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# Evolution Equations with Infinite Delay

To the Memory of Professor T. Yoshizawa

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## 1 Introduction

Suppose that  $A$  is the infinitesimal generator of a  $C_0$  semigroup  $T(t)$  on a Banach space  $E$  with norm  $|\cdot|$ . We consider the evolution equation with infinite delay such that

$$u'(t) = Au(t) + L(u_t), \quad (1.1)$$

where  $u_t$  is a function mapping  $(-\infty, 0]$  into  $E$  defined by  $u_t(\theta) = u(t + \theta)$  for  $\theta \in (-\infty, 0]$ . The operator  $L$  is a bounded linear operator on the phase space  $\mathcal{B}$  into  $E$ .

$\mathcal{B}$  is a Banach space of some functions mapping  $(-\infty, 0]$  into  $E$ . The norm in  $\mathcal{B}$  is denoted by  $\|\cdot\|$ . For a complex number  $\lambda$  and for  $x \in E$  we define a function  $\varepsilon_\lambda \otimes x : (-\infty, 0] \rightarrow E$  by  $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x$ ,  $\theta \in (-\infty, 0]$ . We assume the following axioms on  $\mathcal{B}$ :

(H-1) There exists a constant  $H$  such that  $|\phi(0)| \leq H\|\phi\|$  for every  $\phi \in \mathcal{B}$ .

(H-2) If a function  $u : (-\infty, \sigma + a) \rightarrow E$  is continuous on  $[\sigma, \sigma + a)$ , and if  $u_\sigma \in \mathcal{B}$ , then

(i)  $u_t \in \mathcal{B}$  for all  $t \in [\sigma, \sigma + a)$  and  $u_t$  is a  $\mathcal{B}$  valued continuous function of  $t \in [\sigma, \sigma + a)$ ,

(ii)  $\|u_t\| \leq K(t - \sigma) \sup\{|u(s)| : \sigma \leq s \leq t\} + M(t - \sigma)\|u_\sigma\|$

for all  $t \in [\sigma, \sigma + a)$ , where  $K(r), M(r), r \geq 0$ , are nonnegative, measurable, locally bounded functions which are independent of  $u$ .

(H-3) If  $\{\phi^n\}$  is a Cauchy sequence in  $\mathcal{B}$ , and if the sequence  $\{\phi^n(\theta)\}$  converges to a function  $\phi(\theta)$  uniformly on every compact interval of  $(-\infty, 0]$ , then  $\phi$  lies in  $\mathcal{B}$  and  $\lim_{n \rightarrow \infty} \|\phi^n - \phi\| = 0$ .

(H-4) There exists a constant  $\gamma$  such that  $\varepsilon_\lambda \otimes x \in \mathcal{B}$  for  $\Re \lambda > \gamma$  and  $x \in E$ , and that

$$\|\varepsilon_\lambda\| := \sup\{\|\varepsilon_\lambda \otimes x\| : x \in E, |x| \leq 1\}$$

is finite for each  $\lambda$  with  $\Re \lambda > \gamma$ , and bounded for  $\Re \lambda > \gamma_1$  for some  $\gamma_1 \geq \gamma$ .

We call the constant  $\gamma$  in (H-4) the abscissa of the exponent of the space  $\mathcal{B}$ . The hypothesis (H-3) implies that the integral in  $\mathcal{B}$  is computed from the integral in  $E$  in the following manner.

**Lemma 1.1** *If  $f : [a, b] \rightarrow \mathcal{B}$  is a continuous function such that  $f(t)(\theta)$  is continuous for  $(t, \theta) \in [a, b] \times (-\infty, 0]$ , then*

$$\left[ \int_a^b f(t) dt \right] (\theta) = \int_a^b f(t)(\theta) dt, \quad \theta \in (-\infty, 0].$$

The growth bound  $\omega_s(T)$ , and the essential growth bound  $\omega_e(T)$  are defined by

$$\omega_s(T) := \lim_{t \rightarrow \infty} \frac{\log \|T(t)\|}{t} = \inf_{t > 0} \frac{\log \|T(t)\|}{t},$$

$$\omega_e(T) := \lim_{t \rightarrow \infty} \frac{\log \alpha(T(t))}{t} = \inf_{t > 0} \frac{\log \alpha(T(t))}{t},$$

where  $\alpha(T(t))$  is the measure of noncompactness of  $T(t)$  which is introduced by the Kuratowski measure of noncompactness of bounded sets of  $X$ , cf [2]. Then the spectral radius  $r_s(T(t))$  and the essential spectral radius  $r_e(T(t))$  are given as  $r_s(T(t)) = \exp(t\omega_s(T))$  and  $r_e(T(t)) = \exp(t\omega_e(T))$ . Let  $A$  be the infinitesimal generator of  $T(t)$ ,  $\sigma(A)$  the spectrum of  $A$ ,  $E_\sigma(A)$  the essential spectrum of  $A$ , and set  $N_\sigma(A) := \sigma(A) \setminus E_\sigma(A)$ . The points in  $N_\sigma(A)$  are called normal eigenvalues of  $A$ .

The important fact for our works is that the following inequalities hold:

$$\beta_s(A) := \sup\{\Re \lambda : \lambda \in \sigma(A)\} \leq \omega_s(T)$$

$$\beta_e(A) := \sup\{\Re \lambda : \lambda \in E_\sigma(A)\} \leq \omega_e(T).$$

The first inequality is well known. The second inequality is proved in Webb [7], and it implies the following theorems.

**Theorem 1.2** *Let  $T(t)$  and  $A$  be as in the above, and suppose that  $\omega_e(T) < \omega_s(T)$ . Then the following results hold:*

(i) *There exists at least one point  $\lambda \in N_\sigma(A)$  such that  $\Re\lambda = \omega_s(T)$ : consequently,  $\beta_s(A) = \omega_s(T)$  and  $N_\sigma(A) \neq \emptyset$ .*

(ii) *For any  $b, \omega_e(T) < b < \omega_s(T)$ , the set  $\sigma(A) \cap \{\lambda : \Re\lambda \geq b\}$  consists of finite normal eigenvalues of  $A$ , and  $\sup\{\Re\lambda : \lambda \in \sigma(A), \Re\lambda < b\} < b$ .*

## 2 Semigroup generated by mild solutions

Let  $\phi \in \mathcal{B}$ . The strong solution of (1.1) through  $(0, \phi)$  is a function  $u : (-\infty, \infty) \rightarrow E$  which has the following properties: (i)  $u_0 = \phi$  and  $u$  is continuous, differentiable on  $[0, \infty)$ , and  $u(t) \in D(A)$  for  $t \geq 0$ ; (ii) (1.1) holds for  $t \geq 0$ . The mild solution of (1.1) through  $(0, \phi)$  is a function  $u : (-\infty, \infty) \rightarrow E$  which has the following properties: (i)  $u_0 = \phi$  and  $u$  is continuous on  $[0, \infty)$ ;

$$(ii) \quad u(t) = T(t)\phi(0) + \int_0^t T(t-s)L(u_s) ds, \quad t \geq 0.$$

By the usual method of successive approximation, we can prove that, for every  $\phi \in \mathcal{B}$ , there exists a unique mild solution through  $(0, \phi)$ ; cf. [3],[5],[6], and the references therein.

Denote by  $u(t, \phi)$  this mild solution. Define the solution operator  $U_L(t)$  on  $\mathcal{B}$  by

$$(U_L(t)\phi)(\theta) = u(t + \theta, \phi), \quad \theta \in (-\infty, 0].$$

Then using the axioms of the phase space, we see that  $U_L(t)$  is a  $C_0$  semigroup of bounded linear operators on  $\mathcal{B}$ . Denote by  $A_L$  the infinitesimal generator of  $U_L(t)$ .

In the particular case that  $L \equiv 0$ ,  $U_0(t)$  is given by  $(U_0(t)\phi)(\theta) = T(t + \theta)\phi(0)$ ,  $-t < \theta \leq 0$  and  $(U_0(t)\phi)(\theta) = \phi(t + \theta)$ ,  $\theta \leq -t$ . Set  $K_L(t) = U_L(t) - U_0(t)$ , which is given by  $(K_L(t)\phi)(\theta) = z(t + \theta, \phi)$ ,  $\theta \in (-\infty, 0]$ ,

where

$$z(t, \phi) = \begin{cases} \int_0^t T(t-s)L(u_s) ds & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Taking the Laplace transform of  $U_L(t) = U_0(t) + K_L(t)$ , we can compute the resolvent  $R(\lambda, A_L)$ . To describe the result, we introduce the closed linear operator  $\Delta(\lambda)$  as  $\Delta(\lambda)x = (\lambda I - A - L_\lambda)x$ ,  $x \in D(A)$ , where  $L_\lambda x = L(\varepsilon_\lambda \otimes x)$ . It is well defined for  $\Re \lambda > \gamma$ . If  $\lambda \in \rho(A)$ , then we can write  $(\lambda I - A - L_\lambda) = (I - L_\lambda R(\lambda, A))(\lambda I - A)$ . Hence  $\Delta(\lambda)^{-1}$  exists as a bounded linear operator on  $\mathcal{B}$  as long as  $\Re \lambda$  is sufficiently large. Let  $A_0$  be the infinitesimal generator of  $U_0(t)$ .

**Theorem 2.1** *There exists an  $\omega$  such that, if  $\Re \lambda > \omega$ , then*

$$R(\lambda, A_L)\phi = R(\lambda, A_0)\phi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}L(R(\lambda, A_0)\phi), \quad \phi \in \mathcal{B}.$$

Since  $\phi = R(\lambda, A_L)(\lambda\phi - A_L\phi)$  for  $\phi \in D(A_L)$ , the equation  $A_L\phi = \psi$  holds if and only if  $\phi = R(\lambda, A_L)(\lambda\phi - \psi)$ . Namely, we can compute  $A_L$  itself from the representation of  $R(\lambda, A_L)$ . To do so, we use the infinitesimal generator  $B$  of the trivial  $C_0$  semigroup  $S(t)$  on  $\mathcal{B}$  defined as  $[S(t)\phi](\theta) = \phi(0)$ ,  $t + \theta \geq 0$ , and  $[S(t)\phi](\theta) = \phi(t + \theta)$ ,  $t + \theta < 0$ .

**Theorem 2.2** *A function  $\phi$  lies in  $D(A_L)$  if and only if  $\phi(0) \in D(A)$  and  $\phi - \lambda^{-1}\varepsilon_\lambda \otimes (A\phi(0) + L(\phi)) \in D(B)$  for some  $\lambda > \omega$ , and*

$$A_L\phi = \varepsilon_\lambda \otimes (A\phi(0) + L(\phi)) + B\left(\phi - \lambda^{-1}\varepsilon_\lambda \otimes (A\phi(0) + L(\phi))\right).$$

*In particular,  $(A_L\phi)(0) = A\phi(0) + L(\phi)$  for  $\phi \in D(A_L)$ .*

The second equation above follows from the fact that  $(B\phi)(0) = 0$  for  $\phi \in D(B)$ . As a result, we have the following theorem.

**Theorem 2.3** *If  $\phi \in D(A_L)$ , the mild solution  $u(t, \phi)$  is a strong solution.*

### 3 Growth bound and compact semigroups

The axiom (H-4) says that, if  $\Re\lambda > \gamma$ ,  $\varepsilon_\lambda$  is regarded as an element of  $\mathcal{L}(E, \mathcal{B})$ , the Banach space of bounded linear operators on  $E$  into  $\mathcal{B}$ . We have the following estimate for the abscissa  $\gamma$ . Let  $S_0(t)$  be the restriction of  $S(t)$  to the subspace  $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$ .

**Theorem 3.1** *If  $\Re\lambda > \omega_s(S_0)$ , then  $\varepsilon_\lambda$  lies in  $\mathcal{L}(E, \mathcal{B})$ , and it is holomorphic for  $\lambda$ . Hence the abscissa  $\gamma$  in (H-4) satisfies  $\gamma \leq \omega_s(S_0)$ .*

**Theorem 3.2** *For the semigroup  $S_0(t)$ , the essential growth bound is the same as the growth bound:  $\omega_e(S_0) = \omega_s(S_0)$ .*

Let  $\mathcal{BU}_\gamma$  be the space of continuous functions  $\phi : (-\infty, 0] \rightarrow E$  such that  $e^{-\gamma\theta}\phi(\theta)$  is bounded, uniformly continuous for  $\theta \in (-\infty, 0]$ , and define the norm in this space as  $\|\phi\| = \sup\{e^{-\gamma\theta}|\phi(\theta)| : \theta \in (-\infty, 0]\}$ . Then this space satisfies the axioms (H-1,2,3,4), and  $\gamma$  in the definition is the abscissa of the exponent of this space.

Another space for  $\mathcal{B}$  is made of the set of measurable functions  $\phi : (-\infty, 0] \rightarrow E$  such that  $e^{-\gamma\theta}|\phi(\theta)|$  is integrable on  $(-\infty, 0]$ , where the seminorm is defined by

$$\|\phi\| = |\phi(0)| + \int_{-\infty}^0 e^{-\gamma\theta}|\phi(\theta)| d\theta.$$

Denote by  $E \times \mathcal{L}_\gamma$  the quotient space with respect to this seminorm. Then this space is a Banach space satisfying (H-1,2,3,4), and  $\gamma$  is the abscissa of the exponent.

**Theorem 3.3** *If  $\mathcal{B} = \mathcal{BU}_\gamma$  or  $E \times \mathcal{L}_\gamma$ , then  $\omega_e(S_0) = \omega_s(S_0) = \gamma$ .*

Suppose that the original semigroup  $T(t)$  is a compact semigroup. Then  $K_L(t) = U_L(t) - U_0(t)$  is a compact operator for  $t > 0$ . This implies that  $\alpha(U_L(t)) = \alpha(U_0(t))$  for  $t > 0$ . The following theorem follows from the definition of the essential growth bound.

**Theorem 3.4** *If  $T(t)$  is a compact semigroup, then  $\omega_e(U_L) = \omega_e(U_0)$ .*

We have the estimate of  $\alpha(U_0(t))$  in terms of  $H, K(r), M(r)$  in the axioms of the phase space  $\mathcal{B}$ , and the constant  $\gamma_T = \overline{\lim}_{t \rightarrow 0} \|T(t)\|$ .

**Theorem 3.5** *Let  $T(t)$  be a compact semigroup on  $E$ . Then the following estimates hold for  $t > 0$ .*

(i)  $\alpha(U_0(t)) \leq C_1 \overline{\lim}_{s \rightarrow t-0} M(s)$ , where  $C_1 = H \overline{\lim}_{\epsilon \rightarrow 0} K(\epsilon) \max\{1, \gamma_T\} + \overline{\lim}_{\epsilon \rightarrow 0} M(\epsilon)$ .

(ii) *Suppose that every bounded, continuous function  $\phi : (-\infty, 0] \rightarrow E$  lies in  $\mathcal{B}$  in the manner that  $\sup\{|\phi(\theta)| : \theta \in (-\infty, 0]\} \leq J\|\phi\|$ , where  $J$  is a constant independent of  $\phi$ . Then  $\alpha(U_0(t)) \leq (1 + JH)C_1 \overline{\lim}_{s \rightarrow t-0} \alpha(S_0(s))$ .*

## 4 An example

Let  $E = L^2([0, \pi], C)$ , the set of square integrable functions on  $[0, \pi]$ . Consider the equation

$$u'(t) = Au(t) + b \int_{-\infty}^t e^{-c(t-s)} u(s) ds, \quad (4.1)$$

where  $A$  is defined as  $Af = f''$  for  $f \in E$  such that  $f$  is continuously differentiable, the derivative  $f'$  is absolutely continuous,  $f'' \in E$ , and that  $f(0) = f(\pi) = 0$ . It is well known that  $A$  is a closed linear operator with dense domain. It is self adjoint, the spectrum of  $A$  consists of only point spectrum  $\lambda = -n^2, n = 1, 2, \dots$ ,  $R(\lambda, A)$  has a pole of order 1 at these points, and  $|R(\lambda, A)| \leq 1/|\lambda + 1|$  for  $\Re \lambda > -1$ . Hence  $A$  is the infinitesimal generator of a  $C_0$  semigroup  $T(t)$  such that  $|T(t)| \leq e^{-t}$  for  $t \geq 0$ . Furthermore,  $T(t)$  is a compact semigroup.

Notice that

$$|L(\phi)| := \left| b \int_{-\infty}^0 e^{c\theta} \phi(\theta) d\theta \right| \leq |b| \int_{-\infty}^0 e^{(c+\gamma)\theta} e^{-\gamma\theta} |\phi(\theta)| d\theta.$$

If  $\gamma = -c$ , we have that  $|L(\phi)| \leq |b|\|\phi\|$  for  $\phi \in E \times \mathcal{L}_{-c}$ . If  $\gamma > -c$ , we have that  $|L(\phi)| \leq |b|(c + \gamma)^{-1}\|\phi\|$  for  $\phi \in \mathcal{BU}_\gamma$ . Hence, we can take the space  $E \times \mathcal{L}_{-c}$  or  $\mathcal{BU}_\gamma, \gamma > -c$ , for the phase space of Equation (4.1).

The characteristic operator  $\Delta(\lambda)$  now becomes

$$\Delta(\lambda)f = \lambda f - Af - b \int_{-\infty}^0 e^{(c+\lambda)\theta} d\theta f \quad f \in D(A), \quad \Re \lambda > \gamma \geq -c.$$

If we set  $h(\lambda) = \lambda - b/(c + \lambda)$  for  $\Re \lambda > -c$ , then we can write  $\Delta(\lambda) = h(\lambda)I - A$ . Since, from Theorem 3.5 (i),  $\omega_e(U_L) \leq \gamma$ , the spectrum of  $A_L$  in  $\Re \lambda > \gamma$  consists of only normal eigenvalues. Let  $\lambda$  be such an eigenvalue. Then from [4] it follows that  $N(\Delta(\lambda)) \neq \{0\}$ . Thus we see that  $h(\lambda) = -n^2$  for some  $-n^2 \in P_\sigma(A)$ . Set  $\Lambda_n = \{\lambda : h(\lambda) = -n^2, \Re \lambda > -c\}$  and  $\Lambda = \cup_{n \geq 1} \Lambda_n$ .

The equation  $h(\lambda) = -n^2$  becomes  $(\lambda + c)(\lambda + n^2) = b$ , which has the roots  $\kappa_n = [-(c + n^2) - \sqrt{D}]/2$ ,  $\lambda_n = [-(c + n^2) + \sqrt{D}]/2$ , where  $D = (c + n^2)^2 - 4(cn^2 - b)$ . We are interested in the roots whose real parts are greater than  $-c$ .

### Proposition 4.1

Case (i):  $b > 0$ .  $\Lambda = \{\lambda_n : n \geq 1\}$ , and  $\lambda_1 > \lambda_2 > \dots > \lambda_n \rightarrow -c$  ( $n \rightarrow \infty$ ).

(i-1):  $c \leq 0$ .  $\lambda_n$  are all positive; (i-2):  $c > 0$ . If  $0 < b < c$ ,  $\lambda_n$  are all negative. If  $b = n^2c, n \geq 1$ , then  $\lambda_1 > \lambda_2 > \dots > \lambda_n = 0 > \lambda_{n+1} > \dots$ . If  $n^2c < b < (n + 1)^2c, n \geq 1$ , then  $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0 > \lambda_{n+1} > \dots$ .

Case (ii):  $b \leq 0$ .  $\Lambda$  is an empty set or finite set.

(ii-1):  $c \leq 1$ .  $\Lambda = \emptyset$ ; (ii-2):  $c > 1$ . There exists an integer  $n_c$  such that  $\Lambda = \{\kappa_n, \lambda_n : 1 \leq n \leq n_c\}$ , and the following cases occur. If  $b < -(c - 1)^2/4$ , then  $\kappa_n, \lambda_n$  are all imaginary numbers with real part  $x_n \leq x_1 = -(c + 1)/2$ . If  $-(c - 1)^2/4 \leq b \leq 0$ , then, for some  $k \leq n_c$ , the first  $k$  terms of both  $\{\kappa_n\}$  and  $\{\lambda_n\}$  are real numbers for which  $\lambda_1 < 0$  is the maximum number, and the rest terms are imaginary numbers whose real parts are less than  $\lambda_1$ .

**Theorem 4.2** Suppose that  $\lambda_n \in \Lambda$ . If  $b = 0$  or  $\lambda_n$  is a simple root of  $h(\lambda) = -n^2$ , then  $R(\lambda, A_L)$  has a pole of order 1 at  $\lambda = \lambda_n$ .  $M_{\lambda_0}(A_L) = N(A_L - \lambda_n I)$  consists of functions  $\phi(\theta, x) = a_0 e^{\lambda_n \theta} \sin nx$ , where  $a_0$  is an arbitrary complex constant.

If  $\lambda_n$  is a double root of  $h(\lambda) = -n^2$ , then  $R(\lambda, A_L)$  has a pole of order 2 at  $\lambda = \lambda_n$ .  $M_{\lambda_0}(A_L) = N((A_L - \lambda_n I)^2)$  consists of functions  $\phi(\theta, x) = (a_0 + a_1 \theta) e^{\lambda_n \theta} \sin nx$ , where  $a_0, a_1$  are arbitrary complex constants.

**Proof** From the theorem [4],  $\phi \in N(A_L - \lambda_n)$  if and only if  $\phi = \varepsilon_{\lambda_n} \otimes f$



for some  $f \in E$  such that  $\Delta(\lambda_n)f = 0$ . Recall that

$$\Delta(\lambda_n)f = (\lambda_n I - A - \frac{b}{\lambda_n + c}I)f = (-n^2 - A)f.$$

From the definition of  $A$  it follows that  $f(x) = a_0 \sin nx, 0 \leq x \leq \pi$ .

In the similar manner,  $\phi \in N((A_L - \lambda_n)^2)$  if and only if  $\phi = \varepsilon_{\lambda_n} \otimes f_0 + \varepsilon'_{\lambda_n} \otimes f_1$  for some  $f_0, f_1$  such that  $D_1(\lambda_n)\text{col}[f_0, f_1] = 0$ , i.e.

$$\Delta(\lambda_n)f_0 + \Delta'(\lambda_n)f_1 = 0 \quad \Delta(\lambda_n)f_1 = 0.$$

It is easy to see that  $\Delta'(\lambda_n) = (1 + b(\lambda_n + c)^{-2})I$ .

Now consider the condition that  $h(\lambda) = -n^2$  and  $\Delta'(\lambda) = 0$ , that is,

$$(\lambda + c)(\lambda + n^2) = b \quad 1 + b(\lambda + c)^{-2} = 0.$$

From the second equation,  $b \neq 0$ . Eliminating  $b$ , we have that  $\lambda + n^2 = -(\lambda + c)$ ; hence,  $\lambda = -(c + n^2)/2 = x_n$ , the  $x$  coordinate of the minimum point of  $\Gamma_n$ . This means that,  $\Delta'(\lambda_n) = 0$  if and only if  $b \neq 0$  and  $\lambda_n$  is a double root of  $h(\lambda) = -n^2$ .

Suppose that  $b = 0$  or  $\lambda_n$  is a simple root. From the equation  $\Delta(\lambda_n)f_1 = 0$ , we have  $f_1(x) = a_1 \sin nx$ . Set  $a = (1 + b(\lambda_n + 1)^{-2})a_1$ . Then  $a \neq 0$  if and only if  $a_1 \neq 0$ , and the equation  $\Delta(\lambda_n)f_0 + \Delta'(\lambda_n)f_1 = 0$  becomes

$$-n^2 f(x) - f''(x) + a \sin nx = 0 \quad f(0) = f(\pi) = 0.$$

The solution of the first, differential equation is

$$\begin{aligned} f(x) &= \cos nx \left[ c_1 + (2n)^{-1}a \left( (2n)^{-1} \sin 2nx - x \right) \right] \\ &+ \sin nx \left( c_0 - (2n)^{-1}a \cos 2nx \right). \end{aligned}$$

It satisfies the boundary condition if and only if  $c_1 = a = 0$ . Thus we have that  $f_1(x) = 0, f_0(x) = c_0 \sin nx$ . Thus the function  $\phi$  lies in  $N((A_L - \lambda_n I)^2)$  if and only if

$$\phi(\theta, x) = c_0 e^{\lambda_n \theta} \sin nx.$$

This shows that  $N(A_L - \lambda_n I)^2 = N(A_L - \lambda_n I)$ . Thus the eigenspace is the generalized eigenspace of dimension 1, and  $R(\lambda_n, A_L)$  has a simple pole at  $\lambda_n$ .

Suppose that  $b \neq 0$  and  $\lambda_n$  is a double root. Then  $\Delta(\lambda_n)f_0 = \Delta(\lambda_n)f_1 = 0$ ; hence  $\phi(\theta, x)$  is given as in the theorem. To show that  $N((A_L - \lambda_n)^3) = N((A_L - \lambda_n)^2)$ , consider the equation  $D_2(\lambda_n)\text{col}[f_0, f_1, f_2] = 0$ . Since

$$\Delta''(\lambda_n) = -2b(\lambda_n + c)^{-3}I,$$

the equation becomes

$$\Delta(\lambda_n)f_0 + \alpha f_2 = 0 \quad \Delta(\lambda_n)f_1 = 0 \quad \Delta(\lambda_n)f_2 = 0,$$

where  $\alpha = -2b(\lambda_n + c)^{-3}$ . From the second equation, it follows that  $f_1(x) = a_1 \sin nx$ . Since  $\alpha \neq 0$ , we can apply the result above for the first, and the third equation. As a result, it follows that  $f_0(x) = a_0 \sin nx$ ,  $f_2(x) = 0$ . Hence,  $N((A_L - \lambda_n)^3) \subset N((A_L - \lambda_n)^2)$ , and  $N((A_L - \lambda_n I)^2)$  is the generalized eigenspace of dimension 2, and  $R(\lambda, A)$  has a pole of order 2 at  $\lambda_n$ .

Define a curve  $b = \chi(c)$  in the  $c$ - $b$  plane by  $\chi(c) = 0$ ,  $c \leq 1$ ;  $\chi(c) = -(c-1)^2/4$ ,  $c > 1$ . Following the Proposition 4.1, we divide  $c$ - $b$  plane into the subregions as  $\Pi_1 : b > \chi(c)$ ,  $-\infty < c < \infty$ ,  $\Pi_2 : b \leq \chi(c)$ ,  $c > 1$ ,  $\Pi_3 : b \leq \chi(c)$ ,  $c \leq 1$ .

**Theorem 4.3** Take the space  $\mathcal{B} = \mathcal{B}U_\gamma$ . If  $\gamma > -c$  is sufficiently close to  $-c$ , then the growth bound of  $U_L$  becomes as follows:

$$\omega_s(U_L) = \begin{cases} \lambda_1 & \text{if } (c, b) \in \Pi_1 \\ -(c+1)/2 & \text{if } (c, b) \in \Pi_2. \end{cases}$$

If  $(c, b) \in \Pi_3$ , then  $\omega_s(U_L) \leq \gamma$  whenever  $\gamma > -c$ .

**Theorem 4.4** Take the space  $\mathcal{B} = E \times \mathcal{L}_{-c}$ . Then the growth bound, and the essential growth bound of  $U_L$  becomes as follows:

$$\omega_e(U_L) = -c, \quad \omega_s(U_L) = \begin{cases} \lambda_1 & \text{if } (c, b) \in \Pi_1 \\ -(c+1)/2 & \text{if } (c, b) \in \Pi_2 \\ -c & \text{if } (c, b) \in \Pi_3. \end{cases}$$

**Theorem 4.5** If  $\mathcal{B} = E \times \mathcal{L}_{-c}$ , then the following assertions hold:

- (i) If  $(c, b) \in \Pi_1$  and  $b > c$ , then  $\lambda_1 > 0$ , and  $\|U_L(t)\| \geq e^{\lambda_1 t}$  for  $t \geq 0$ ; (ii) If  $(c, b) \in \Pi_1$  and  $b = c$ , then there exists an  $M$  such that  $1 \leq \|U_L(t)\| \leq M$

for  $t \geq 0$ . If  $(c, b) \in \Pi_1$  and  $b < c$ , then  $\lambda_1 < 0$ , and  $\|U_L(t)\| \leq M_\epsilon e^{t(\lambda_1 + \epsilon)}$  for  $t \geq 0$ ; (iii) If  $(c, b) \in \Pi_2$ , then  $\|U_L(t)\| \leq M_\epsilon e^{t(-(c+1)/2 + \epsilon)}$  for  $t \geq 0$ .

If  $\mathcal{B} = \mathcal{BU}_\gamma$ ,  $\gamma > -c$ , then the assertions above hold provided  $\gamma$  is sufficiently close to  $-c$ .

**Theorem 4.6** In the case  $(c, b) \in \Pi_3$ , we have different estimates of  $\|U_L(t)\|$  according to the choice of  $\mathcal{B}$ .

Choose  $\mathcal{B} = \mathcal{BU}_\gamma$ ,  $\gamma > -c$ . If  $c > 0$ , then we can take a negative  $\gamma$  and  $\|U_L(t)\| \leq M_\epsilon e^{t(\gamma + \epsilon)}$  for  $t \geq 0$ . If  $c \leq 0$ , then  $\gamma$  becomes positive, and we only know that  $\|U_L(t)\| \leq M_\epsilon e^{t(\gamma + \epsilon)}$  for  $t \geq 0$ .

Choose  $\mathcal{B} = E \times \mathcal{L}_{-c}$ . If  $c > 0$ , then  $\|U_L(t)\| \leq M_\epsilon e^{t(-c + \epsilon)}$  for  $t \geq 0$ . If  $c < 0$ , then  $\|U_L(t)\| \geq e^{-ct}$  for  $t \geq 0$ . If  $c = 0$ ,  $b \leq 0$ , then  $\|U_L(t)\| \geq 1$ .

**Corollary 4.7** Take the phase space as  $\mathcal{B} = \mathcal{BU}_\gamma$ ,  $\gamma > -c$ . If  $\gamma$  is sufficiently close to  $-c$ , then the null solution of Equation (4.1) has the following stability: if  $b = c > 0$ , it is stable but not asymptotically stable; if  $c > 0$ ,  $c > b$ , it is exponentially asymptotically stable. If  $(c, b) \in \Pi_1$ ,  $b > c$ , then the null solution of Equation (4.1) is not stable for any choice of  $\gamma > -c$ .

**Corollary 4.8** Take the phase space as  $\mathcal{B} = E \times \mathcal{L}_{-\gamma}$ . The null solution of Equation (4.1) is exponentially asymptotically stable if and only if  $c > 0$  and  $b < c$ . If  $c > 0$  and  $b = c$ , it is stable but not asymptotically stable. If  $c < 0$ , or if  $c \geq 0$  and  $b > c$ , then it is not stable.

In the case  $c = 0$  the equation becomes

$$u'(t) = Au(t) + b \int_{-\infty}^0 u(t + \theta) d\theta.$$

Since  $|T(t)f| \leq e^{-t}|f|$  for  $t \geq 0$ ,  $f \in E$ , it follows that

$$\begin{aligned} \|U_0(t)\phi\| &= |T(t)\phi(0)| + \int_{-t}^0 |T(t + \theta)\phi(0)| d\theta + \int_{-\infty}^{-t} |\phi(t + \theta)| d\theta \\ &\leq e^{-t}|\phi(0)| + \int_0^t e^{-s}|\phi(0)| ds + \int_{-\infty}^0 |\phi(s)| ds \\ &= |\phi(0)| + \int_{-\infty}^0 |\phi(\theta)| d\theta = \|\phi\|. \end{aligned}$$

Hence, we have that  $\|U_0(t)\| \leq 1$  for  $t \geq 1$ . Since  $\|U_0(t)\| \geq \alpha(U_0(t)) \geq 1$ , it follows that  $\|U_0(t)\| = \alpha(U_0(t)) \equiv 1$  in the space  $E \times \mathcal{L}_0$ .

If  $b = 0$ , then  $U_L(t) = U_0(t)$ , and the null solution is stable. If  $b > 0$ , the null solution is not stable from Corollary 4.8. If  $b < 0$ , from Theorem 4.6 we only know that  $\|U_L(t)\| \geq 1$ . Is the null solution stable or not? About this interesting problem, Murakami has informed us of the following stability result.

**Theorem 4.9** *If  $c = 0, b < 0$ , the null solution of Equation (4.1) is  $\mathcal{B} - E$  uniformly asymptotically stable: that is, there exists a constant  $M$  such that  $|u(t, \phi)| \leq M\|\phi\|$  for  $t \geq 0, \phi \in \mathcal{B}$ , and for any  $\epsilon > 0$  there exist a  $\tau(\epsilon) > 0$  such that  $|u(t, \phi)| \leq \epsilon\|\phi\|$  for  $t > \tau(\epsilon), \phi \in \mathcal{B}$ .*

**Proof.** Let  $\{f_n\}, n = 1, 2, \dots$ , be the complete orthonormal system of the self adjoint operator  $A$ , and set  $u^n(t) = \langle u(t), f^n \rangle$ . Then  $u(t) = \sum_{n \geq 1} u^n(t) f^n$ , and  $T(t)u(0) = \sum_{n \geq 1} e^{-n^2 t} u^n(0) f^n, t \geq 0$ . Since

$$L(u_s) = \sum_{n \geq 1} \langle L(u_s), f^n \rangle f^n,$$

it follows that

$$T(t-s)L(u_s) = \sum_{n \geq 1} e^{-n^2(t-s)} \langle L(u_s), f^n \rangle f^n.$$

From the definition of  $L$ , it follows that

$$\langle L(u_s), f^n \rangle = \langle b \int_{-\infty}^0 u(s+\theta) d\theta, f^n \rangle = b \int_{-\infty}^s \langle u(r), f^n \rangle dr.$$

Hence  $u^n(t)$  satisfies the equation

$$u^n(t) = e^{-n^2 t} u^n(0) + \int_0^t e^{-n^2(t-s)} b \int_{-\infty}^s u^n(r) dr ds.$$

Taking the derivatives successively, we know that  $u^n(t)$  satisfies the equation

$$x'(t) = -n^2 x(t) + b \int_{-\infty}^t x(r) dr$$

with the initial condition  $x(\theta) = \phi^n(\theta) := \langle \phi(\theta), f^n \rangle$ ,  $\theta \in (-\infty, 0]$ , and the equation  $x''(t) = -n^2 x'(t) + bx(t)$  with the initial conditions

$$x(0) = \phi^n(0), \quad x'(0) = -n^2 \phi^n(0) + b \int_{-\infty}^0 \phi^n(\theta) d\theta.$$

Set  $\lambda_{\pm}^n = (-n^2 \pm \sqrt{D_n})/2$ ,  $D_n = n^4 + 4b$ . If  $D_n \neq 0$ , the solution is given as

$$\begin{aligned} u^n(t) &= x(t) \\ &= \frac{-\lambda_-^n x(0) + x'(0)}{\sqrt{D_n}} e^{\lambda_+^n t} + \frac{\lambda_+^n x(0) - x'(0)}{\sqrt{D_n}} e^{\lambda_-^n t} \\ &= \frac{(n^2 + \sqrt{D_n})e^{\lambda_-^n t} - (n^2 - \sqrt{D_n})e^{\lambda_+^n t}}{2\sqrt{D_n}} \phi^n(0) \\ &\quad + \frac{(e^{\lambda_+^n t} - e^{\lambda_-^n t})b}{\sqrt{D_n}} \int_{-\infty}^0 \phi^n(\theta) d\theta. \end{aligned}$$

If  $D_n = 0$ , then  $\lambda_{\pm}^n = -n^2/2$ , and the solution is given as

$$\begin{aligned} u^n(t) &= x(t) \\ &= \left[ (1 + (n^2 t/2))x(0) + tx'(0) \right] e^{-n^2 t/2} \\ &= \left[ (1 - (n^2 t/2))\phi^n(0) + bt \int_{-\infty}^0 \phi^n(\theta) d\theta \right] e^{-n^2 t/2}. \end{aligned}$$

Since  $b < 0$ , it follows that  $\Re \lambda_{\pm}^n < 0$  for  $n \geq 1$ , and there exists a constant  $c_1$  such that, for  $D_n \neq 0$ ,

$$|u^n(t)| \leq c_1 \left[ |\phi^n(0)| + \frac{|b|}{\sqrt{|D_n|}} \int_{-\infty}^0 |\phi^n(\theta)| d\theta \right],$$

and for  $D_n = 0$ ,

$$|u^n(t)| \leq c_1 \left[ |\phi^n(0)| + |b| \int_{-\infty}^0 |\phi^n(\theta)| d\theta \right].$$

Since

$$\int_{-\infty}^0 |\phi^n(\theta)| d\theta \leq \int_{-\infty}^0 |\phi(\theta)| d\theta$$

for  $n \geq 1$ , it follows that

$$\begin{aligned} \left( \sum_{n=1}^{\infty} |u^n(t)|^2 \right)^{1/2} &\leq c_1 \left( \sum_{n=1}^{\infty} |\phi^n(0)|^2 \right)^{1/2} \\ &\quad + c_1 |b| \left( 1 + \sum_{n^4+4b \neq 0} \frac{1}{|D_n|} \right)^{1/2} \int_{-\infty}^0 |\phi(\theta)| d\theta. \end{aligned}$$

for  $t \geq 0$ . Thus there exists a constant  $M$  such that

$$|u(t, \phi)| \leq M \|\phi\| \quad \text{for } t \geq 0, \phi \in \mathcal{B}.$$

Take an  $N$  such that  $D_n > 0$  for  $n \geq N$ . For  $n \geq N$ , set

$$h_n = (n^2 + \sqrt{D_n})/2\sqrt{D_n}, \quad k_n = (n^2 - \sqrt{D_n})/2\sqrt{D_n}.$$

Since  $e^{\lambda_-^n t} \leq e^{-t}$ ,  $e^{\lambda_+^n t} \leq 1$ , it follows that

$$|u^n(t)| \leq (|h_n|e^{-t} + |k_n|)|\phi^n(0)| + \frac{2|b|}{\sqrt{D_n}} \int_{-\infty}^0 |\phi(\theta)| d\theta.$$

Set  $H_m = \sup\{|h_n| : n \geq m\}$ ,  $K_m = \sup\{|k_n| : n \geq m\}$ ,  $m \geq N$ . Since  $\lim_{n \rightarrow \infty} |h_n| = 1$ , and since  $\lim_{n \rightarrow \infty} k_n = 0$ ,  $H_m$  is bounded,  $\lim_{m \rightarrow \infty} K_m = 0$ , and

$$\begin{aligned} \left( \sum_{n \geq m} |u^n(t)|^2 \right)^{1/2} &\leq (H_m e^{-t} + K_m) \left( \sum_{n \geq m} |\phi^n(0)|^2 \right)^{1/2} \\ &\quad + \left( \sum_{n \geq m} \frac{4|b|^2}{D_n} \right)^{1/2} \int_{-\infty}^0 |\phi(\theta)| d\theta \\ &\leq \left( H_m e^{-t} + K_m + \left( \sum_{n \geq m} \frac{4|b|^2}{D_n} \right)^{1/2} \right) \|\phi\|. \end{aligned}$$

Let  $\epsilon > 0$ . Then there exist  $m = m(\epsilon) \geq N$  and  $\tau_1(\epsilon)$  such that

$$\left( \sum_{n \geq m} |u^n(t)|^2 \right)^{1/2} < \epsilon \|\phi\| \quad \text{for } t > \tau_1(\epsilon), \phi \in \mathcal{B}.$$

Since  $|u^n(t, \phi)| \rightarrow 0$  as  $t \rightarrow \infty$  uniformly for  $1 \leq n < m$  and for  $\phi$  such that  $\|\phi\| \leq 1$ , there exists a  $\tau_2(\epsilon) > 0$  such that  $(\sum_{n < m} |u^n(t)|^2)^{1/2} < \epsilon$  provided  $t \geq \tau_2(\epsilon)$ ,  $\|\phi\| \leq 1$ . Consequently, it follows that, if  $t > \tau_3(\epsilon) := \max\{\tau_1(\epsilon), \tau_2(\epsilon)\}$  and if  $\|\phi\| \leq 1$ , then  $|u(t, \phi)| < \sqrt{2}\epsilon$ . Since  $u(t, \phi)$  is linear in  $\phi$ , it follows that  $|u(t, \phi)| \leq \epsilon \|\phi\|$  for  $t > \tau(\epsilon) := \tau_3(\epsilon/\sqrt{2})$ ,  $\phi \in \mathcal{B}$ .

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