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Title	Weyl's Relation on a Doubly Connected Space and the Aharonov-Bohm
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Citation	数理解析研究所講究録 (1997), 982: 240-257
Issue Date	1997-03
URL	http://hdl.handle.net/2433/60905
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

数理解析研究所講究録
982 巻 1997 年 240-257

Weyl's Relation on a Doubly Connected Space and the Aharonov-Bohm Effect

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1 Introduction.

There are cases in which quantum particles move in a multiply connected space. A remarkable example is the Aharonov-Bohm effect[1], where an experiment is set up to keep electrons from penetrating into a region of non-vanishing magnetic field (e.g. Ref. [2, 3, 4]). However, in quantum mechanics, the momentum is represented by the generator of a translation operator, so that it takes a special consideration when the underlying space has "holes" inaccessible to the particles.

More precisely, in the standard quantum mechanics in $\mathbb{R}^N$, the momentum and position operators are defined to be self-adjoint operators, p_j and q_j, acting on a Hilbert space $\mathcal{H}$ and satisfying Heisenberg's canonical commutation relations (CCR):

$$\begin{cases} [p_j, q_{j'}] = -i\hbar\delta_{jj'} \\ [p_j, p_{j'}] = 0 = [q_j, q_{j'}], \quad j, j' = 1, 2, \cdots, N;\ N \in \mathbb{N}. \end{cases} \tag{1}$$

on a dense subspace in $\mathcal{H}$.

It is well-known that the Weyl's CCR for strongly continuous one-parameter groups, $\{e^{isp_j}\}_{-\infty<s<\infty}, \{e^{itq_j}\}_{-\infty<t<\infty}$,

$$\begin{cases} e^{itq_j}e^{isp_k} = \exp[-ist\hbar\delta_{jk}]\, e^{isp_k}e^{itq_j}, \\ e^{itq_j}e^{isq_k} = e^{isq_k}e^{itq_j}, \quad e^{itp_j}e^{isp_k} = e^{isp_k}e^{itp_j} \end{cases} \tag{2}$$

($s, t \in \mathbb{R}$; $j, k = 1, \cdots, N$) determines $\{p_j, q_j\}$ uniquely up to unitary equivalence. The irreducible representation of p_j, q_j are unitary equivalent to the Schrödinger representation by von Neumann's uniqueness theorem (see Theorem VIII.14 in Ref.[5]). Thus, these p_j, q_j satisfy Heisenberg's CCR. Conversely, those self-adjoint operators p_j, q_j satisfying Heisenberg's CCR lead to Weyl's. In this sense, the two CCR's are equivalent.

The equivalence of the two CCR's, Heisenberg's and Weyl's, presents a problem when the underlying space $\mathbb{R}^N$ is replaced by a multiply connected one, $\mathbb{R}^N \setminus$ holes. While Weyl's CCR implies Heisenberg's (see Corollary of Theorem VIII.14 in Ref.[5]), the converse is not necessarily true. Nelson gave a mathematical example in non-Euclidean space to show it is not true (see Corollary and Nelson's example on p.275 in Ref.[5]). More realistic examples were given by Reeh[6] and Arai[7] for the case of the Aharonov-Bohm effect with a string of magnetic field of zero radius. They considered a particle moving on a plane with holes of the 0-radius, $R = 0$. Though it must be remarked that what they called momenta were actually the mass-times-velocity operators, $p_j - \mathsf{q}A_j$, which is called the mv-momentum (kinetic momentum) by Feynman (see Ref.[8, (21.14)]) where $j = x, y$, i.e. $N = 2$, $p_x \equiv p_1, p_y \equiv p_2$; and q is a charge, these operators do satisfy Heisenberg's CCR but do not Weyl's unless the magnetic flux going through the interior of every rectangular closed curves in $\mathbb{R}^2$, has a value of an integer multiple of $2\pi\hbar c/\mathsf{q}$.

In this paper, we deal with a underlying space with a disc-shaped hole of a finite radius R, $\Omega_R \equiv \mathbb{R}^2 \setminus D_R$, $D_R \equiv \{(x,y) \mid x^2+y^2 \leq R^2\}$ $(R>0)$. We shall extend Reeh's and Arai's result with $R=0$ to the case of $R>0$. Our method is to reduce the disc by a conformal mapping to a line segment, and invoke Arai's argument using the fact that the segment has Lebesgue measure zero as dose Arai's point hole.

In 2, we shall show that there are uncountably many different self-adjoint extensions of $\frac{\hbar}{i}\frac{\partial}{\partial x}$ and $\frac{\hbar}{i}\frac{\partial}{\partial y}$ with suitable boundary conditions on ∂D_R. However, in 3 it turns out that none of the self-adjoint extensions of $\frac{\hbar}{i}\frac{\partial}{\partial x}$ and $\frac{\hbar}{i}\frac{\partial}{\partial y}$ satisfy Weyl's CCR.

In 4, therefore, we define momentum operators as generators of shifts along the stream lines of an incompressible vortex-free flow passing by the disc D_R. The position operators are defined in the standard way as multiplication by stream-line coordinates. Such a construction using stream lines was once made by Tomonaga[9] to extract a collective mode of motion of a many-particle system.

To establish that the canonical pairs so defined have unique self-adjoint extensions satisfying Weyl's CCR, we use a conformal (Joukowski) transformation to reduce the disc D_R to line segment $[-2R, 2R]$ and follow Arai's argument using the fact that the holes have Lebesgue measure zero. Of course, the canonical pairs satisfy Heisenberg's CCR also.

In 5, we shall introduce magnetic flux to show that Weyl's CCR for the mv-momentum with respect to the new coordinates is destroyed by the Aharonov-Bohm phase, while Heisenberg's remains valid. The canonical pairs are the same as those in 4, and both CCR's, Weyl's and Heisenberg's, are valid. We shall watch how the Joukowski transformation maps poles of the gauge potential. Then, the point we wish to make here is that the Aharonov-Bohm effect shows itself in the algebra of operators besides the well-known change in the interference pattern. As a conclusion of our assertion in this paper, we shall show in 5 that the Aharonov-Bohm effect appears in Weyl's CCR for just coordinates by the Joukowski transformation, not (x,y)-coordinates, which is caused by inequivalence between Weyl's CCR and Heisenberg's.

Naturally, the results in 4 and 5 and be extended to the case where the underlying space is Riemann surface conformally equivalent to $\mathbb{R}^2 \setminus [-2R, 2R]$.

2 Realistic Case: Hole of Finite Size.

We deal with the case of $\Omega_R \stackrel{\text{def}}{=} \mathbb{R}^2 \setminus D_R$, where $D_R \stackrel{\text{def}}{=} \{(x,y) \mid x^2+y^2 \leq R^2\}$, for the fixed radius $R>0$. We denote $\mathbb{R}^2 \setminus \{(0,0)\}$ by Ω_0. We set $\mathsf{m} = \hbar = c = 1$.

We consider a spinless charged particle with the charge $q \neq 0$ moving in the plane Ω_R under the influence of a magnetic field which goes through D_R perpendicularly to the plane and vanishes outside. Let $\mathbf{A}(x,y) \stackrel{\text{def}}{=} (A_x(x,y), A_y(x,y))$ be a gauge potential of the magnetic field. A_j may be singular at points inside D_R, but we assume that

$$A_j \in C^\infty(\overline{\Omega_R}), \qquad j = x, y, \tag{3}$$

$$B(x,y) \stackrel{\text{def}}{=} \frac{\partial}{\partial x}A_y(x,y) - \frac{\partial}{\partial y}A_x(x,y) = 0, \qquad (x,y) \in \Omega_R, \tag{4}$$

where $\overline{\Omega_R}$ denote the closure of Ω_R.

In this section, we shall find that there are uncountably many self-adjoint extensions of $-i\partial/\partial x$ and $-i\partial/\partial y$ on $L^2(\Omega_R)$ although $-i\partial/\partial x$ and $-i\partial/\partial y$ are essentially self-adjoint on

$L^2(\Omega_0)$ (see Refs.[6, 7]).

For $y \in [-R, R]$, we define two functions $w_{1,\pm} : [-R, R] \to \mathbb{R}$ by $w_{1,\pm}(y) \stackrel{\text{def}}{=} \pm\sqrt{R^2 - y^2}$. Similarly, for $x \in [-R, R]$, we define two functions $w_{2,\pm} : [-R, R] \to \mathbb{R}$ by $w_{2,\pm}(x) \stackrel{\text{def}}{=} \pm\sqrt{R^2 - x^2}$.

Then, we define y-section X_y of $\mathbb{R}$ for every $y \in \mathbb{R}$ by $X_y \stackrel{\text{def}}{=} (-\infty, \infty)$ if $R < |y|$; $(-\infty, w_{1,-}(y)) \cup (w_{1,+}(y), \infty)$ if $|y| \le R$, and x-section Y_x of $\mathbb{R}$ for every $x \in \mathbb{R}$ by $Y_x \stackrel{\text{def}}{=} (-\infty, \infty)$ if $R < |x|$; $(-\infty, w_{2,-}(x)) \cup (w_{2,+}(x), \infty)$ if $|x| \le R$.

We define two sets $AC^x_{loc}(\Omega_R)$ and $AC^y_{loc}(\Omega_R)$ of functions on Ω_R by

$$AC^x_{loc}(\Omega_R) \stackrel{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \text{for almost all } y \in \mathbb{R},\ f(\cdot, y) \text{ is absolutely continuous on arbitrary closed interval } [c, c'] \text{ contained inside } X_y \text{ such that } \frac{\partial f}{\partial x} \in L^2(\Omega_R) \right\},$$

and $AC^y_{loc}(\Omega_R)$ is defined similarly by replacing $f(\cdot, y)$ by $f(x, \cdot)$.

Let f be in $L^2(\Omega_R)$. Then we make a function $f^{ext} \in L^2(\mathbb{R}^2)$ as $f^{ext}(x, y) \stackrel{\text{def}}{=} f(x, y)$ if $(x, y) \in \Omega_R$; 0 if $(x, y) \in \mathbb{R}^2 \setminus \Omega_R = D_R$.

Remark 2.1: Let f_1 be in $AC^x_{loc}(\Omega_R)$, and f_2 be in $AC^y_{loc}(\Omega_R)$. Then, we obtain $D_x f_1 = \partial f_1/\partial x$, and $D_y f_2 = \partial f_2/\partial y$, where $D_j\,(j = x, y)$ denotes derivative in the sense of distribution with test functions in $C_0^\infty(\Omega_R)$ which denotes the set of all $C^\infty(\Omega_R)$-functions with compact support in Ω_R.

We define two operators p_j $(j = x, y)$ by

$$p_x \stackrel{\text{def}}{=} \frac{1}{i} D_x \left(= \frac{1}{i}\frac{\partial}{\partial x} \right), \tag{5}$$

$$D(p_x) \stackrel{\text{def}}{=} \left\{ f \in AC^x_{loc}(\Omega_R) \,|\, \text{for almost all } |y| \le R \ \lim_{x \to w_{1,-}(y)} f(x, y) = 0 = \lim_{x \to w_{1,+}(y)} f(x, y) \right\},$$

and p_y is defined similarly.

In general, we denote by $D(T)$ the domain of operators T.

Remark 2.2: $C_0^\infty(\Omega_R) \subset \mathrm{D}(p_x),\ \ \mathrm{D}(p_y)$.

The position operators q_j $(j = x, y)$ are realized as self-adjoint operators:

$$q_j \stackrel{\text{def}}{=} x_j \quad (\text{the multiplication by } x_j),$$

$$D(q_j) \stackrel{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,|\, \int_{\Omega_R} dxdy |x_j f(x, y)|^2 < \infty \right\},$$

where $x_j \equiv x$ if $j = x$; and $x_j \equiv y$ if $j = y$ (see Example 5.11 and its remark 1 in [12] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [5]).

First of all, we note fundamental facts concerning functions in $L^2(X_y)$, $L^2(Y_x)$, and $AC^j_{loc}(\Omega_R)$.

Lemma 2.1:

(a) If $f \in L^2(\Omega_R)$, then $f \in L^2(X_y)$ for almost all $y \in \mathbb{R}$, and $f \in L^2(Y_x)$ for almost all $x \in \mathbb{R}$.

(b) If $f \in AC^x_{loc}(\Omega_R)$, then $\lim_{x \to \pm\infty} f(x,y) = 0$ for almost all $y \in \mathbb{R}$, and $\lim_{x \to w_{1,\pm}(y)} f(x,y)$ exists for almost all $|y| \le R$.

(c) If $f \in AC^y_{loc}(\Omega_R)$, then $\lim_{y \to \pm\infty} f(x,y) = 0$ for almost all $x \in \mathbb{R}$, and $\lim_{y \to w_{2,\pm}(x)} f(x,y)$ exists for almost all $|x| \le R$.

By Lemmas 2.1(b) and (c), from now on, we set

$$f(w_{1,-}(y),y) \stackrel{\text{def}}{=} \lim_{x \to w_{1,-}(y)} f(x,y), \quad f(w_{1,+}(y),y) \stackrel{\text{def}}{=} \lim_{x \to w_{1,+}(y)} f(x,y),$$

for $f \in AC^x_{loc}(\Omega_R)$, and similarly we use a notation $f(x, w_{2,-}(x))$ for $f \in AC^y_{loc}(\Omega_R)$.

There are many ways to extend p_j $(j = x,y)$ to self-adjoint operators. We shall show it in the following theorem for comprehending all self-adjoint extensions of p_j $(j = x,y)$.

Theorem 2.2: p_j is closed symmetric, but not essentially self-adjoint. Furthermore, p_j has uncountably many different self-adjoint extensions.

By Theorem 2.2 above, we knew that there are many self-adjoint extensions of p_j $(j = x,y)$. In order to momentum operators generate shifts, we wish there were a pair of self-adjoint extensions of p_j $(j = x,y)$ satisfying Weyl's CCR. As a matter of fact, we shall realize that there no such pair in the self-adjoint extensions of p_j $(j = x,y)$. After here and in the next section, we shall see it.

The following corollary follows immediately from Corollary on p.141 in Ref.[11], which is the first step for comprehending the domain of self-adjoint extensions of p_j $(j = x,y)$:

Corollary: There is a one-one correspondence between self-adjoint extension of p_j and unitary operators from $\text{Ker}\left(p_j^* - i\right)$ onto $\text{Ker}\left(p_j^* + i\right)$. Let $U_j : \text{Ker}\left(p_j^* - i\right) \to \text{Ker}\left(p_j^* + i\right)$ $(j = x,y)$ be an arbitrary unitary operator, and p_{U_j} be the self-adjoint extension of p_j corresponding to U_j $(j = x,y)$. Then,

$$\begin{aligned} D(p_{U_j}) &= \left\{\varphi_0 + \varphi_+ + U_j\varphi_+ \,|\, \varphi_0 \in D(p_j), \varphi_+ \in \text{Ker}(p_j^* - i)\right\}, \\ p_{U_j}(\varphi_0 + \varphi_+ + U_j\varphi_+) &= p_j\varphi_0 + i\varphi_+ - iU_j\varphi_+. \end{aligned}$$

Remark 2.3: We can show easily that $\{p_j, q_j\}_{j=x,y}$ satisfies Heisenberg's CCR on $C_0^\infty(\Omega_R)$. In order to investigate Weyl's, we need to show the behavior of $\exp[itp_j]$, which will be investigated in the following section.

For getting description of domains of self-adjoint extensions p_{U_j} $(j = x,y)$ appropriate for the boundary conditions on ∂D_R, we get exactly adjoint operators of p_j as the following proposition. Its proof follows from Example in VIII.2 in Ref.[5] with introducing an arbitrary function $j^2 \in C_0^\infty(\mathbb{R})$ for applying the example in Ref.[5] to our case:

Proposition 2.3:

(a) $p_x^* = -i\partial/\partial x$ with $D(p_x^*) = AC'^x_{loc}(\Omega_R)$.

(b) $p_y^* = -i\partial/\partial y$ with $D(p_y^*) = AC'^y_{loc}(\Omega_R)$.

We will prepare some lemmas for a while in order to investigate boundary conditions on ∂D_R for functions in $D(p_{U_j})$ $(j = x, y)$.

We define $WS_x^{\pm}(\Omega_R)$, the vector space of weak solutions for $D_x f = \pm f$, by

$$WS_x^{\pm}(\Omega_R) \equiv \{f \in L^2(\Omega_R) \,|\, D_x f = \pm f\}.$$

It is evident that

Lemma 2.4: If $\varphi \in \mathrm{Ker}(p_x^* \pm i)$, then $\overline{\varphi} \in WS_x^{\pm}(\Omega_R)$.

Since Ω_R is open, for every $(x, y) \in \Omega_R$, there exists $\delta_{x,y} > 0$ such that $Ball\,((x,y), \delta_{x,y}) \subset \Omega_R$, where $Ball\,((x,y), \delta_{x,y})$ denotes the open ball with center (x, y) and radius $\delta_{x,y}$. And $Ball\,((x,y), \delta_{x,y}/2) \subset Ball\,((x,y), \delta_{x,y})$. There exists an open perfect square $J_{x,y}$ with center (x, y) such that $\overline{J_{x,y}} \subset Ball\,((x,y), \delta_{x,y}/2)$, where $\overline{J_{x,y}}$ denotes the closure of $J_{x,y}$. So, we have $\overline{J_{x,y}} \subsetneq \Omega_R$ and $\Omega_R = \bigcup_{(x,y)\in\Omega_R} J_{x,y}$. We denote by $\mathbf{J}$ the set $\{J_{x,y} \,|\, (x,y) \in \Omega_R\}$.

Let $\rho_\varepsilon *$ $(\varepsilon > 0)$ be the Friedrichs mollifier. And we set $\varphi_\varepsilon \equiv \rho_\varepsilon * \varphi$ for $\varphi \in WS_x^{\pm}(\Omega_R)$. To be exact, let φ^{ext} be a function which is defined by φ on Ω_R, and 0 on $\mathbb{R}^2 \setminus \Omega_R$. And we define φ_ε by $\varphi_\varepsilon(x,y) \stackrel{\text{def}}{=} \int\int_{\mathbb{R}^2} \rho_\varepsilon(x - x', y - y')\varphi^{ext}(x', y')dx'dy'$.

The following fact can be proved easily.

Lemma 2.5:

(a) $\varphi_\varepsilon \to \varphi$ as $\varepsilon \downarrow 0$ in $L^2(\Omega_R)$.

(b) $\varphi_\varepsilon \in C^\infty(\Omega_R)$.

(c) For every $J = J_1 \times J_2 \in \mathbf{J}$ and $\varphi^{\pm} \in WS_x^{\pm}(\Omega_R)$, there exists $g_{J,\varepsilon} \in C^\infty(J_2)$ such that $\varphi_\varepsilon^{\pm}(x,y) = \exp[\pm x]\, g_{J,\varepsilon}(y)$ for $(x,y) \in J$. Here $g_{J,\varepsilon}$ may be the zero-valued function or not.

Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of $L^2((-R,R))$. We define functions $f_n^{\pm}$ $(n \in \mathbb{N})$ on $\overline{\Omega_R}$ by $f_n^{\pm}(x,y) \stackrel{\text{def}}{=} \sqrt{2}e^{\mp x}\chi_{X_y^{\pm}}(x)e^{\sqrt{R^2-y^2}}e_n(y)$, where $\chi_{X_y^+}(x) \stackrel{\text{def}}{=} 1$ if $|y| \le R$ and $w_{1,+}(y) \le x$; 0 otherwise, and $\chi_{X_y^-}(x) \stackrel{\text{def}}{=} 1$ if $|y| \le R$ and $x \le w_{1,-}(y)$; 0 otherwise.

Similarly, we define functions $g_n^{\pm}$ $(n \in \mathbb{N})$ on $\overline{\Omega_R}$ by $g_n^{\pm}(x,y) \stackrel{\text{def}}{=} \sqrt{2}e^{\mp y}\chi_{Y_x^{\pm}}(y)e^{\sqrt{R^2-x^2}}e_n(x)$, where $\chi_{Y_x^+}(y) \stackrel{\text{def}}{=} 1$ if $|x| \le R$ and $w_{2,+}(x) \le y$; 0 otherwise, and $\chi_{Y_x^-}(y) \stackrel{\text{def}}{=} 1$ if $|x| \le R$ and $y \le w_{2,-}(x)$; 0 otherwise.

Then we get the following proposition:

Lemma 2.6:

(a)

(a-1) $\{f_n^{\pm}\}_{n\in\mathbb{N}}$ is a complete orthonormal basis of $\mathrm{Ker}(p_{U_x} \mp i)$.

(a-2) For every $\varphi \in \mathrm{D}(p_{U_x})$,

$$\varphi(w_{1,+}(y), y) = \gamma_{U_x}(\varphi_+; y)\varphi(w_{1,-}(y), y), \tag{6}$$

where

$$\gamma_{U_x}(\varphi_+; y) = \frac{\sum_{m=1}^{\infty} < f_m^+, \varphi_+ >_{L^2(\Omega_R)} e_m(y)}{\sum_{n=1}^{\infty} < f_n^-, U_x\varphi_+ >_{L^2(\Omega_R)} e_n(y)}.$$

(a-3) For every $f_+, g_+ \in \mathrm{Ker}(p_{U_x} - i)$,

$$\sum_{m=1}^{\infty} \overline{< f_m^+, g_+ >_{L^2(\Omega_R)}} < f_m^+, f_+ >_{L^2(\Omega_R)} = \sum_{m=1}^{\infty} \overline{< f_m^-, U_x g_+ >_{L^2(\Omega_R)}} < f_m^-, U_x f_+ >_{L^2(\Omega_R)} .$$

(a-4) For every $\varphi \in \mathrm{D}(p_{U_x})$,

$$\int_{-R}^{R} dy |\varphi(w_{1,+}(y), y)|^2 = \int_{-R}^{R} dy |\varphi(w_{1,-}(y), y)|^2 .$$

(b)

(b-1) $\{g_n^\pm\}_{n\in\mathbb{N}}$ is a complete orthonormal basis of $\mathrm{Ker}(p_{U_y} \mp i)$.

(b-2) For every $\varphi \in \mathrm{D}(p_{U_y})$,

$$\varphi(x, w_{2,+}(x)) = \gamma_{U_y}(\varphi_+; x)\varphi(x, w_{2,-}(x)),$$

where

$$\gamma_{U_y}(\varphi_+; x) = \frac{\sum_{m=1}^{\infty} < g_m^+, \varphi_+ >_{L^2(\Omega_R)} e_m(x)}{\sum_{n=1}^{\infty} < g_n^-, U_y \varphi_+ >_{L^2(\Omega_R)} e_n(x)} .$$

(b-3) For every $f_+, g_+ \in \mathrm{Ker}(p_{U_y} - i)$,

$$\sum_{m=1}^{\infty} \overline{< g_m^+, g_+ >_{L^2(\Omega_R)}} < g_m^+, f_+ >_{L^2(\Omega_R)} = \sum_{m=1}^{\infty} \overline{< g_m^-, U_y g_+ >_{L^2(\Omega_R)}} < g_m^-, U_y f_+ >_{L^2(\Omega_R)} .$$

(b-4) For every $\varphi \in \mathrm{D}(p_{U_y})$,

$$\int_{-R}^{R} dx |\varphi(x, w_{2,+}(x))|^2 = \int_{-R}^{R} dx |\varphi(x, w_{2,-}(x))|^2 .$$

Remark 2.4: We here note that γ_{U_j} $(j = x, y)$ is depend on φ_+, which is different from the 1-dimensional case (see Example 1 in X.1 in Ref.[11]. Then, why does p_{U_j} keep symmetry? The reason is as follows: For instance,
let $f = f_0 + f_+ + U_x f_+$, $g = g_0 + g_+ + U_x g_+ \in \mathrm{D}(p_{U_x})$. Then we have by Proposition 2.3(a) and Lemma 2.6(a)

$$\begin{aligned}
& < g, p_{U_x} f >_{L^2(\Omega_R)} - < p_{U_x} g, f >_{L^2(\Omega_R)} \\
= {} & \frac{1}{i} \int_{-R}^{R} dy \left(\overline{\gamma_{U_x}(g_+; y)^{-1}} \gamma_{U_x}(f_+; y)^{-1} - 1 \right) \overline{g_+(w_{1,+}(y), y)} f_+(w_{1,+}(y), y) \\
= {} & \frac{2}{i} \left(\sum_{m=1}^{\infty} \overline{< f_m^-, U_x g_+ >_{L^2(\Omega_R)}} \sum_{n=1}^{\infty} < f_n^-, U_x f_+ >_{L^2(\Omega_R)} \delta_{mn} \right. \\
& \left. - \sum_{m=1}^{\infty} \overline{< f_m^+, g_+ >_{L^2(\Omega_R)}} \sum_{n=1}^{\infty} < f_n^+, f_+ >_{L^2(\Omega_R)} \delta_{mn} \right) \\
= {} & \frac{2}{i} \sum_{m=1}^{\infty} \left(\overline{< f_m^-, U_x g_+ >_{L^2(\Omega_R)}} < f_m^-, U_x f_+ >_{L^2(\Omega_R)} - \overline{< f_m^+, g_+ >_{L^2(\Omega_R)}} < f_m^+, f_+ >_{L^2(\Omega_R)} \right) \\
= {} & 0.
\end{aligned}$$

Now that we have Proposition 2.3 and Lemmas 2.4-2.6, we can characterize the domains of p_{U_x} and p_{U_y} with the suitable boundary conditions on ∂D_R, respectively:

Theorem 2.7:

(a) $p_{U_x} = \frac{1}{i}\frac{\partial}{\partial x}$ with

$$\begin{aligned} D(p_{U_x}) = \Big\{ f \in AC^x_{loc}(\Omega_R) \Big| & \int_{-R}^{R} dy |f(w_{1,\pm}(y),y)|^2 < \infty, \\ & \text{and there exists } f_{\text{pls}} \in \text{Ker}(p_x^* - i) \text{ such that} \\ & f(w_{1,+}(y),y) = f_{\text{pls}}(w_{1,+}(y),y), \\ & f(w_{1,+}(y),y) = \gamma_{U_x}(f_{\text{pls}};y) f(w_{1,-}(y),y) \\ & \text{for almost all } -R < y < R \Big\}. \end{aligned}$$

(b) $p_{U_y} = \frac{1}{i}\frac{\partial}{\partial y}$ with

$$\begin{aligned} D(p_{U_y}) = \Big\{ f \in AC^y_{loc}(\Omega_R) \Big| & \int_{|x|<R} dx |f(x, w_{2,\pm}(x))|^2 < \infty \\ & \text{and there exists } f_{\text{pls}} \in \text{Ker}(p_y^* - i) \text{ such that} \\ & f(x, w_{2,+}(x)) = f_{\text{pls}}(x, w_{2,+}(x)), \\ & f(x, w_{2,+}(x)) = \gamma_{U_y}(f_{\text{pls}};x) f(x, w_{2,-}(x)) \\ & \text{for almost all } -R < x < R \Big\}. \end{aligned}$$

Now that we obtain exactly the boundary conditions on ∂D_R depending on the domain of p_{U_j} ($j = x, y$), we can comprehend the behavior of $\exp\left[itp_{U_x}\right]$ and $\exp\left[itp_{U_y}\right]$, which implies that no pair, $\exp\left[itp_{U_x}\right]$ and $\exp\left[itp_{U_y}\right]$, satisfies Weyl's CCR. It will be shown in the next section.

3 Behavior of $\exp[itp_x]$ and $\exp[itp_y]$.

In this section, we investigate behavior of $\exp[itp_x]$ and $\exp[itp_y]$. Here arises difficulties. In fact, we shall find it turns out in this section that any pair of self-adjoint extensions of $-i\partial/\partial x$ and $-i\partial/\partial y$ on $L^2(\Omega_R)$ does not satisfy Weyl's CCR. This is not a case with $L^2(\Omega_0)$, which was studied by Reeh[6] and Arai[7].

Let $\chi_{B^+}(x,y) \stackrel{\text{def}}{=} 1$ if $|y| \le R$ and $w_{1,+}(y) \le x$; 0 otherwise, and $\chi_{B^-}(x,y) \stackrel{\text{def}}{=} 1$ if $|y| \le R$ and $x \le w_{1,-}(y)$; 0 otherwise. For $f \in AC^x_{loc}(\Omega_R)$ with $\int_{-R}^{R} dy|f(w_{1,\pm}(y),y)|^2 < \infty$, we can give explicit construction of f_{pls} and f_{mns} by

$$\begin{aligned} f_{\text{pls}}(x,y) &\stackrel{\text{def}}{=} e^{-x} f(w_{1,+}(y),y) e^{w_{1,+}(y)} \chi_{B^+}(x,y), \\ f_{\text{mns}}(x,y) &\stackrel{\text{def}}{=} e^{x} f(w_{1,-}(y),y) e^{w_{1,-}(y)} \chi_{B^-}(x,y). \end{aligned}$$

Then, we have $\int\int_{\Omega_R} dxdy|f_{\rm pls}(x,y)|^2 = \frac{1}{2}\int_{-R}^{R} dy|f(w_{1,+}(y),y)|^2 < \infty$, So, $f_{\rm pls} \in L^2(\Omega_R)$. It is clear that $f_{\rm pls} \in {\rm Ker}(p_x^* - i)$ by Proposition 2.3. Similarly, $f_{\rm mns} \in {\rm Ker}(p_x^* + i)$. In the same way, we define for $f \in AC_{loc}^y(\Omega_R)$ with $\int_{|x|<R} dy|f(x, w_{2,\pm}(x))|^2 < \infty$,

$$
\begin{aligned}
f_{\rm pls}(x,y) &\stackrel{\rm def}{=} e^{-y} f(x, w_{2,+}(x)) e^{w_{2,+}(x)} \chi_{A+}(x,y), \\
f_{\rm mns}(x,y) &\stackrel{\rm def}{=} e^{y} f(x, w_{2,-}(x)) e^{w_{2,-}(x)} \chi_{A-}(x,y),
\end{aligned}
$$

where $\chi_{A+}(x,y) \stackrel{\rm def}{=} 1$ if $|x| < R$ and $w_{2,+}(x) \le y$; 0 otherwise, and $\chi_{A-}(x,y) \stackrel{\rm def}{=} 1$ if $|x| < R$ and $y \le w_{2,-}(x)$; 0 otherwise.

Remember Lemma 2.6 (a-4) and (b-4). Then, for $f \in {\rm D}(p_{U_j})$ $(j = x, y)$, we define f_0 by

$$
f_0(x,y) \stackrel{\rm def}{=} f(x,y) - f_{\rm pls}(x,y) - f_{\rm mns}(x,y).
$$

Then, it is clear that $f_0 \in {\rm D}(p_j)$ $(j = x, y)$.

We will clarify the meaning of the functions $f_{\rm pls}$ and mns, which gives an important decomposition.

Lemma 3.1:

(a) Fix U_x. Then, $U_x f_{\rm pls} = f_{\rm mns}$ for every $f \in {\rm D}(p_{U_x})$, and

$$
f = f_0 + f_{\rm pls} + U_x f_{\rm pls} \quad \in {\rm D}(p_x) + {\rm Ker}(p_x^* - i) + {\rm Ker}(p_x^* + i)
$$

is a unique decomposition.

(b) Fix U_y. Then, $U_y f_{\rm pls} = f_{\rm mns}$ for every $f \in {\rm D}(p_{U_y})$, and

$$
f = f_0 + f_{\rm pls} + U_y f_{\rm pls} \quad \in {\rm D}(p_y) + {\rm Ker}(p_y^* - i) + {\rm Ker}(p_y^* + i)
$$

is a unique decomposition.

Fix U_j $(j = x, y)$ and $t \in \mathbb{R}$. Since p_{U_x} is self-adjoint, there exists a dense set $\mathcal{A}(p_{U_j})$ of analytic vectors of p_{U_j} (see Corollary1 on p203 in Ref.[11]).

We can give behaviors of $\exp\left[itp_{U_j}\right]$ $(j = x, y)$ as the following proposition, which tell us that any pair of $\exp\left[itp_{U_j}\right]$ $(j = x, y)$ destroy Weyl's CCR:

Proposition 3.2:

(a) For any $f \in \mathcal{A}(p_{U_x})$,

(a-1) if (x, y) satisfies one of the following conditions at least, (i) $|y| > R$; (ii) $|y| \le R$ with $x, x+t \le w_{1,-}(y)$; or (iii) $|y| \le R$ with $w_{1,+}(y) \le x, x+t$, then

$$
\left(e^{itp_{U_x}} f\right)(x,y) = f(x+t, y),
$$

(a-2) if $t > 0$ and $|y| \le R$, then

$$
\left(e^{itp_{U_x}} f\right)(w_{1,-}(y), y) = \sum_{n=0}^{\infty} \gamma_{U_x}(f_{\rm pls}^{(n)}; y)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(w_{1,+}(y) + t, y)\lceil_{t=0}\right) t^n,
$$

where $g^{(n)}$ denotes $\partial^n g/\partial x^n$, (a-3) if $t > 0$, $|y| \le R$, $x < w_{1,-}(y)$ and $w_{1,-}(y) < x + t$, then

$$\left(e^{itp_{U_x}} f\right)(x, y)$$
$$= \sum_{n=0}^{\infty} \gamma_{U_x}(f_{\text{pls}}^{(n)}; y)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x + t + 2w_{1,+}(y), y) \lceil_{t=-x-w_{1,+}(y)} \right) (t + x + w_{1,+}(y))^n,$$

(a-4) if $t < 0$ and $|y| \le R$, then

$$\left(e^{itp_{U_x}} f\right)(w_{1,+}(y), y) = \sum_{n=0}^{\infty} \gamma_{U_x}(f_{\text{pls}}^{(n)}; y) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(w_{1,-}(y) + t, y) \lceil_{t=0} \right) t^n,$$

(a-5) if $t < 0$, $|y| \le R$, $w_{1,+}(y) < x$ and $x + t < w_{1,+}(y)$, then

$$\left(e^{itp_{U_x}} f\right)(x, y)$$
$$= \sum_{n=0}^{\infty} \gamma_{U_x}(f_{\text{pls}}^{(n)}; y) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x + t + 2w_{1,-}(y), y) \lceil_{t=-x-w_{1,-}(y)} \right) (t + x + w_{1,-}(y))^n.$$

(b) For any $f \in \mathcal{A}(p_{U_y})$,

(b-1) if (x, y) satisfies one of the following conditions at least, (i) $|x| > R$; (ii) $|x| \le R$ with $y, y + t \le w_{2,-}(x)$; or (iii) $|x| \le R$ with $w_{2,+}(x) \le y, y + t$, then

$$\left(e^{itp_{U_y}} f\right)(x, y) = f(x, y + t),$$

(b-2) if $t > 0$ and $|x| \le R$, then

$$\left(e^{itp_{U_y}} f\right)(x, w_{2,-}(x)) = \sum_{n=0}^{\infty} \gamma_{U_y}(f_{\text{pls}}^{(n)}; x)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x, w_{2,+}(x) + t) \lceil_{t=0} \right) t^n,$$

where $g^{(n)}$ denotes $\partial^n g / \partial y^n$,

(b-3) if $t > 0$, $|x| \le R$, $y < w_{2,-}(x)$ and $w_{2,-}(x) < y + t$, then

$$\left(e^{itp_{U_y}} f\right)(x, y)$$
$$= \sum_{n=0}^{\infty} \gamma_{U_y}(f_{\text{pls}}^{(n)}; x)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x, y + t + 2w_{2,+}(x)) \lceil_{t=-y-w_{2,+}(x)} \right) (t + y + w_{2,+}(x))^n,$$

(b-4) if $t < 0$ and $|x| \le R$, then

$$\left(e^{itp_{U_y}} f\right)(x, w_{2,+}(x)) = \sum_{n=0}^{\infty} \gamma_{U_y}(f_{\text{pls}}^{(n)}; x) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x, w_{2,-}(x) + t) \lceil_{t=0} \right) t^n,$$

(b-5) if $t < 0$, $|x| \le R$, $w_{2,+}(x) < y$ and $y + t < w_{2,+}(x)$, then

$$\left(e^{itp_{U_y}} f\right)(x, y)$$
$$= \sum_{n=0}^{\infty} \gamma_{U_y}(f_{\text{pls}}^{(n)}; x) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x, y + t + 2w_{2,-}(x)) \lceil_{t=-y-w_{2,-}(x)} \right) (t + y + w_{2,-}(x))^n.$$

Since $\mathcal{A}(p_{U_j})$ $(j = x, y)$ is dense in $L^2(\Omega_R)$, for arbitrary $f \in L^2(\Omega_R)$ we can approximate elements in $\mathcal{A}(p_{U_j})$ to f in the sense of the almost everywhere convergence on Ω_R. So, Proposition 3.2 means that $\exp\left[itp_{U_x}\right] f$ $(f \in L^2(\Omega_R))$ jumps at the boundary of the hole D_R in a moment. When this f goes across the hole D_R, the equality between $(\exp[isp_x]\exp[itp_y]f)(x,y)$ and $(\exp[itp_y]\exp[isp_x]f)(x,y)$, which would normally be valid, must be destroyed. Roughly speaking, for instance, let $-2R < x, y < -R$, $s = R$, and $t = 3R$. Defining $F_s^{U_y} \stackrel{\text{def}}{=} e^{isp_{U_y}} f$, since $|y| > R$,

$$\left(e^{itp_{U_x}} F_s^{U_y}\right)(x,y) = F_s^{U_y}(x+t,y) = \left(e^{isp_{U_y}} f\right)(x+t,y)$$

holds by Proposition 3.2 (a-1). Since $R < x + t$,

$$\left(e^{isp_{U_y}} f\right)(x+t,y) = f(x+t, y+s)$$

holds by Proposition 3.2 (b-1). Thus, we have

$$\left(e^{itp_{U_x}} e^{itp_{U_y}} f\right)(x,y) = f(x+t, y+s)$$

for almost all $(x,y) \in (-2R,-R) \times (-2R,-R)$.

Now, defining $G_t^{U_x} \stackrel{\text{def}}{=} e^{itp_{U_x}} f$, since $|x| > R$,

$$\left(e^{isp_{U_y}} G_t^{U_x}\right)(x,y) = G_t^{U_x}(x, y+s) = \left(e^{itp_{U_x}} f\right)(x, y+s)$$

holds by Proposition 3.2 (b-1). Since $-R < y + s < 0$, and $x < w_{1,-}(y) < w_{1,+}(y) < x + t$ for all $|y| \le R$,

$$\begin{aligned}
&\left(e^{itp_{U_x}} f\right)(x, y+s) \\
= &\sum_{n=0}^{\infty} \gamma_{U_x}(f_{\text{pls}}^{(n)}; y+s)^{-1} \frac{1}{n!} \\
&\qquad \times \left(\frac{d^n f}{dt^n}(x+t+2w_{1,+}(y+s), y+s)\lceil_{t=-x-w_{1,+}(y+s)}\right)(t + x + w_{1,+}(y+s))^n
\end{aligned}$$

by Proposition 3.2 (a-3). Thus, we have

$$\begin{aligned}
&\left(e^{isp_{U_y}} e^{itp_{U_x}} f\right)(x,y) \\
= &\sum_{n=0}^{\infty} \gamma_{U_x}(f_{\text{pls}}^{(n)}; y+s)^{-1} \frac{1}{n!} \\
&\qquad \times \left(\frac{d^n f}{dt^n}(x+t+2w_{1,+}(y+s), y+s)\lceil_{t=-x-w_{1,+}(y+s)}\right)(t + x + w_{1,+}(y+s))^n
\end{aligned}$$

for almost all $(x,y) \in (-2R,-R) \times (-2R,-R)$.

Therefore, we realize that

$$\left(e^{itp_{U_x}} e^{isp_{U_y}} f\right)(x,y) \neq \left(e^{isp_{U_y}} e^{itp_{U_x}} f\right)(x,y)$$

for almost all $(x,y) \in (-2R,-R) \times (-2R,-R)$. So, Weyl's CCR is destroyed but remember that Heisenberg's holds for p_{U_x} (see Remark 2.3).

Remember the case of $\Omega_0 \equiv \mathbb{R}^2 \setminus \{(0,0)\}$. In Reeh's and Arai's case, they essentially used the hole, the origin, has the 0-Lebesgue measure.

Since Weyl's CCR is destroyed in our case, the momentum operators defined by using $-i\partial/\partial x$ and $-i\partial/\partial y$ as (5) can not give any representation which is equivalent to the Schrödinger one even if we consider any boundary condition on ∂D_R. Thus, we redefine the momentum and the position operators such that they are equivalent to the Schrödinger representation, and are useful for discussing inequivalence[6, 7] in our case.

4 The Definition of the Momentum and Position Operators Using Streamlines.

In order that the momentum operators generate shift in a space Ω_R having a hole of the shape of a disc of radius R, we introduce streamlines.

We take a coordinate given by a velocity potential $\xi \stackrel{\text{def}}{=} \phi(x,y)$ and a flow function $\eta \stackrel{\text{def}}{=} \psi(x,y)$. Then we first define $\phi(x,y)$ and $\psi(x,y)$ by the Joukowski transformation $\zeta(z)$: For $z \stackrel{\text{def}}{=} x+iy$, we set $\zeta(z) \stackrel{\text{def}}{=} z+R^2/z$ and $\zeta \stackrel{\text{def}}{=} \xi+i\eta$. So $\phi(x,y)$ and $\psi(x,y)$ are determined as

$$\xi = \phi(x,y) \stackrel{\text{def}}{=} x\left(1+\frac{R^2}{x^2+y^2}\right), \qquad \eta = \psi(x,y) \stackrel{\text{def}}{=} y\left(1-\frac{R^2}{x^2+y^2}\right).$$

By ϕ^{-1} and ψ^{-1}, we denote functions satisfying $x = \phi^{-1}(\xi,\eta)$ and $y = \psi^{-1}(\xi,\eta)$.

By the Joukowski transformation $\zeta(z)$, we get two conformal mappings $J_{int} : \mathrm{Int}D_R \xrightarrow{1-1} \mathbb{R}^2_{2R}$ and $J_{out} : \Omega_R \xrightarrow{1-1} \mathbb{R}^2_{2R}$, where $\mathbb{R}^2_{2R} \stackrel{\text{def}}{=} \mathbb{R}^2 \setminus \{(\xi,0) \,|\, -2R \le \xi \le 2R\}$, and $\mathrm{Int}D_R \stackrel{\text{def}}{=} \{(x,y) \,|\, x^2+y^2 < R^2\}$.

We note here the Cauchy-Riemann relations:

$$\frac{\partial\phi}{\partial x} = a(x,y) \stackrel{\text{def}}{=} 1 - R^2\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial\psi}{\partial y} \tag{7}$$

$$\frac{\partial\phi}{\partial y} = b(x,y) \stackrel{\text{def}}{=} -2R^2\frac{xy}{(x^2+y^2)^2} = -\frac{\partial\psi}{\partial x} \tag{8}$$

By the change of variables, we have

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} a^u & -b^u \\ b^u & a^u \end{pmatrix} \begin{pmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{pmatrix},$$

where $a^u(\xi,\eta) \stackrel{\text{def}}{=} a(\phi^{-1}(\xi,\eta),\psi^{-1}(\xi,\eta))$, and $b^u(\xi,\eta) \stackrel{\text{def}}{=} b(\phi^{-1}(\xi,\eta),\psi^{-1}(\xi,\eta))$.

Here we set $c(x,y) \stackrel{\text{def}}{=} \sqrt{a(x,y)^2+b(x,y)^2}$. We define two operators, p_ξ and p_η, acting in $L^2(\Omega_R)$ by

$$p_\xi \stackrel{\text{def}}{=} \frac{1}{ic}\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)\frac{1}{c}, \qquad D(p_\xi) \stackrel{\text{def}}{=} C_0^\infty(\Omega_R),$$

$$p_\eta \stackrel{\text{def}}{=} \frac{1}{ic}\left(-b\frac{\partial}{\partial x}+a\frac{\partial}{\partial y}\right)\frac{1}{c}, \qquad D(p_\eta) \stackrel{\text{def}}{=} C_0^\infty(\Omega_R).$$

We can define two self-adjoint operators, q_ξ^S and q_η^S, by

$$q_\xi^S \stackrel{\text{def}}{=} c\phi\frac{1}{c}, \qquad D(q_\xi^S) \stackrel{\text{def}}{=} \left\{f \in L^2(\Omega_R) \,\middle|\, \int\int_{\Omega_R} dxdy|\phi(x,y)f(x,y)|^2 < \infty\right\},$$

$$q_\eta^S \stackrel{\text{def}}{=} c\psi\frac{1}{c}, \qquad D(q_\eta^S) \stackrel{\text{def}}{=} \left\{f \in L^2(\Omega_R) \,\middle|\, \int\int_{\Omega_R} dxdy|\psi(x,y)f(x,y)|^2 < \infty\right\}$$

(see Example 5.11 and its remark 1 in [12] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [5]).

For functions f of $(x,y) \in \Omega_R$, we define functions f^u of $(\xi,\eta) \in \mathbb{R}^2_{2R}$ by $f^u(\xi,\eta) \stackrel{\text{def}}{=} f(\phi^{-1}(\xi,\eta),\psi^{-1}(\xi,\eta))$. So, $f(x,y) = f^u(\phi(x,y),\psi(x,y))$.

We define a Hilbert space $L^2_c(\mathbb{R}^2_{2R})$ by

$$L^2_c(\mathbb{R}^2_{2R}) \stackrel{\text{def}}{=} \left\{ f : \text{functions of } (\xi,\eta) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|f(\xi,\eta)|^2}{c^u(\xi,\eta)^2} < \infty \right\}$$

with an inner product $< f,g >_{L^2_c(\mathbb{R}^2_{2R})} \stackrel{\text{def}}{=} \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{\overline{f(\xi,\eta)}g(\xi,\eta)}{c^u(\xi,\eta)^2}$. And we define a linear operator $U : L^2(\Omega_R) \longrightarrow L^2_c(\mathbb{R}^2_{2R})$ by $(Uf)(\xi,\eta) \stackrel{\text{def}}{=} f^u(\xi,\eta)$ for every $f \in L^2(\Omega_R)$. Then, it is clear that U is a unitary operator, and

$$\begin{aligned}
Up_\xi U^{-1} &= \frac{1}{i}c^u\frac{\partial}{\partial\xi}\frac{1}{c^u}, & \mathrm{D}(Up_\xi U^{-1}) &= C_0^\infty(\mathbb{R}^2_{2R}),\\
Up_\eta U^{-1} &= \frac{1}{i}c^u\frac{\partial}{\partial\eta}\frac{1}{c^u}, & \mathrm{D}(Up_\xi U^{-1}) &= C_0^\infty(\mathbb{R}^2_{2R}),\\
Uq_\xi U^{-1} &= c^u\xi\frac{1}{c^u}, & \mathrm{D}(Uq_\xi U^{-1}) &= \left\{ f \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|\xi|^2|f(\xi,\eta)|^2}{c^u(\xi,\eta)^2} < \infty \right\},\\
Uq_\eta U^{-1} &= c^u\eta\frac{1}{c^u}, & \mathrm{D}(Uq_\eta U^{-1}) &= \left\{ f \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|\eta|^2|f(\xi,\eta)|^2}{c^u(\xi,\eta)^2} < \infty \right\}.
\end{aligned}$$

Of course, we have

Lemma 4.1: The operators p_ξ and p_η are symmetric.

Thus, since p_ξ and p_η are closable, we can denote by $\overline{p}_\xi$ and $\overline{p}_\eta$ the closures of p_ξ and p_η. As we expect, we have

Lemma 4.2: $\{\overline{p}_j, q_j\}_{j=\xi,\eta}$ satisfies Heisenberg's CCR on $C_0^\infty(\Omega_R)$.

Since the hole $\{(\xi,0) \mid -2R \le \xi \le 2R\}$ intercepts the differential $-i\partial/\partial\xi$ at only $\eta = 0$, we can easily get the following proposition by investigating the deficiency index of $-i\partial/\partial\xi$:

Proposition 4.3: $\overline{p}_\xi$ is self-adjoint.

From now on, we denote $\overline{p}_\xi$ by p^S_ξ. In order to get self-adjoint extensions of p_η and exact descriptions of their domains, we will prepare some lemmas for a while.

In the same way as Theorem 2.2 and its corollary, we have the following lemma:

Lemma 4.4: $U\overline{p}_\eta U^{-1}$ has uncountably many different self-adjoint extensions in $L^2_c(\mathbb{R}^2_{2R})$. And let $U_\eta : \mathrm{Ker}\left(\left(U\overline{p}_\eta U^{-1}\right)^* - i\right) \subset L^2_c(\mathbb{R}^2_{2R}) \to \mathrm{Ker}\left(\left(U\overline{p}_\eta U^{-1}\right)^* + i\right) \subset L^2_c(\mathbb{R}^2_{2R})$ be an arbitrary unitary operator, and $U\overline{p}_{U_\eta}U^{-1}$ be the self-adjoint extension of $U\overline{p}_\eta U^{-1}$ corresponding to U_η. Then,

$$\mathrm{D}(U\overline{p}_{U_\eta}U^{-1}) = \left\{\varphi_0 + \varphi_+ + U_\eta\varphi_+ \mid \varphi_0 \in \mathrm{D}(U\overline{p}_\eta U^{-1}),\ \varphi_+ \in \mathrm{Ker}\left(\left(U\overline{p}_\eta U^{-1}\right)^* - i\right)\right\},$$

$$U\overline{p}_{U_\eta}U^{-1}(\varphi_0 + \varphi_+ + U_\eta\varphi_+) = U\overline{p}_\eta U^{-1}\varphi_0 + i\varphi_+ - iU_\eta\varphi_+$$

Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of $L^2((-2R,2R))$. We define functions $h_n^\pm$ ($n \in \mathbb{N}$) on $\mathbb{R}^2_{2R}$ by $h_n^\pm(\xi,\eta) \stackrel{\text{def}}{=} \sqrt{2}e^{\mp\eta}c^u(\xi,\eta)\chi_{H_\xi^\pm}(\eta)e_n(\xi)$, where $\chi_{H_\xi^+}(\eta) \stackrel{\text{def}}{=} 1$ if $|\xi| < 2R$ and $0 < \eta$; 0 otherwise.
and $\chi_{H_\xi^-}(\eta) \stackrel{\text{def}}{=} 1$ if $|\xi| < 2R$ and $\eta < 0$; 0 otherwise.

In the same way as Lemma 2.6, we have
Lemma 4.5: $\{h_n^\pm\}_{n\in\mathbb{N}}$ is a complete orthonormal basis of Ker $\left(U\overline{p}_{U_\eta}U^{-1} \mp i\right)$ $\subset L_c^2(\mathbb{R}^2_{2R})$. For every $\varphi \in \mathrm{D}\left(U\overline{p}_{U_\eta}U^{-1}\right)$,

$$\lim_{\eta\downarrow 0}\varphi(\xi,\eta) = \gamma_{U_\eta}(\varphi_+;\xi)\lim_{\eta\uparrow 0}\varphi(\xi,\eta). \tag{9}$$

where

$$\gamma_{U_\eta}(\varphi_+;\xi) = \frac{\sum_{m=1}^{\infty} < h_m^+, \varphi_+ >_{L_c^2(\mathbb{R}^2_{2R})} e_m(\xi)}{\sum_{n=1}^{\infty} < h_n^-, U_\eta\varphi_+ >_{L_c^2(\mathbb{R}^2_{2R})} e_n(\xi)}.$$

And

$$\sum_{m=1}^{\infty}\overline{< h_m^+, g_+ >_{L_c^2(\mathbb{R}^2_{2R})}} < h_m^+, f_+ >_{L_c^2(\mathbb{R}^2_{2R})}$$
$$= \sum_{m=1}^{\infty}\overline{< h_m^-, (U_\eta g_+) >_{L_c^2(\mathbb{R}^2_{2R})}} < h_m^-, (U_\eta f_+) >_{L_c^2(\mathbb{R}^2_{2R})}$$

holds for every $f_+, g_+ \in \mathrm{Ker}(U\overline{p}_{U_\eta}U^{-1} - i)$.

We denote by U_η^I the unitary operator U_η satisfying $\gamma_{U_\eta}(\varphi_+;y) = 1$ for all $\varphi_+ \in$ Ker $\left(\left(U\overline{p}_\eta U^{-1}\right)^* - i\right)$ and $\xi \in (-2R, 2R)$. And let p_η^S be $\overline{p}_{U_\eta^I} \equiv U^{-1}(U\overline{p}_{U_\eta^I}U^{-1})U$.

We define a set H_ξ for $\xi \in \mathbb{R}$ by
$H_\xi \stackrel{\text{def}}{=} \{\eta \in \mathbb{R} \,|\, \text{there exists } \xi_\eta \text{ such that } (\xi,\eta_\xi) \in \mathbb{R}^2_{2R}\}$.
Then we define a set $AC^\eta_{loc}(\mathbb{R}^2_{2R})$ of functions on $\mathbb{R}^2_{2R}$ by

$$AC^\eta_{loc}(\mathbb{R}^2_{2R}) \stackrel{\text{def}}{=} \left\{ f \in L_c^2(\mathbb{R}^2_{2R}) \,\middle|\, \begin{array}{l}\text{for almost all } \xi \in \mathbb{R},\ f(\xi,\cdot) \text{ is absolutely continuous} \\ \text{on arbitrary closed interval } [c,c'] \text{ contained inside } H_\xi \text{ such that} \\ \dfrac{\partial f}{\partial\eta} \in L_c^2(\mathbb{R}^2_{2R})\end{array}\right\}.$$

Now that we have Lemmas 4.5 and 4.6, we can describe exactly the domain of p_η^S. In the same way as Theorem 2.7, we can prove the following lemma:

Lemma 4.6: $Up_\eta^S U^{-1} = \frac{1}{i} c^u \frac{\partial}{\partial \eta} \frac{1}{c^u}$ with

$$\begin{aligned} D(Up_\eta^S U^{-1}) &= \Big\{ f \in AC_{loc}^{\eta}(\mathbb{R}_{2R}^2) \Big| \int_{|\xi|<2R} |\frac{f(\xi,0)}{c^u(\xi,0)}|^2 < \infty, \\ &\quad \text{and } \lim_{\eta\downarrow 0} f(\xi,\eta) = \lim_{\eta \uparrow 0} f(\xi,\eta) \\ &\quad \text{for almost all } \xi \in [-2R, 2R] \Big\}. \end{aligned}$$

For the selected self-adjoint extension p_η^S, the behavior of $\exp\left[itp_\eta^S\right]$ is given by the following lemma in the same way as Proposition 3.2 below, we have

Lemma 4.7: For $f^u \in L_c^2(\mathbb{R}_{2R}^2)$,

$$\left(Ue^{itp_\eta^S}U^{-1}f^u\right)(\xi,\eta) = \left(e^{itUp_\eta^S U^{-1}}f^u\right)(\xi,\eta) = f^u(\xi,\eta+t)\frac{c^u(\xi,\eta)}{c^u(\xi,\eta+t)}.$$

5 Inequivalence in the Aharonov-Bohm Effect between Heisenberg's CCR and Weyl's

For discussing the Aharonov-Bohm effect, we give a gauge potential $\tilde{\mathbf{A}}(x,y) \stackrel{\text{def}}{=} (A_\xi(x,y), A_\eta(x,y))$ by

$$A_\xi(x,y) \stackrel{\text{def}}{=} \frac{1}{c(x,y)^2}\left(a(x,y)A_x(x,y) + b(x,y)A_y(x,y)\right), \quad (10)$$

$$A_\eta(x,y) \stackrel{\text{def}}{=} \frac{1}{c(x,y)^2}\left(-b(x,y)A_x(x,y) + a(x,y)A_y(x,y)\right). \quad (11)$$

We denote $(A_\xi^u(\xi,\eta), A_\eta^u(\xi,\eta))$ by $\tilde{\mathbf{A}}^u(\xi,\eta)$.

Using $\partial a/\partial x = -\partial b/\partial y$ and $\partial a/\partial y = \partial b/\partial x$ by the Cauchy-Riemann relations (7) and (8), we know that the magnetic field determined by the new vector potentials vanishes on Ω_R:

Lemma 5.1: For $(\xi,\eta) = (\phi(x,y), \psi(x,y)) \in \mathbb{R}_{2R}^2$ $((x,y) \in \Omega_R)$,

$$0 = \frac{1}{c^2(x,y)}\left(\frac{\partial A_y}{\partial x}(x,y) - \frac{\partial A_x}{\partial y}(x,y)\right) = \frac{\partial A_\eta^u}{\partial \xi}(\xi,\eta) - \frac{\partial A_\xi^u}{\partial \eta}(\xi,\eta).$$

And, on $C_0^\infty(\mathbb{R}_{2R}^2)$,

$$\left[Up_\xi^S U^{-1} - \mathrm{q}A_\xi^u\,,\ Up_\eta^S U^{-1} - \mathrm{q}A_\eta^u\right] = -i\mathrm{q}\left(\frac{\partial A_\eta^u}{\partial \xi} - \frac{\partial A_\xi^u}{\partial \eta}\right) = 0.$$

Remark 5.1: We defined (A_ξ, A_η) so that 1-form $A_x dx + A_y dy$ is equal to $A_\xi^u d\xi + A_\eta^u d\eta$, i.e., $A_x dx + A_y dy = A_\xi^u d\xi + A_\eta^u d\eta$.

Functions, A_j $(j = x, y)$ and A_j^u $(j = \xi, \eta)$, are real-valued measurable functions with respect to measure spaces, $(\Omega_R, dxdy)$ and $(\mathbb{R}_{2R}^2, d\xi d\eta/c^u(\xi,\eta)^2)$ respectively. And besides, A_j $(j = x, y)$ and A_j^u $(j = \xi, \eta)$ are finite almost everywhere with respect to $dxdy$ and

$d\xi d\eta/c^u(\xi,\eta)^2$ respectively. So, we can define self-adjoint operators acting in $L^2(\Omega_R)$ and $L^2_c(\mathbb{R}^2_{2R})$, respectively, as multiplication operators (see Example 5.11 and its remark 1 in [12] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [5]). We denote them by the same symbols, i.e.,

$$(A_j f)(x,y) \stackrel{\text{def}}{=} A_j(x,y)f(x,y),$$
$$\mathrm{D}(A_j) \stackrel{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int\int_{\Omega_R} |A_j(x,y)f(x,y)|^2 < \infty \right\},$$
$$\left(A_j^u f^u\right)(\xi,\eta) \stackrel{\text{def}}{=} A_j^u(\xi,\eta)f^u(\xi,\eta),$$
$$\mathrm{D}(A_j^u) \stackrel{\text{def}}{=} \left\{ f^u \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} \frac{|A_j^u(\xi,\eta)f^u(\xi,\eta)|^2}{c^u(\xi,\eta)^2} < \infty \right\}.$$

Here we modify Ω_R. We set $\Omega_x^{\mathrm{mod}} \stackrel{\text{def}}{=} \Omega_R \setminus \{(x,0) \mid R \le x\}$, and $\Omega_y^{\mathrm{mod}} \stackrel{\text{def}}{=} \Omega_R \setminus \{(0,y) \mid R \le y\}$. Furthermore, we modify $\mathbb{R}^2_{2R}$ as $(\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}} \stackrel{\text{def}}{=} \mathbb{R}^2_{2R} \setminus \{(x,0) \mid 2R < \xi\}$, and $(\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}} \stackrel{\text{def}}{=} \mathbb{R}^2_{2R} \setminus \{(0,y) \mid 0 < \eta\}$.

Remark 5.2: $J_{out}(\Omega_x^{\mathrm{mod}}) = (\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}}$, and $J_{out}\left(\Omega_y^{\mathrm{mod}}\right) = (\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}}$.

Then, since $(\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}}$ and $(\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}}$ are simply connected, and Lemma 5.1 holds, we can use Poincaré's lemma, and there exist two functions $\Lambda_\xi^u \in C_0^\infty\left((\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}}\right)$ and $\Lambda_\eta^u \in C_0^\infty\left((\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}}\right)$ such that $\tilde{\mathbf{A}}^u = \nabla\Lambda_\xi^u$ on $(\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}}$, and $\tilde{\mathbf{A}}^u = \nabla\Lambda_\eta^u$ on $(\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}}$.

The following lemmas, Lemmas 5.2 and 5.3, are technical ones for our purpose that we show the Aharonov-Bohm effect in Weyl's CCR caused by inequivalence between Weyl's CCR and Heisenberg's.

Lemma 5.2:
(a) $C_0^\infty(\Omega_x^{\mathrm{mod}})$ and $C_0^\infty(\Omega_y^{\mathrm{mod}})$ are dense in $L^2(\Omega_R)$.
(b) $C_0^\infty((\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}})$ and $C_0^\infty((\mathbb{R}^2_{2R})_\eta^{\mathrm{mod}})$ are dense in $L^2_c(\mathbb{R}^2_{2R})$.

Lemma 5.3: $C_0^\infty((\mathbb{R}^2_{2R})_\xi^{\mathrm{mod}})$ is a core for $Up_\xi^S U^{-1}$ in $L^2_c(\mathbb{R}^2_{2R})$.

Now, we consider $\Lambda_j(x,y) \stackrel{\text{def}}{=} (U^{-1}\Lambda_j^u)(x,y) \equiv \Lambda_j^u(\phi(x,y),\psi(x,y))$ $(j = \xi,\eta)$. It is clear that Λ_j is measurable (actually $\Lambda_j \in C^\infty(\Omega_j^{\mathrm{mod}})$). So, by well-known fact, we can define the following self-adjoint operators, Λ_j $(j = \xi,\eta)$:

$$(\Lambda_j f)(x,y) \stackrel{\text{def}}{=} \Lambda_j(x,y)f(x,y),$$
$$\mathrm{D}(\Lambda_j) \stackrel{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int\int_{\Omega_R} |\Lambda_j(x,y)f(x,y)|^2 < \infty \right\}$$

(see Example 5.11 and its remark 1 in [12] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [5]).

The following theorem means that the mv-momentum w.r.t. ξ-axis is realized as a self-adjoint operator in the same way as Arai's proof[7].

Theorem 5.4: $P_\xi \stackrel{\text{def}}{=} p_\xi^S - \mathrm{q}A_\xi$ is essentially self adjoint on $C_0^\infty(\Omega_x^{\text{mod}})$.

The following theorem means that the mv-momentum w.r.t. η-axis is realized as a self-adjoint operator with a small difference from Arai's proof[7].

Theorem 5.5: $P_\eta \stackrel{\text{def}}{=} p_\eta^S - \mathrm{q}A_\eta$ is essentially self adjoint on $e^{i\mathrm{q}\Lambda_\eta}\mathrm{D}(p_\eta^S)$.

For $\xi, \eta, s, t \in \mathbb{R}$, we define two curves $C_\pm(\xi,\eta;s,t)$ in $\mathbb{R}^2$ from $(\xi,\eta) \in \mathbb{R}^2_{2R}$ to $(\xi+s, \eta+t)$ by $C_-(\xi,\eta;s,t) \stackrel{\text{def}}{=} \{(\xi+\theta s, \eta)\,|\,0 \le \theta \le 1\} \cup \{(\xi+s, \eta+\theta t)\,|\,0 \le \theta \le 1\}$, and $C_+(\xi,\eta;s,t) \stackrel{\text{def}}{=} \{(\xi, \eta+\theta t)\,|\,0 \le \theta \le 1\} \cup \{(\xi+\theta s, \eta+t)\,|\,0 \le \theta \le 1\}$. Then we define a rectangle $C(\xi,\eta;s,t)$ by $C(\xi,\eta;s,t) \stackrel{\text{def}}{=} C_-(\xi,\eta;s,t) - C_+(\xi,\eta;s,t)$, which is the rectangular closed curve: $(\xi,\eta) - (\xi+s,\eta) \to (\xi+s,\eta+t) \to (\xi,\eta+t) \to (\xi,\eta)$.

In the same way as Arai's, for every $s, t \in \mathbb{R}$, we define a function $\Phi^u_{s,t}$ on $\mathbb{R}^2_{2R}$ by

$$\Phi^u_{s,t}(\xi,\eta) \stackrel{\text{def}}{=} \int_{C(\xi,\eta;s,t)} \widetilde{\mathbf{A}}^u(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{r}},$$

where $\tilde{\mathbf{r}} \stackrel{\text{def}}{=} (\xi,\eta)$. For $(\xi_0,\eta_0) \in \Delta_{s,t} \equiv (\mathbb{R} \setminus \{\pm 2R, \pm 2R - s\}) \times (\mathbb{R} \setminus \{0, -t\})$, if $C(\xi,\eta;s,t) \subsetneqq \mathbb{R}^2_{2R}$, i.e., $C(\xi_0,\eta_0;s,t) \cap [-2R, 2R] = \phi$, then it is evident that $\int_{C(\xi_0,\eta_0;s,t)} \widetilde{\mathbf{A}}^u(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{r}}$ is finite. For the other cases, namely $C(\xi_0,\eta_0;s,t) \cap [-2R, 2R] = \{(\xi_0, \eta_0 + \theta_0 t)\}$, $\{(\xi_0 + s, \eta_0 + \theta_0 t)\}$ or $\{(\xi_0, \eta_0 + \theta_0 t), (\xi_0 + s, \eta_0 + \theta_0 t)\}$ for some $0 < \theta_0 < 1$, $A^u_\eta(\xi_0, \eta')$ and $A^u_\eta(\xi_0 + s, \eta')$ have discontinuous points, $(\xi_0, \eta_0 + \theta_0 t)$ and $(\xi_0 + s, \eta_0 + \theta_0 t)$ respectively. However $A^u_\eta(\xi_0, \eta')$ and $A^u_\eta(\xi_0 + s, \eta')$ are integrable functions of η' on $\eta' \in [\eta_0, \eta_0 + t]$ by using (3), (7), (8), (10), (11), and the fact that $J_{out}(\Omega_R) = \mathbb{R}^2_{2R}$. So we know that $\int_{C(\xi_0,\eta_0;s,t)} \widetilde{\mathbf{A}}^u(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{r}}$ is also finite. Thus, the function $\Phi^u_{s,t}(\xi,\eta)$ is defined on $\Delta_{s,t}$. Then, $\Phi^u_{s,t}(\xi,\eta)$ is a real-valued measurable function with respect to the measure space $(\mathbb{R}^2_{2R}, d\xi d\eta / c^u(\xi,\eta)^2)$. And besides, $\Phi^u_{s,t}(\xi,\eta)$ is finite almost everywhere with respect to $d\xi d\eta / c^u(\xi,\eta)^2$. Here we note that the poles, $(\pm 2R, 0)$, of $1/c^u$ are outside $\mathbb{R}^2_{2R}$. So, we can define a self-adjoint operator $\Phi^u_{s,t}$ as multiplication operator on $L^2_c(\mathbb{R}^2_{2R})$, i.e.,

$$\mathrm{D}(\Phi^u_{s,t}) \stackrel{\text{def}}{=} \left\{ f^u \in L^2_c(\mathbb{R}^2_{2R}) \,|\, \Phi^u_{s,t} f^u \in L^2_c(\mathbb{R}^2_{2R}) \right\},$$

$$\left(\Phi^u_{s,t} f^u\right)(\xi,\eta) \stackrel{\text{def}}{=} \Phi^u_{s,t}(\xi,\eta) f^u(\xi,\eta).$$

(see Example 5.11 and its remark 1 in [12] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [5]).

Remark 5.3: If $C(\xi,\eta;s,t) \subsetneqq \mathbb{R}^2_{2R}$ ($(\xi,\eta) \in \Delta_{s,t}$), then it is clear that $J_{out}^{-1} C(\xi,\eta;s,t) \subsetneqq \Omega_R$. Thus, $U^{-1} \Phi^u_{s,t} U$ means the flux going through the interior of $J_{out}^{-1} C(\xi,\eta;s,t)$ because the Stokes theorem holds. If $C(\xi,\eta;s,t) \cap [-2R, 2R] \neq \phi$ ($(\xi,\eta) \in \Delta_{s,t}$), then, by the definition of $\Delta_{s,t}$, the points in $C(\xi,\eta;s,t) \cap [-2R, 2R]$ are not poles of $A^u_j(\xi,\eta)$. So, in this case, although $J_{out}^{-1} C(\xi,\eta;s,t)$ is not a loop in Ω_R, if we have $A_j \in C(D_R \setminus \mathrm{S})$ ($j = x, y$) and $D_R \setminus \mathrm{S}$ is connected where we denote by S the set of all singularities of A_j ($j = x, y$) with $\mathrm{S} \subsetneqq D_R$, then by adding a curve in D_R to $J_{out}^{-1} C(\xi,\eta;s,t)$, we can make a loop C in $\mathbb{R}^2$.

So, $U^{-1}\Phi^{u}_{s,t}U$ can be extended an operator acting in $L^2(\mathbb{R}^2)$, and it means the flux going through the interior C.

Defining $\Phi^{\widetilde{\mathbf{A}}}_{s,t}(x,y) \stackrel{\text{def}}{=} \Phi^{u}_{s,t}(\phi(x,y),\psi(x,y))$, in the sense of operator $\Phi^{\widetilde{\mathbf{A}}}_{s,t} \stackrel{\text{def}}{=} U^{-1}\Phi^{u}_{s,t}U$, we obtain our desired theorem. By this theorem, we can show that the Aharonov-Bohm effect appears in Weyl's CCR for just $\exp\left[itp^S_j\right]$ $(j=\xi,\eta)$, not $\exp\left[itp_{U_j}\right]$ $(j=x,y)$, simultaneously inequivalence between Weyl's CCR and Heisenberg's.

Theorem 5.6: For all $s,t\in\mathbb{R}$,

$$e^{is\overline{P}_\xi}e^{it\overline{P}_\eta} = \exp\left[-i\mathsf{q}\Phi^{\widetilde{\mathbf{A}}}_{s,t}\right]e^{it\overline{P}_\eta}e^{is\overline{P}_\xi}.$$

As a conclusion of our assertion in this paper,

Theorem 5.7: The system $\left\{\overline{P}_j, q^S_j\right\}_{j=\xi,\eta}$ of self-adjoint operators fulfills Heisenberg's CCR on a certain dense domain in $L^2(\Omega_R)$, on the other hand $\left\{\overline{P}_j, q^S_j\right\}_{j=\xi,\eta}$ fulfills Weyl's CCR if and only if $\Phi^{\widetilde{\mathbf{A}}}_{s,t}(x,y)$ is a function having a value of an integer multiple of $2\pi/\mathsf{q}$ for every $C(x,y;s,t)$ in Ω_R.

Corollary: The system $\left\{p^S_j, q^S_j\right\}_{j=\xi,\eta}$ of self-adjoint operators fulfills Heisenberg's CCR and Weyl's.

6 Discussion for Riemann surfaces.

We extended Reeh's and Arai's results to the case where the "hole" has a finite radius. We used a disc for the hole, but the disc is not essential for our argument. Our method is valid over Riemann surfaces conformally equivalent to $\mathbb{R}^2_{2R}$ with a conformal map which plays a role of J_{out}. Namely, for a Riemann surface $\mathcal{R}$ in $\mathbb{R}^2$ which is conformally equivalent to $\mathbb{R}^2_{2R}$, it is evident that there exists a biholomorphic function $J_{\mathcal{R}}$ such that $J_{\mathcal{R}}(\mathcal{R}) = \mathbb{R}^2_{2R}$, so we can use $J_{\mathcal{R}}$ instead of the Joukowski transformation J_{out}.

For some general holes, we realize that our method will be valid over Riemann surfaces conformally equivalent to the space removed finite segments parallel to ξ-axis and finite points from $\mathbb{R}^2$, which reminds us the following theorem: *Let $\mathcal{R}$ be a non-compact Riemann surface of planar character. Then, there exists a biholomorphic function $J:\mathcal{R}\to J(\mathcal{R})$ and $x_0\in\mathcal{R}$ such that $f(x_0)=\infty$, and $F=(\mathbb{C}\cup\{\infty\})\setminus J(\mathcal{R})$ is a bounded closed set in $\mathbb{C}$ and each connected components of F is a segment parallel to the real axis or a point* (see Theorem 7.3 in Ref.[13]).

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