$$-\int_a^b \xi(v,y)^T \left[\lambda^T f_y(t,x,x',y,y') - \frac{d}{dt}\lambda^T f_{y'}(t,x,x',y,y')\right] dt$$

$$\geqq -\int_a^b [\xi(v,y)+y(t)]^T \left[\lambda^T f_y(t,x,x',y,y') - \frac{d}{dt}\lambda^T f_{y'}(t,x,x',y,y')\right] dt \geqq 0.$$

Since $-\int_a^b \lambda^T f$ is pseudo-invex in y and y', we have

$$-\int_a^b \lambda^T f(t,x,x',v,v')dt + \int_a^b \lambda^T f(t,x,x',y,y')dt \geqq 0. \tag{7}$$

From (6) and (7), $\int_a^b \lambda^T f(t,x,x',y,y')dt \geqq \int_a^b \lambda^T f(t,u,u',v,v')dt$, a contradiction to (5). Thus the result holds. □

In the following theorems and proofs, $\lambda^{*T} f^*$ represents $\lambda^{*T} f(t,x^*,x^{*\prime},y^*,y^{*\prime})$ and partial derivatives are similarly denoted.

Theorem 3.2. (Strong duality) *Let (x^*,y^*,λ^*) be an efficient solution for (MSP). Suppose that*

$$\left[p(t)^T\left(\lambda^{*T} f^*_{yy} - \frac{d}{dt}\lambda^{*T} f^*_{yy'}\right) + \frac{d}{dt}\left(p(t)^T\frac{d}{dt}\lambda^{*T} f^*_{y'y'}\right)\right.$$

$$\left. + \frac{d^2}{dt^2}\left(-\,p(t)^T\lambda^{*T} f^*_{y'y'}\right)\right] p(t) = 0 \tag{8}$$

only has the solution $p(t)=0$ for all $t\in[a,b]$, and the set

$$\left\{f_y^{i*} - \frac{d}{dt}f_{y'}^{i*} : i=1,2,\cdots,p\right\} \text{ is linearly independent.} \tag{9}$$

Then (x^*,y^*,λ^*) is feasible for (MSD). If, in addition, the pseudo-invexity conditions of Theorem 3.1 are satisfied, then (x^*,y^*,λ^*) is an efficient solution for (MSD), and the optimal values of (MSP) and (MSD) are equal.

Proof: Applying the necessary conditions of Valentine [10], if (x^*, y^*, λ^*) is an efficient solution of (MSP), then there exist $\alpha \in R^p$, $\beta : [a,b] \to R^m$, $\gamma \in R$ and $\delta \in R^p$ such that

$$H^* \equiv \alpha^T f^* - \beta(t)^T \left(\frac{d}{dt}\lambda^{*T} f^*_{y'} - \lambda^{*T} f^*_y\right) - \gamma y^{*T}\left(\lambda^{*T} f^*_y - \frac{d}{dt}\lambda^{*T} f^*_{y'}\right) - \delta^T \lambda^*$$

satisfies

$$H^*_y - \frac{d}{dt}H^*_{y'} + \frac{d^2}{dt^2}H^*_{y''} = 0, \tag{10}$$

$$H^*_x - \frac{d}{dt}H^*_{x'} + \frac{d^2}{dt^2}H^*_{x''} = 0, \tag{11}$$

$$(\beta - \gamma y^*)^T\left(f^*_y - \frac{d}{dt}f^*_{y'}\right) - \delta = 0, \tag{12}$$

$$\beta^T\left(\frac{d}{dt}\lambda^{*T} f^*_{y'} - \lambda^{*T} f^*_y\right) = 0, \tag{13}$$

$$\gamma y^{*T}\left(\lambda^{*T} f^*_y - \frac{d}{dt}\lambda^{*T} f^*_{y'}\right) = 0, \tag{14}$$

$$\delta^T \lambda^* = 0, \tag{15}$$

$$(\alpha, \beta, \gamma, \delta) \geq 0, \tag{16}$$

throughout $[a,b]$ (except at coners of $(x^{*\prime}(t), y^{*\prime}(t))$ where (10) and (11) hold for unique right-and left-hand limits). $\alpha, \beta(t), \gamma$ and δ cannot be simultaneously zero at any $t \in [a,b]$, and β is continuous except perhaps at corners of $(x^{*\prime}(t), y^{*\prime}(t))$.

From (10), we have

$$\begin{aligned}&(\beta - \gamma y^*)^T\left(\lambda^{*T} f^*_{yy} - \frac{d}{dt}\lambda^{*T} f^*_{y'y}\right) + \frac{d}{dt}\left((\beta - \gamma y^*)^T \frac{d}{dt}\lambda^{*T} f^*_{y'y'}\right) \\ &+ \frac{d^2}{dt^2}\left(-(\beta - \gamma y^*)^T \lambda^{*T} f^*_{y'y'}\right) + (\alpha - \gamma\lambda^*)^T\left(f^*_y - \frac{d}{dt}f^*_{y'}\right) = 0.\end{aligned} \tag{17}$$

From (11), we have

$$\begin{aligned}&\alpha^T f^*_x + (\beta - \gamma y^*)^T\left(\lambda^{*T} f^*_{yx} - \frac{d}{dt}\lambda^{*T} f^*_{y'x}\right) - \frac{d}{dt}\alpha^T f^*_{x'} \\ &- \frac{d}{dt}\left((\beta - \gamma y^*)^T(\lambda^{*T} f^*_{yx'} - \frac{d}{dt}\lambda^{*T} f^*_{y'x'} - \lambda^{*T} f^*_{y'x})\right) \\ &+ \frac{d^2}{dt^2}\left(-(\beta - \gamma y^*)^T \lambda^{*T} f^*_{y'x'}\right) = 0.\end{aligned} \tag{18}$$

Multiplying (17) by $\beta - \gamma y^*$ and then using (12), (13), (14) and (15) gives

$$\left[(\beta - \gamma y^*)^T \left(\lambda^{*T} f^*_{yy} - \frac{d}{dt}\lambda^{*T} f^*_{y'y}\right) + \frac{d}{dt}\left((\beta - \gamma y^*)^T \frac{d}{dt}\lambda^{*T} f^*_{y'y'}\right)\right.$$

$$\left. + \frac{d^2}{dt^2}\left(-(\beta - \gamma y^*)^T \lambda^{*T} f^*_{y'y'}\right)\right](\beta - \gamma y^*) = 0.$$

Thus by the assumption (8),

$$\beta = \gamma y^*. \tag{19}$$

From (17), we have

$$(\alpha - \gamma\lambda^*)^T \left(f^*_y - \frac{d}{dt} f^*_{y'}\right) = 0.$$

By the assumption (9),

$$\alpha = \gamma\lambda^*. \tag{20}$$

This gives $\gamma > 0$, since if $\gamma = 0$, then by (12), (18) and (19) $\alpha = \beta(t) = \delta = 0$ for all $t \in [a, b]$, contradicting the necessary condition (16). The equation (18) with (20) now becomes

$$\lambda^{*T} f^*_x - \frac{d}{dt}\lambda^{*T} f^*_{x'} = 0 \text{ and } x^{*T}\left(\lambda^{*T} f^*_x - \frac{d}{dt}\lambda^{*T} f^*_{x'}\right) = 0. \tag{21}$$

By (21), (x^*, y^*, λ^*) is feasible for (MSD). If the pseudo-invexity conditions of Theorem 3.1 are satisfied, then by weak duality, (x^*, y^*, λ^*) is an efficient solution for (MSD), and the optimal values of (MSP) and (MSD) are equal. □

Remark 3.2. If f does not explicitly depend on y', the system reduces to $p(t)^T \lambda^{*T} f^*_{yy}$ $p(t) = 0$, which has only a zero solution iff $\lambda^{*T} f^*_{yy}$ is positive or negative definite for all $t \in [a, b]$.

A converse duality theorem may be stated; the proof would be analogous to that of Theorem 3.2.

Theorem 3.3. (Converse duality) *Let (x^*, y^*, λ^*) be an efficient solution for (MSD). Assume that the system*

$$\left[p(t)^T \left(\lambda^{*T} f^*_{xx} - \frac{d}{dt}\lambda^{*T} f^*_{xx'}\right) + \frac{d}{dt}\left(p(t)^T \frac{d}{dt}\lambda^{*T} f^*_{x'x'}\right)\right.$$
$$\left. + \frac{d^2}{dt^2}\left(-\, p(t)^T \lambda^{*T} f^*_{x'x'}\right)\right] = 0$$

only has the solution $p(t) = 0$, $t \in [a, b]$ and the set $\{f^{i*}_x - \frac{d}{dt} f^{i*}_{x'} : i = 1, 2, \cdots, p\}$ is linearly independent. Then (x^*, y^*, λ^*) is feasible for (MSP). If, in addition, the pseudo-invexity conditions of Theorem 3.1 are satisfied, then (x^*, y^*, λ^*) is an efficient solution for (MSP), and the optimal values of (MSP) and (MSD) are equal.

4. SELF DUALITY

Assume that $m = n, f(t, x, x', y, y') = -f(t, y, y', x, x')$ (i.e., f skew-symmetric) for all $(x(t), y(t)), t \in [a, b]$ such that $(x'(t), y'(t))$ is piecewise smooth on $[a, b]$ and that $x_0 = y_0, x_1 = y_1$.

It follows that (MSD) may be rewritten as a minimization problem :

(MSD')

$$\begin{array}{ll} \text{Minimize} & \displaystyle\int_a^b f(t, y, y', x, x')dt \\ \text{subject to} & x(a) = x_0, \;\; x(b) = x_1, \;\; y(a) = x_0, \;\; y(b) = x_1, \\ & \lambda^T f_x(t, y, y, x, x') - \dfrac{d}{dt}\lambda^T f_{x'}(t, y, y', x, x') \leqq 0, \\ & x^T \left[\lambda^T f_x(t, y, y', x, x') - \dfrac{d}{dt}\lambda^T f_{x'}(t, y, y', x, x')\right] \geqq 0, \\ & \lambda > 0, \;\; \lambda^T e = 1. \end{array}$$

(MSD') is formally identical to (MSP); that is, the objective and constraint functions and initial conditions of (MSP) and (MSD') are identical. This problem is said to be self-dual.

It is easily seen that whenever (x, y, λ) is feasible for (MSP), then (y, x, λ) is feasible for (MSD), and vice versa.

Theorem 4.1. *Assume that (MSP) is self-dual and the pseudo-invexity conditions of Theorem 3.1 are satisfied. If (x^*, y^*, λ^*) is an efficient solution for (MSP) and the assumptions of Theorem 3.2 hold, then (y^*, x^*, λ^*) is an efficient solution for both (MSP) and (MSD), and the common optimal value is 0.*

Proof: By Theorem 3.2, (x^*, y^*, λ^*) is an efficient solution for (MSD), and the optimal values of (MSP) and (MSD) are equal to $\int_a^b f(t, x^*, x^{*\prime}, y^*, y^{*\prime})dt$. From self-duality, (y^*, x^*, λ^*) is feasible for both (MSP) and (MSD), so Theorems 3.1 and 3.2 give optimality in both problems, and thus objective values of $\int_a^b f(t, y^*, y^{*\prime}, x^*, x^{*\prime})dt$. But $\int_a^b f(t, y^*, y^{*\prime}, x^*, x^{*\prime})dt = -\int_a^b f(t, x^*, x^{*\prime}, y^*, y^{*\prime})dt$ by skew-symmetry of f. Hence

$$\int_a^b f(t, x^*, x^{*\prime}, y^*, y^{*\prime})dt = -\int_a^b f(t, x^*, x^{*\prime}, y^*, y^{*\prime})dt = 0. \qquad \square$$

5. STATIC SYMMETRIC DUAL PROGRAMS

If the time dependency of programs (MSP) and (MSD) is removed and f is considered to have domain $R^n \times R^m$, we obtain the symmetric dual pair given by

(SP) Minimize $f(x, y)$

subject to

$$(\lambda^T f)_y(x, y) \leqq 0,$$
$$y^T(\lambda^T f)_y(x, y) \geqq 0,$$
$$\lambda > 0, \quad \lambda^T e = 1.$$

(SD) Maximize $f(u, v)$

subject to

$$(\lambda^T f)_x(u, v) \geqq 0,$$
$$u^T(\lambda^T f)_x(u, v) \leqq 0,$$
$$\lambda > 0, \quad \lambda^T e = 1.$$

The following duality theorems can be proved along the lines of Theorems 3.1, 3.2 and 3.3.

Theorem 5.1. *Let (x, y, λ) be feasible for (SP) and (u, v, λ) be feasible for (SD). Assume that $\lambda^T f$ is pseudo-invex in x, and $-\lambda^T f$ is pseudo-invex in y, with $\eta(x,u)+u \geqq 0$ and $\xi(v,y)+y \geqq 0$.*
Then $f(x,y) \nleq f(u,v)$.

Theorem 5.2. *Let (x^*, y^*, λ^*) be an efficient solution for (SP). Assume that ${\lambda^*}^T f^*_{yy}$ is positive or negative definite, and the set $\{f^{i*}_y : i = 1, 2, \cdots, p\}$ is linearly independent. Then (x^*, y^*, λ^*) is feasible for (SD). If, in addition, the pseudo-invexity conditions of Theorem 5.1 are satisfied, then (x^*, y^*, λ^*) is an efficient solution for (SD), and the optimal values of (SP) and (SD) are equal.*

Theorem 5.3. *Let (x^*, y^*, λ^*) be an efficient solution for (SD). Assume that ${\lambda^*}^T f^*_{xx}$ is positive or negative definite, and the set $\{f^{i*}_x : i = 1, 2, \cdots, p\}$ is linearly independent. Then (x^*, y^*, λ^*) is feasible for (SP). If, in addition, the pseudo-invexity conditions of Theorem 5.1 are satisfied, then (x^*, y^*, λ^*) is an efficient solution for (SP), and the optimal values of (SP) and (SD) are equal.*

The pair (SP) and (SD) will be self-dual when $m = n$ and f is skew-symmetric (i.e., $f(x,y) = -f(y,x)$ for all $x, y \in R^n$).

We state without proof a static version of Theorem 4.1.

Theorem 5.4. *Assume that (SP) is self-dual and the pseudo-invexity conditions of Theorem 5.1 are satisfied. If (x^*, y^*, λ^*) is an efficient solution for (SP), ${\lambda^*}^T f^*_{yy}$ is positive or negative definite and the set $\{f^{i*}_y : i = 1, 2, \cdots, p\}$ is linearly independent,* *then (y^*, x^*, λ^*) is an efficient solution for both (SP) and (SD), and the common optimal value is* 0.

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