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A period-doubling bifurcation for the Duffing equation

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1 Introduction

In this paper, we briefly mention the results showed in [4]. We consider the periodic solutions of the Duffing equation which describes the nonlinear forced oscillation;

$$u''(t)+\mu u'(t)+\kappa u(t)+\alpha u^{3}(t)=f_{\lambda}(t), \quad t\in R \tag{1.1}$$

where μ, α are positive constants and κ is a nonnegative constant, and $f_{\lambda}(t)$ is a given family of T-periodic external forces parameterized by λ which somehow represents the magnitude of f_{λ} (e.g., $f_{\lambda}=\lambda\sin(t)$). It is well-known that for any λ there exists at least one T-periodic solution of (1.1), and furthermore if the magnitude λ is suitably small, then its solution is unique and asymptotically stable. As λ increases, we can observe by numerical computations that the solution loses its stability and various bifurcation phenomena take place. In particular, the period-doubling bifurcations are observed as very important phenomena along the route toward a so called "Chaos". However, it is surprising that there have been no rigorous proofs of these bifurcation phenomena. Recently, Komatsu-Kano-Matsumura [3] tried to detect a bifurcation phenomenon around a "linear probe" $\{(\lambda, u_{\lambda})\}_{\lambda>0}$ inserted into the product space (λ, u), which is defined by

$$\begin{cases} u_{\lambda}(t):=\lambda U(t), \quad U(t): \text{given } T\text{-periodic fucntion} \\ f_{\lambda}(t):=u_{\lambda}''(t)+\mu u_{\lambda}'(t)+\kappa u_{\lambda}(t)+\alpha u_{\lambda}^{3}(t). \end{cases} \tag{1.2}$$

Here we should note that $u=u_{\lambda}$ is a trivial solution of (1.1) corresponding to f_{λ} for any λ. Then, in the particular case $U(t)=\sin(2\pi t)\,(T=1)$, studying the linearized equation of (1.1) at $u=u_{\lambda}$

$$v''(t)+\mu v'(t)+\kappa v(t)+3\alpha\lambda^{2}U^{2}(t)v(t)=0 \tag{1.3}$$

by the arguments of continued fractions, they showed that T-periodic solution bifurcates at least three points from the probe $\{u_{\lambda}\}_{\lambda>0}$ under some condition on μ. They also made a conjecture by numerical computations that there are infinitely many bifurcation points of T-periodic solution. However, they could not obtain any results on period-doubling bifurcations. On the other hand, numerical computations in the case $U(t)=\sin(2\pi t)+0.5$, indicate that there might be infinitely many bifurcation points of both T-periodic and $2T$-periodic solutions, and $2T$-periodic solution bifurcates at first as λ increases. Tracing this first branch, we also observe that $2^{n}T$-periodic solutions bifurcate and strange attractor appears. In this paper,we show that for more general T-periodic functions $U(t)$, only T-periodic and $2T$-periodic solutions can bifurcate from $\{u_\lambda\}_{\lambda>0}$, and under some condition on μ there exist infinitely many bifurcation points of T-periodic solution, and also do exist infinitely many bifurcation points of $2T$-periodic solution (period-doubling bifurcations) except some paticular cases. Furthermore, we show the asymptotic stability and unstability of the trivial solution $u_\lambda(t)$ alternates at each these bifurcation points. We also show that the case $U(t) = \sin(2\pi t)$ is really a particular one where only T-periodic solutions bifurcate from $\{u_\lambda\}_{\lambda>0}$. The precise conditions and main Theorem are stated in Section 2. In Section 3, we reformulate the problem in order to apply Crandall-Rabinowitz's Theorem [2] on bifurcation theory. In this process, eigenvalue problem of (1.3) plays an essential role. We relate it to the Lyapunov exponent in Section 4 and show the properties of the Lyapunov exponent, making use of the expansion theory by generalized eigen-functions established by Titschmarsh-Kodaira in Section 5. From these properties and asymptotic analysis with respect to λ, which details are stated in Section 7, we prove main Theorem in Section 6.

2 Main Theorem

To state the main Theorem precisely, we assume that

(2.1) $\quad U^2(t)$ has $N+1$ zero points $\{t_i\}_{i=0}^N$ of n-th order on $[t_0, t_0+T]$,

where $t_0 < t_1 < \cdots < t_N = t_0 + T$. We define $\nu = \frac{1}{n+2}$ and also define $S_i = \int_{t_{i-1}}^{t_i} |U(s)|ds$.

Theorem 2.1 *Suppose (2.1) and*

$$\frac{\mu}{2} < \frac{N}{T}\log(\cot\frac{\nu\pi}{2}), \tag{2.2}$$

then it holds the followings.

(1) The case N=1 :

There exist λ^* and $\{\lambda_i\}_{i=0}^\infty$ ($\lambda^* < \lambda_0 < \lambda_1 \cdots \to \infty$) such that the sequence of bifurcation points for $\lambda > \lambda^*$ is coincident with $\{\lambda_i\}_{i=0}^\infty$, where $\{\lambda_{4m}\}, \{\lambda_{4m+1}\}$ are T-periodic bifurcation points and $\{\lambda_{4m+2}\}, \{\lambda_{4m+3}\}$ are $2T$-periodic bifurcation points. Moreover, it holds that if $\lambda \in (\lambda_{2m+1}, \lambda_{2m})$, then u_λ is asymptotically stable, if $\lambda \in (\lambda_{2m}, \lambda_{2m+1})$, then u_λ is unstable.

(2) The case N=2 :

There exist infinitely many T-periodic bifurcation points, and also exist infinitely many $2T$-periodic bifurcation points except for the following cases.

(i) When $S_1 = S_2$, there does not exist $2T$-periodic bifurcation point for large λ.

(ii) When $\frac{S_1}{S_2} = \frac{2p+1}{2q+1}$ ($p, q \in \mathbb{N}$, $S_1 \neq S_2$), we assume

$$\frac{\mu}{2} < \frac{1}{T}\log(\frac{|\tilde{\Delta}| + \sqrt{|\tilde{\Delta}|^2 - 4}}{2}), \tag{2.3}$$

where

$$\tilde{\Delta} = \inf_\lambda \frac{2\{\cos(S_1+S_2)\lambda + \cos(S_1-S_2)\lambda\cos^2\nu\pi\}}{\sin^2\nu\pi}$$

instead of (2.2), then there also exist infinitely many 2T-periodic bifurcation points. The stability of u_λ changes at any above bifurcation points.

(3) The case $N \geq 3$

There exist infinitely many T-periodic bifurcation points. Furthermore, if $\{S_i\}_{i=1}^N$ are irrationaly independent, there also exist infinitely many 2T-periodic bifurcation points. The stability of u_λ changes at any these bifurcation points.

Remark 1 If $\frac{S_1}{S_2} \neq \frac{2p+1}{2q+1}$ $(p, q \in N)$, it holds that $\tilde{\Delta} = \frac{-2(1+\cos^2 \nu\pi)}{\sin^2 \nu\pi}$. Then we have

$$\log\left(\frac{|\tilde{\Delta}| + \sqrt{|\tilde{\Delta}|^2 - 4}}{2}\right) = 2\log(\cot \frac{\nu\pi}{2}),$$

which is consistent to the condition (2.2) .

Example 1 In the case $U(t) = \sin 2\pi t \pm 1$, $U^2(t)$ has two zero points of forth order. Applying Theorem, if $\frac{\mu}{2} < \frac{1}{2}\log(\frac{2+\sqrt{3}}{2-\sqrt{3}})$, there exist infinitely many 1-periodic bifurcation points and infinitely many 2-periodic bifurcation points.

Example 2 In the case $U(t) = \sin 2\pi t + 0.5$, $U^2(t)$ has three zero points of second order. So, if $\frac{\mu}{2} < \frac{1}{2}\log(\frac{2+\sqrt{2}}{2-\sqrt{2}})$, there exist infinitely many 1-periodic bifurcation points and infinitely many 2-periodic bifurcation points.

Remark 2 When $U(t) = \sin 2\pi t$, there does not exist 2-periodic bifurcation points. Because the period of $U(t)$ is 1 but the period of $U^2(t)$ is 1/2, the period of any bifurcation points is 1 or 1/2.

3 Reformulation of the problem

To prove the Theorem, we make use of a following bifurcation Theorem proved by Crandall-Rabinowitz [2].

Theorem 3.1 (Crandall and Rabinowitz) *Let X, Y be Banach spaces, V a neighborhood of 0 in X and*

$$F : (0, \infty) \times V \to Y$$

have the properties for a $\lambda_0 > 0$

(a) $F(\lambda, 0) = 0$ for $\lambda \in (0, \infty)$,

(b) The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous,

(c) $N(F_x(\lambda_0, 0))$ and $Y/R(F_x(\lambda_0, 0))$ are one dimensional.

(d) $F_{\lambda x}(\lambda_0, 0)x_0 \notin R(F_x(\lambda_0, 0))$, where $N(F_x(\lambda_0, 0)) = span\{x_0\}$.

If Z is any complement of $N(F_x(\lambda_0, 0))$ in X, then there is a neighborhood U of (λ_0 ,0) in $R \times X$, an interval (-a, a), and continuous functions $\varphi : (-a, a) \to R$, $\psi(-a, a) \to Z$ such that $\varphi(0) = \lambda_0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \{\varphi(\epsilon), \epsilon x_0 + \epsilon\psi(\epsilon) : |\epsilon| < a\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}. \quad (3.1)$$

If F_{xx} is also continuous, the function φ and ψ are once continuously differentiable.

We first note that any periodic solution of (1.1) should have the period $\widetilde{T} = mT$ for a $m \in \mathbb{N}$. So, for a fixed m, we look for the periodic solution of (1.1) in the form:

$$u(t) = u_\lambda(t) + \lambda v(t), \tag{3.2}$$

where $v(t)$ is a $\widetilde{T}$-periodic function. Then $v(t)$ satisfies

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) + \Lambda(U^2(t)v(t) + U(t)v^2(t) + \frac{1}{3}v^3(t)) = 0 \\ v(t+\widetilde{T}) = v(t), \quad t \in R, \end{cases} \tag{3.3}$$

where $\Lambda = 3\alpha\lambda^2$.

We define Banach spaces X and Y by

$$X = \{u \in C^2(R); u(t) = u(t+\widetilde{T}), t \in R\}$$

$$Y = \{u \in C(R); u(t) = u(t+\widetilde{T}), t \in R\}$$

with norm

$$||u||_X = \max_{0\le t\le \widetilde{T}} |u''(t)| + \max_{0\le t\le \widetilde{T}} |u'(t)| + \max_{0\le t\le \widetilde{T}} |u(t)|$$

and

$$||u||_Y = \max_{0\le t\le \widetilde{T}} |u(t)|$$

and define $F : (0,\infty) \times V \to Y$ by

$$F(\Lambda, v) = v'' + \mu v' + \kappa v + \Lambda(U^2 v + U v^2 + \frac{1}{3}v^3). \tag{3.4}$$

Then the following holds.

Lemma 3.2 *The hypotheses (a) - (d) of Theorem(3.1) are equivalent to the following three conditions.*

(i) There exist Λ_0 and a nontrivial solution v_0 which satisfies the linearized problem of (3.3) at $v = 0$:

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) + \Lambda_0 U^2(t)v(t) = 0 \\ v(t+\widetilde{T}) = v(t), \quad t \in R \end{cases} \tag{3.5}$$

(ii) span$\{v_0\}$is one dimensional.

(iii) $\int_0^{\widetilde{T}} v_0(t)v_0^*(t)U^2(t)dt \neq 0$, where $v_0^*(t)$ is a nontrivial solution of the adjoint equation for (3.5)

$$\begin{cases} v''(t) - \mu v'(t) + \kappa v(t) + \Lambda_0 U^2(t)v(t) = 0, \\ v(t+\widetilde{T}) = v(t), \quad t \in R \end{cases} \tag{3.6}$$

4 Eigenvalue problem of the linearized equation

We study the linearized equation:

$$v''(t) + \mu v'(t) + \kappa v(t) + \Lambda U^2(t)v(t) = 0. \tag{4.1}$$

We put $v(t) = e^{-\mu t/2}w(t)$,then (4.1) becomes

$$w''(t) - \frac{\mu^2}{4}w(t) + \kappa w(t) + \Lambda U^2(t)w(t) = 0 \tag{4.2}$$

We seek the solution in the form $e^{\mu t/2}\tilde{w}(t)$, where $\tilde{w}$ is periodic of period $\tilde{T} = mT$. Let $\Phi_\Lambda(t)$ be a fundamental matrix for (4.2):

$$\Phi_\Lambda(t) = \begin{pmatrix} \phi_1(t,\Lambda) & \phi_2(t,\Lambda) \\ \phi_1'(t,\Lambda) & \phi_2'(t,\Lambda) \end{pmatrix}$$

From the Froquet's Theory, if characteristic root of $\Phi_\Lambda(T)$ has a form $e^{\mu T/2}\omega_m$, where ω_m is the primitive m-th root of 1, then (4.1) has mT-periodic solution. Here, the characteristic roots of $\Phi_\Lambda(T)$ are given by the roots of

$$\sigma^2 - \Delta(\Lambda)\sigma + 1 = 0, \tag{4.3}$$

where $\Delta(\Lambda) = \phi_1(T,\Lambda) + \phi_2'(T,\Lambda)$. is a trace of $\Phi_\Lambda(T)$
If $|\Delta(\Lambda)| \leq 2$, then the roots of (4.3) are complex conjugates of magnitude 1 or ± 1. Therefore, there does not exist the root of the form $e^{\mu T/2}\omega_m$. If $|\Delta(\Lambda)| > 2$, then the roots of (4.3) are real and one root is always larger than 1 in magnitude and the other less than 1. Therefore the following result can be proved.

Lemma 4.1 *For the eigenvalue problem of the linearized equation, it holds that*

(i) (4.1) has T-periodic solution at Λ_0 if and only if $\Delta(\Lambda_0) = e^{\mu T/2} + e^{-\mu T/2}$,

(ii) (4.1) has 2T-periodic solution at Λ_0 if and only if $\Delta(\Lambda_0) = -(e^{\mu T/2} + e^{-\mu T/2})$,

(iii) (4.1) does not have $mT(m \geq 3)$ periodic solution,

(iv) The dimension of eigenspace of T or 2T periodic solution is 1.

5 Lyapunov exponent

As stated in the previous section, the existence or the nonexistence of the eigenvalue of (4.1) are determined by $\Delta(\Lambda)$. In this section, we investigate $\Delta(\Lambda)$ in detail. We define

$$\Sigma = \{\Lambda > 0, \quad |\Delta(\Lambda)| \leq 2\}.$$

Then for $\Lambda \notin \Sigma$, we can well define

$$z(\Lambda) = \frac{1}{T}\cosh^{-1}\frac{\Delta(\Lambda)}{2}, \tag{5.1}$$

such that $Rez(\Lambda) > 0$. We note that $Imz(\Lambda)$ is equal 0 if $\Delta(\Lambda) > 2$, and equals $i\pi$ if $\Delta(\Lambda) < -2$. $Rez(\Lambda)$ is so called Lyapunov exponent. Concerning $z(\Lambda)$, the following Lemma holds.

Lemma 5.1 *$z(\Lambda)$ can be represented in the form*

(5.2)
$$\frac{dz}{d\Lambda} = -\frac{1}{T}\int_0^T G_\Lambda(\tau,\tau)U^2(\tau)d\tau.$$

Here Green's function $G_\Lambda(t,s)$ is given by

$$G_\Lambda(t,s) = G_\Lambda(s,t) = \frac{w^+(t)w^-(s)}{[w^+,w^-]} \quad ; \quad t \geq s,$$

where w^+ stands for a solution of (4.2) in $L^2_{U^2}(0,\infty)$, w^- stands for a solution of (4.2) in $L^2_{U^2}(-\infty,0)$, and $[w^+,w^-]$ is the Wronskian.

Remark 3 $L^2_{U^2}$ denotes the function space L^2 weighted U^2 i.e.

$$L^2_{U^2}(R) = \{h(t); \int_R |h(s)|^2U^2(s)ds < \infty\}$$

Let the operator $L = \frac{1}{U^2}(-\frac{d^2}{dt^2} + (\frac{\mu^2}{4} - \kappa))$, then L is a self adjoint operator in $L^2_{U^2}$ and $G_\Lambda(t,s)$ is a integral kernel of the resolvent $(L - \Lambda I)^{-1}$.

According to the expansion theory by generalized eigen-functions established by Titschmarsh-Kodaira, $G_\Lambda(s,t)$ has the following representation;

(5.3)
$$G_\Lambda(s,t) = \int_R \frac{\Sigma_{1\leq i,j\leq 2}\phi_1(s,\xi)\phi_2(t,\xi)\sigma_{ij}(d\xi)}{\xi - \Lambda},$$

where $\{\sigma_{ij}\}$ is a matrix valued stiltjes measure which is nonnegative definite. Substituting this to (5.2), we have the following Lemma.

Lemma 5.2 *For any $\Lambda \notin \Sigma$, it holds that*

(5.4)
$$\frac{d^2z}{d\Lambda^2} = -\int_{\xi\in\Sigma} \frac{\sigma(d\xi)}{(\xi-\Lambda)^2},$$

where $\sigma(d\xi)$ is a nonnegative stiltjes measure. That is $\frac{d^2z}{d\Lambda^2} < 0$ for any $\Lambda \notin \Sigma$.

Lemma 5.3

(5.5)
$$\frac{dz}{d\Lambda}(\Lambda_0) \neq 0 \Longleftrightarrow \int_0^T v_0(t)v_0^*(t)U^2(t)dt \neq 0$$

Proof. Put $v_0(t) = e^{-\mu t/2}w_0(t)$, then $w_0(t)$ satisfies (4.2). So, $v_0(t)$ is equal to $e^{-\mu t/2}w^-(t)$ except for constant factor. In the same way, $v_0^*(t)$ is equal to $e^{\mu t/2}w^+(t)$ up to constant. Therefore

(5.6)
$$\int_0^T v_0(t)v_0^*(t)U^2(t)dt \neq 0 \Longleftrightarrow \int_0^T w^+(t)w^-(t)U^2(t)dt \neq 0$$

Thus,from Lemma 5.1, the proof is completed. □

6 Proof of Theorem

From the previous arguments and the following results concerning the asymptotic behavior of $\Delta(\Lambda)$ as $\Lambda \to \infty$, we can prove Theorem 2.1.

Proposition 6.1 *Suppose $U(t)$ satisfies the hypotheses of Theorem 2.1. Then it holds the followings.*
(1) The case N=1 :

(6.1) $$\Delta(\Lambda) = \frac{2\cos(S_1\sqrt{\Lambda})}{\sin\nu\pi}(1+o(1)) \quad as\ \Lambda \to \infty$$

(2) The case N=2 :

(6.2)
$$\Delta(\Lambda) = \frac{2\{\cos((S_1+S_2)\sqrt{\Lambda}) + \cos((S_1-S_2)\sqrt{\Lambda})\cos^2\nu\pi\}}{\sin^2\nu\pi}(1+o(1)) \quad as\ \Lambda \to \infty$$

(3) The case $N \geq 3$:

(6.3) $$\limsup_{\Lambda\to\infty}\Delta(\Lambda) \geq \frac{1}{\sin^N\nu\pi}\{(1+\cos\nu\pi)^N + (1-\cos\nu\pi)^N\}$$

and if $\{S_i\}_{i=1}^N$ are irrationaly indepedent, then

(6.4) $$\liminf_{\Lambda\to\infty}\Delta(\Lambda) \leq \frac{-1}{\sin^N\nu\pi}\{(1+\cos\nu\pi)^N + (1-\cos\nu\pi)^N\}$$

We only show the rough sketch of the proof of (1). For details, see [4].
We define $\rho(t) = U^2(t)$. Then there exists $\beta \geq 1$ such that

(6.5) $$\rho(t) = C_1 t^n(1 + C_2 t^\beta + O(t^{2\beta})) \quad as\ t \to 0$$

We want to decide $\{\phi_i\}_{i=1,2}$ on [0, T], but $\rho(t)$ has two zero points on [0, T]. So we define $\{\widehat{\phi}_i\}_{i=1,2}$ as the fundamental solutions for $\widehat{\rho}(t) = \rho(T-t)$, we get

(6.6) $$\Delta(\Lambda) = \phi_1(\frac{T}{2})\widehat{\phi_2}'(\frac{T}{2}) + \phi_1'(\frac{T}{2})\widehat{\phi_2}(\frac{T}{2}) + \phi_2(\frac{T}{2})\widehat{\phi_1}'(\frac{T}{2}) + \phi_2'(\frac{T}{2})\widehat{\phi_1}(\frac{T}{2}),$$

First we consider $\{\phi_i(\frac{T}{2})\}_{i=1,2}$. Changing the following variable and function:

(6.7) $$variable: \quad x = \int_0^t \sqrt{\rho(s)}ds,$$

(6.8) $$function: \quad g(x) = \rho(t)^{1/4}w(t),$$

then (4.2)is rewritten

(6.9) $$g''(x) + (\Lambda - Q(x))g(x) = 0,$$

where $Q(x) = (\mu^2/4 - \kappa)\rho^{-1}(t) - \rho^{-3/4}(t)(\rho^{-1/4}(t))''$.

From (6.5), it holds that

$$(6.10)\quad Q(x)=Q_0(x)\{1-\frac{8\beta(\beta^2-1)C_2}{n(n+4)(n+2\beta+2)}C_1^{-\nu\beta}(2\nu)^{-2\nu\beta}x^{2\nu\beta}+0(x^{4\nu\beta})\},$$

where $Q_0(x)=-n(n+4)\nu^2x^{-2}/4$ and $\nu=\frac{1}{n+2}$.
Putting $\Phi_1(x)=\rho^1/4(t)\phi_1(t)$ and $\Phi_2(x)=\rho^1/4(t)\phi_2(t)$, then $\Phi_1(x),\Phi_2(x)$ satisfy (6.9), we should note that $\Phi_1(x),\Phi_2(x)$ also satisfy

(6.11)
$$\begin{cases}\Phi_1(x)=\Lambda^{-\frac{1-2\nu}{4}}A(\sqrt{\Lambda}x)+\frac{1}{\sqrt{\Lambda}}\int_0^x(A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x)-(A(\sqrt{\Lambda}x)B(\sqrt{\Lambda}s))\widetilde{Q}(s)\Phi_1(s)ds,\\ \Phi_2(x)=\Lambda^{-\frac{1+2\nu}{4}}B(\sqrt{\Lambda}x)+\frac{1}{\sqrt{\Lambda}}\int_0^x(A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x)-(A(\sqrt{\Lambda}x)B(\sqrt{\Lambda}s))\widetilde{Q}(s)\Phi_2(s)ds.\end{cases}$$

Here $\widetilde{Q}(x)=Q(x)-Q_0(x)$, $A(y)=A_n\sqrt{y}J_{-\nu}(y)$ and $B(y)=B_n\sqrt{y}J_\nu(y)$, where J_ν is a ν-th Bessel function and

$$(6.12)\quad \begin{aligned}A_n&=\frac{1}{\sqrt{2}}\Gamma(1-\nu)(n+2)^{n\nu/2}C_1^{\nu/2},\\ B_n&=\frac{1}{\sqrt{2}}\Gamma(1+\nu)(n+2)^{n\nu/2}C_1^{\nu/2}.\end{aligned}$$

Using successive approximations with $\Phi_1^{(0)}(x)=\Lambda^{-\frac{1-2\nu}{4}}A(\sqrt{\Lambda}x)$ and

$$\Phi_1^{(n)}(x)=\frac{1}{\sqrt{\Lambda}}\int_0^x(A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x)-(A(\sqrt{\Lambda}x)B(\sqrt{\Lambda}s))\widetilde{Q}(s)\Phi_1^{(n-1)}(s)ds,$$

there follows

$$(6.13)\quad |\Phi_1(x)-\Lambda^{-\frac{1-2\nu}{4}}A(\sqrt{\Lambda}x)|=o(\Lambda^{-\frac{1-2\nu}{4}})\quad \Lambda\to\infty,$$

for any fixed x.
In the same way,

$$(6.14)\quad |\Phi_2(x)-\Lambda^{-\frac{1+2\nu}{4}}B(\sqrt{\Lambda}x)|=o(\Lambda^{-\frac{1+2\nu}{4}})\quad \Lambda\to\infty,$$

for any fixed x. From the fact:

$$(6.15)\quad \begin{aligned}A(y)&=A_n\sqrt{\frac{2}{\pi}}\cos(y-\frac{1-2\nu}{4}\pi)(1+o(1)),\quad y\to\infty,\\ B(y)&=B_n\sqrt{\frac{2}{\pi}}\cos(y-\frac{1+2\nu}{4}\pi)(1+o(1)),\quad y\to\infty,\end{aligned}$$

it holds that

$$(6.16)\quad \begin{aligned}\phi_1(\frac{T}{2})&=\rho(\frac{T}{2})^{-\frac{1}{4}}\Lambda^{-\frac{1-2\nu}{4}}A_n\sqrt{\frac{2}{\pi}}\cos(\int_0^{\frac{T}{2}}\sqrt{\rho(y)}dy\sqrt{\Lambda}-\frac{1-2\nu}{4}\pi)(1+o(1))\\ \phi_2(\frac{T}{2})&=\rho(\frac{T}{2})^{-\frac{1}{4}}\Lambda^{-\frac{1+2\nu}{4}}B_n\sqrt{\frac{2}{\pi}}\cos(\int_0^{\frac{T}{2}}\sqrt{\rho(y)}dy\sqrt{\Lambda}-\frac{1+2\nu}{4}\pi)(1+o(1))\end{aligned}$$

as $\Lambda \to \infty$. In the same way, we have estimate of $\{\widehat{\phi}_i(\frac{T}{2}\}_{i=1,2}$. From (6.6), we have

$$\Delta(\Lambda) = \frac{2\cos(S_1\sqrt{\Lambda})}{\sin \nu\pi}(1 + o(1)) \quad as\ \Lambda \to \infty \tag{6.17}$$

The proof is completed.

References

[1] E.Coddington and N.Levinson, Theory of Ordinary Differential Equations. New York: McGraw-Hill 1955

[2] H.G.Crandall, P.H.Rabinowitz, Bifurcation from simple eigenvalues, Jour. Func. Anal. 8 (1971) 321-340.

[3] Y.Komatsu, T.kano and A.matsumura, A Bifurcation phenomenon for the periodic solution of the Duffing equation,to appear

[4] Y.Komatsu, S.Kotani and A.matsumura, A Period-doubling Bifurcation for the Duffing equation, preprint

[5] Yoshisuke Ueda, The Road to Chaos Aerial Press, Inc.