Title	One-sided phase constraint $x(t) \ge 0$ forms an envelope(Continuous and Discrete Mathematical Optimization)
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Citation	数理解析研究所講究録 (1997), 981: 193-198
Issue Date	1997-03
URL	http://hdl.handle.net/2433/60880
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
981 巻 1997 年 193-198

One-sided phase constraint $x(t) \geq 0$ forms an envelope

相条件 $x(t) \geq 0$ は包絡線を生成する[1]

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Introduction

In this paper, we are concerned with the following max-type function:

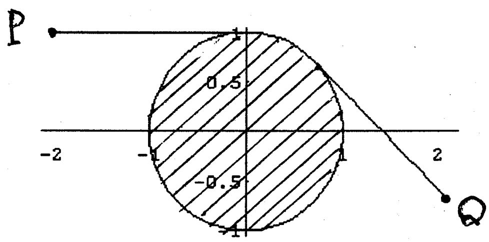
$$S(x) := \max_{t \in T} G(x(t), t) \quad x \in X, \tag{1}$$

where T is a compact metric space, X is a subspace of the set of all n-dimensional vector-valued continuous functions $C(T)^n$ equipped with the uniform norm. We denote by G_x and G_{xx} the gradient (row) vector and the Hesse matrix of f w.r.t. x, respectively, and assume them to be continuous on $R^n \times T$. This max-type function is induced from a phase constraint

$$G(x(t), t) \leq 0 \quad {}^{\forall} t \in T,$$

which appears in variational problems and optimal control problems. For instance, a variational problem to find the shortest path in R^2 joining two given points P and Q that does not transverse the unit ball is formulated as follows:

$$\begin{aligned} &\text{Minimize} && \int_0^1 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt \\ &\text{subject to} && (x_1(0), x_2(0)) = P, \quad (x_1(1), x_2(1)) = Q, \\ &&& 1 - x_1^2(t) - x_2(t)^2 \leq 0 \quad {}^{\forall} t \in [0, 1]. \end{aligned}$$



There are two aims in this paper. First one is to give formulae for first- and second-order directional derivatives of $S(x)$. Second one is to show that one-sided phase constraint $x(t) \geq a(t)$, where $a(t)$ is a given continuous function, always forms an envelope except two trivial cases:

$$x(t) \equiv a(t),$$

$$x(t) > a(t) \quad \text{for every } t.$$

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[1]This research is partially supported by the Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture, No. 08640294

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By the way, there are a lot of papers that delt with another max-type function:

$$S_0(x) := \max_{t\in T} G(x,t) \quad x \in R^n, \tag{2}$$

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Demyanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9] [10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this max-type function, for example, in Tcheby-cheff approximation. The latter max-type function $S_0(x)$ is a special case of $S(x)$. Indeed, if we take as X $\{x \in C(T)^n \mid x(t) \equiv \text{constant vector} \in R^n\}$, then $S(x)$ reduces to $S_0(x)$. So $S(x)$ inherits a lot of properties from $S_0(x)$.

論文の概要

次の max-型関数の１次と２次の方向微分について考察する。

$$S(x) := \max_{t\in T} G(x(t),t) \quad x \in X, \tag{3}$$

ただし T はコンパクト距離空間, X は n 次元ベクトル値連続関数全体 $C(T)^n$ の部分空間とする。この max-型関数は変分問題や最適制御問題の相条件

$$G(x(t),t) \leq 0 \quad {}^{\forall}t \in T$$

を考察するとき出会う。本論文では、この相条件から包絡線が生成されるかどうかを調べるために、max-型関数 $S(x)$ の２次の方向微分を表す公式を与える。

ところで、従来よく研究されてきた max-型関数は次の関数である。

$$S_0(x) := \max_{t\in T} G(x,t) \quad x \in R^n, \tag{4}$$

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 T が x に依存してよいとすれば、$S_0(x)$ の最小化問題はパラメトリック最適化問題になる。$S(x)$ が $S_0(x)$ と本質的に異なる点は、後者は x と t が独立に動けるのに対し、前者は x が t に依存することである。しかしながら、$S_0(x)$ は $S(x)$ のスペシャルケースと見なすこともできる。つまり、X として n 次元ベクトル値定数関数全体 $\{x(t) \equiv a \mid a \in R^n\}$ をとればよい。従って、$S(x)$ は $S_0(x)$ の多くの性質を受け継ぐことになる。その結果、相条件も包絡線を生成する。より正確に言えば、片側相制約 $x(t) \geq a(t)$ について、二つの自明なケース：

$$\bar{x}(t) \equiv a(t),$$

$$\bar{x}(t) > a(t) \quad \text{for every } t.$$

を除いて、点 $\bar{x}$ において包絡線を生成する方向 y を選ぶ事が出来る。

In the following, we denote by $T(x)$ the set of all extreme points $G(x(\cdot),\cdot)$, that is,

$$T(x) := \{t \in T\ ;\ G(x(t),t) = S(x)\}, \quad x \in C(T)^n.$$

THEOREM 1 *The function $S(x)$ is continuous.*

THEOREM 2 *The function $S(x)$ is directionally differentiable in any direction $y \in X$, and its directional derivative is given by*

$$S'(x;y) = \max\{G_x(x(t),t)y(t);\ t \in T(x)\}. \tag{5}$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 2 , we get Danskin's formula.

COROLLARY 1 *(Danskin[3]) The function $S_0(x)$ is directionally differentiable in any direction $y \in R^n$ and its directional derivative is given by*

$$S_0'(x;y) = \max\{G_x(x,t)y;\ t \in T(x)\}. \tag{6}$$

Next, we consider a second-order directional derivative of $S(x)$.

DEFINITION 1 *The upper second-order directional derivative of $S(x)$ at x in the direction y is defined by*

$$\overline{S}''(x;y) = \limsup_{\varepsilon \to +0} \frac{S(x+\varepsilon y) - S(x) - \varepsilon S'(x;y)}{\varepsilon^2} \tag{7}$$

DEFINITION 2 *([9]) For any functions $u,\ v \in C(T)$ satisfying*

$$\begin{cases} u(t) \geq 0 & {}^{\forall}t \in T, \\ v(t) \geq 0 & \text{if } u(t) = 0, \end{cases} \tag{8}$$

we define a function $E : T \to [-\infty, +\infty]$ by

$$E(t) := \begin{cases} \sup\left\{\limsup \frac{v(t_n)^2}{4u(t_n)};\ \{t_n\} \text{ satisfies } (10)\right\}, & \text{if } t \in T_0, \\ 0 & \text{if } u(t) = v(t) = 0 \text{ and } t \notin T_0, \\ -\infty & \text{otherwise,} \end{cases} \tag{9}$$

$$T_0 := \left\{t \in T;\ {}^{\exists}t_n \to t \text{ s.t. } u(t_n) > 0,\ -\frac{v(t_n)}{u(t_n)} \to +\infty\right\}. \tag{10}$$

THEOREM 3 *Let x and y be arbitrary functions in $C(T)^n$. Then it holds that*

$$\overline{S}''(x;y) = \max\left\{\frac{y(t)^T G_{xx}(x(t),t)y(t)}{2} + E(t)\ ;\ t \in T(x;y)\right\}, \tag{11}$$

where $T(x;y) := \{t \in T(x)\ ;\ S'(x;y) = G_x(x(t),t)y(t)\}$ and $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x(t),t), \quad v(t) = S'(x;y) - G_x(x(t),t)y(t). \tag{12}$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 3 , we get the following formula due to [9].

COROLLARY 2 *Let x and y be arbitrary points in R^n. Then it holds that*

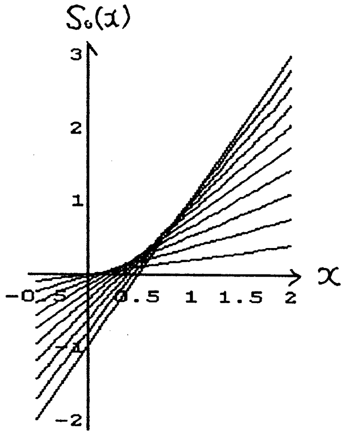
$$\overline{S}''(x;y) = \max\left\{\frac{y^T G_{xx}(x,t)y}{2} + E(t) \ ; \ t \in T(x;y)\right\}, \tag{13}$$

where $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x,t), \quad v(t) = S'(x;y) - G_x(x,t)y. \tag{14}$$

We proved in [9] [10] that an envelope is formed from $G(x,t)$ when $E(t) > 0$ at some point $t \in T(x;y)$.

EXAMPLE 1 *Let us consider a family of straight lines $f(x,t) = 2tx - t^2$, $t \in [0,1]$, $x \in R$. It is evident that it forms an envelope $S_0(x) = x^2$, $(0 \le x \le 1)$.*



Hence $S_0''(0;y) = y^2$ for any $y \ge 0$ and $f_{xx}(0,t) \equiv 0$. Thus there is a gap between the second-order directional derivatives of the max-type function $S_0(x)$ and its constituent functions $f(x,t)$. On the other hand, it is directly computed from the definition that $E(0) = y^2$ for every $y > 0$, which fills the gap.

$$\begin{aligned}\overline{S}_0''(0;y) &= \max\left\{\frac{1}{2}y^T f_{xx}(0,t)y + E(t) \ ; \ t \in T(0;y)\right\} \\ &= E(0) = y^2\end{aligned}$$

It is reasonable to guess that an envelope is formed from the phase constraint $x(t) \ge a(t)$ when the function $E(t)$ is positive at some $t \in T(x;y)$ as well as the max-type function $S_0(x)$. However this is not a proof but a guess. So we next give a proof that the one-sided phase constraint $x(t) \ge a(t)$ certainly forms an envelope for a certain direction $y(t)$ except two trivial cases.

THEOREM 4 *Let T be a connected compact metric space. Assume that $\bar{x}(t)$ does not satisfy neither*

$$x(t) \equiv a(t), \tag{15}$$

nor

$$x(t) > a(t) \quad \text{for every } t. \tag{16}$$

Then there exists a function $y \in C(T)$ such that the one-sided phase constraint $x(t) \geq a(t)$ forms an envelope in the direction y.

Proof. Let $y(t) := -2\sqrt{x(t) - a(t)}$ and put for $\xi \in R$

$$\begin{aligned} s(\xi) &:= S(\bar{x} + \xi y) = \max_t \{a(t) - \bar{x}(t) - \xi y(t)\} \\ &= \max_t \{a(t) - \bar{x}(t) + 2\xi\sqrt{\bar{x}(t) - a(t)}\} \end{aligned}$$

Then $s(\xi)$ becomes a standard max-function.

$$s(\xi) = \max_{\tau \in T'} \{2\xi\tau - \tau^2\}$$

Furthermore, from the assumption, the image of T by the continuous function $\sqrt{x(t) - a(t)}$ is a compact interval $T' := [0, t_1]$ with $t_1 > 0$. Hence $s(\xi)$ is same with Example 1, so that an envelope is formed.

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