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The Fourier Coefficients and the Singular Moduli of the Elliptic Modular Function $j(\tau)$

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Abstract

We give a closed formula for the Fourier coefficients of the elliptic modular function $j(\tau)$ expressed in terms of singular moduli, *i.e.,* the values at imaginary quadratic arguments. The formula is a consequence of a theorem of D. Zagier [6] which is intimately related to a recent result of R. Borcherds [2] on a construction of modular forms as infinite products.

Key Words: *Elliptic modular function; Fourier coefficients; complex multiplication; modular forms of half integral weight.*

1. Introduction

The elliptic modular function $j(\tau)$, often referred to as the modular invariant, exhibits many beautiful properties. In particular, each singular modulus, *i.e.,* the value at each imaginary quadratic argument (a CM point), is algebraic and generates a certain abelian extension referred to as the ring class field over the imaginary quadratic field of the argument. On the other hand, the Fourier coefficients of $j(\tau)$ have a mysterious connection with the degrees of irreducible representations of the largest sporadic simple group "Monster"; this surprising connection is known as (a part of) the "moonshine", which was established by I. Frenkel-J. Lepowsky-A. Meurman [4] and R. Borcherds [1].

Since CM points are dense in the complex upper half-plane $\mathfrak{H}$, the domain of definition of the j-function, the function $j(\tau)$ as an analytic (or even continuous) function is completely determined by its values at such points. It would therefore not be unreasonable to expect a formula for the Fourier coefficients of $j(\tau)$ expressed in terms of the singular moduli. The aim of the present paper is to show that there indeed exists such a formula. A different kind of exact formula for the Fourier coefficients of $j(\tau)$ has been known since the work of H. Petersson [5] and H. Rademacher [6]. This formula expresses the coefficients as an infinite series involving a Kloosterman sum and the modified Bessel function of the first kind. It is said to be an analytical formula, whereas our formula is essentially arithmetical.

Thus the idea of explaining the moonshine via complex multiplication theory might not be sheer nonsense.

2. Theorem

The elliptic modular function $j(\tau)$ is invariant under the action of the modular group $SL_2(\mathbb{Z})$; in particular, it has a Fourier series expansion:

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n \quad (q = e^{2\pi i \tau},\ \tau \in \mathfrak{H}),$$

the first few coefficients being $c_1 = 196884, c_2 = 21493760, c_3 = 864299970, \ldots$. All the c_n are positive integers.

After D. Zagier, we define for each natural number $d > 0,\ d \equiv 0, 3 \pmod 4$, an integer $J_1(d)$ by

$$J_1(d) = \sum_{\mathcal{O} \supseteq \mathcal{O}_d} \frac{2}{w_{\mathcal{O}}} \sum_{[\mathfrak{a}_{\mathcal{O}}]} (j(\mathfrak{a}_{\mathcal{O}}) - 744),$$

where the first sum runs over all imaginary quadratic orders $\mathcal{O}$ that contain the order $\mathcal{O}_d$ of discriminant $-d$, $w_{\mathcal{O}}$ is the number of units in $\mathcal{O}$, and the second sum is over a representative of the proper $\mathcal{O}$-ideal class. Note that here $j(\tau)$ is viewed in the standard manner as a function on the equivalence classes of lattices in $\mathbb{C}$. In addition, we set

$$J_1(0) = 2, J_1(-1) = -1 \text{ and } J_1(d) = 0 \text{ for } d < -1 \text{ or } d \equiv 1, 2 \pmod 4.$$

In remark 3) appearing after the following theorem, it is shown that $J_1(d)$ is in fact an integer. Our formula is then given as

Theorem. *For any* $n \geq 1$,

$$c_n = \frac{1}{n} \left\{ \sum_{r \in \mathbb{Z}} J_1(n - r^2) + \sum_{r \geq 1, \text{odd}} ((-1)^n J_1(4n - r^2) - J_1(16n - r^2)) \right\}.$$

Examples.

$$\begin{aligned} c_1 &= 2J_1(0) - J_1(3) - J_1(15) - J_1(7) \\ &= 2 \times 2 - (-248) - (-192513) - (-4119) \\ &= 196884. \\ c_2 &= \frac{1}{2}(J_1(7) + J_1(-1) - J_1(31) - J_1(23) - J_1(7)) \\ &= (J_1(-1) - J_1(31) - J_1(23))/2 \\ &= (-1 - (-39493539) - (-3493982))/2 \\ &= 21493760. \end{aligned}$$

Several remarks are in order:

1) In each sum in the formula, only finitely many terms are not 0.

2) By using relation (3) in the next section, the formula can also be written as

$$c_n = \frac{1}{n}\sum_{r\in\mathbb{Z}}\left\{J_1(n-r^2) - \frac{(-1)^{n+r}}{4}J_1(4n-r^2) + \frac{(-1)^r}{4}J_1(16n-r^2)\right\}. \qquad (1)$$

3) As is well known from the theory of complex multiplication, the sum $\sum_{[\mathfrak{a}_\mathcal{O}]}(j(\mathfrak{a}_\mathcal{O})-744)$ in the definition of $J_1(d)$ is the (absolute) trace of the algebraic integer $j(\mathcal{O})-744$, from which it follows that the summand $\frac{2}{w_\mathcal{O}}\sum_{[\mathfrak{a}_\mathcal{O}]}(j(\mathfrak{a}_\mathcal{O})-744)$ is an integer if $\mathcal{O}\neq\mathcal{O}_3,\ \mathcal{O}_4$. On the other hand, using the well known values $j(\mathcal{O}_3)=0$ and $j(\mathcal{O}_4)=1728$, as well as $w_{\mathcal{O}_3}=6$ and $w_{\mathcal{O}_4}=4$, and the fact that the class numbers of $\mathcal{O}_3$ and $\mathcal{O}_4$ are both 1, we obtain $\frac{2}{w_\mathcal{O}}\sum_{[\mathfrak{a}_\mathcal{O}]}(j(\mathfrak{a}_\mathcal{O})-744) = -248,\ 492$ for $\mathcal{O}=\mathcal{O}_3,\ \mathcal{O}_4$, respectively. Hence $J_1(d)$ is always a rational integer. Values of $J_1(d)$ up to $d=100$ are given in the table at the end of the paper.

4) The values of $J_1(d)$ can be calculated recursively and in an elementary way (without knowing anything about complex multiplication) using

$$J_1(4n-1) = -a_n - \sum_{2\le r\le\sqrt{4n+1}} r^2 J_1(4n-r^2),$$

$$J_1(4n) = -2\sum_{1\le r\le\sqrt{4n+1}} J_1(4n-r^2)$$

for $n\ge 0$, where $a_0=1,\ a_n=240\sum_{d|n}d^3\ (n\ge 1)$, and an empty sum is understood to be 0. This is due to D. Zagier (see the next section).

5) In the language of binary quadratic forms, the definition of $J_1(d)$ reads as follows:

$$J_1(d) = \sum_{[Q]}\frac{2}{|\mathrm{Aut}(Q)|}(j(\alpha_Q)-744),$$

where the sum is over a set of representatives of the $SL_2(\mathbb{Z})$-equivalence classes of integral, not necessarily primitive, positive-definite quadratic forms of discriminant $-d$, $|\mathrm{Aut}(Q)|$ denotes the order of the automorphism group of Q in $SL_2(\mathbb{Z})$, and α_Q is the imaginary quadratic irrationality in $\mathfrak{H}$ that corresponds to Q.

3. Proof

The crucial point in the proof of the theorem is provided by the following result due to Don Zagier.

Theorem (D. Zagier [7]). *The series*

$$g_1(\tau) = \sum_{\substack{d\ge -1\\ d\equiv 0,3(4)}} J_1(d)q^d$$

is a modular form of weight $\frac{3}{2}$ *on* $\Gamma_0(4) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ 4|c\}$*, holomorphic in* $\mathfrak{H}$ *and meromorphic at cusps. Specifically,*

$$g_1(\tau) = -\frac{E_4(4\tau)\theta_1(\tau)}{\eta(4\tau)^6}, \tag{2}$$

where $E_4(\tau) = \sum_{n=0}^{\infty} a_n q^n$ *is the normalized Eisenstein series of weight* 4 *(*a_n *being as in the preceding remark 4)),* $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ *is the Dedekind eta function, and* $\theta_1(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$ *is one of the standard theta series of Jacobi.*

He proved this by showing

$$\sum_{r\in\mathbb{Z}} J_1(4n-r^2) = 0 \quad n \geq 0 \tag{3}$$

and

$$\sum_{r\in\mathbb{Z}} (n-r^2) J_1(4n-r^2) = 2a_n \quad n \geq 0. \tag{4}$$

Since it is easy to check that the coefficients of the expression on the right-hand side of (2) satisfy the same recursions, and since the recursions clearly determine the coefficients uniquely, this proves (2) and hence the theorem. (See the book of Eichler-Zagier [3] for these kinds of recursions and a connection with the theory of Jacobi forms.) The relations (3) and (4) were deduced from a classical formula on the diagonal of the Kronecker modular equation and from a similar formula due to M. Eichler. For details and discussion on the relation between the present discussion and a theorem of R. Borcherds [2], see Zagier's forthcoming paper [7].

By virtue of this theorem, we can unify our formula, or rather the equivalent formula (1), into an identity between modular forms (of weight 2) as

$$\frac{1}{2\pi i}\frac{d}{d\tau} j(\tau) = g_1(\tau)\theta_0(\tau) - \frac{1}{4}((g_1\theta_1)|U_4)(\tau + \frac{1}{2}) + \frac{1}{4}((g_1\theta_1)|U_4^2)(\tau),$$

where $\theta_0(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$, and U_4 is the operator $\sum b_n q^n \mapsto \sum b_{4n} q^n$, which, as well as the translation $\tau \mapsto \tau + \frac{1}{2}$, sends a modular form to a modular form of the same weight (but possibly on a different group). Hence, owing to the finite-dimensionality of the space of modular forms of a given weight and a group which are holomorphic except possible poles of bounded order at cusps, the equality holds if the first several Fourier coefficients coincide, which is indeed the case. Thereby the proof of our theorem is completed.

Incidentally, the relations (3) and (4) give us a formula for quick and elementary calculation of $J_1(d)$, as mentioned in the preceding section; we can also calculate $J_1(d)$ by (2) or by the following formulas:

$$\sum_{d\geq 0, \equiv 0(4)} J_1(d) q^{d/4} = 2\frac{E_4(\tau)}{\theta_0(\tau)\theta_1(\tau)^4},$$

$$\sum_{d\geq -1, \equiv 3(4)} J_1(d) q^{d/4} = -2\frac{E_4(\tau)}{\theta_2(\tau)\theta_1(\tau)^4},$$

where $\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}$ is the other standard theta series of Jacobi.

A more "natural" proof of the theorem is provided by taking into account the action of the Hecke operators. Specifically, an argument like the one used to prove (3) shows that

$$\sum_{r\in\mathbb{Z}} J_2(4n-r^2) = 2nc_n \ (n \geq 0), \tag{6}$$

where, in general, we define

$$J_m(d) = \sum_{\mathcal{O}\supseteq\mathcal{O}_d} \frac{2}{w_{\mathcal{O}}} \sum_{[\mathfrak{a}_{\mathcal{O}}]} ((j-744)|T_m)(\mathfrak{a}_{\mathcal{O}}) \quad (T_m : \text{the Hecke operator of weight } 0)$$

for any $m \geq 1$. The relation (6) is then transformed into our theorem using the relations

$$J_2(d) = J_1(4d) + \left(\frac{-d}{2}\right) J_1(d) + 2J_1(\frac{d}{4}) \tag{7}$$

and (3), where $\left(\frac{-d}{2}\right)$ is Kronecker's symbol and $J_1(\frac{d}{4}) = 0$ if $\frac{d}{4}$ is not an integer. The relation (7) and similar relations for $J_m(d)$ can be interpreted as saying that the Hecke actions on $g_1(\tau)$ and on $j(\tau)$ are compatible, as discussed in Zagier [7].

Table Values of $J_1(d)$ for $-1 \leq d \leq 100$.

d	$J_1(d)$	d	$J_1(d)$	d	$J_1(d)$	d	$J_1(d)$
−1	−1	24	4833456	51	−5541103056	76	784073551152
0	2	27	−12288992	52	6896878512	79	−1339190286960
3	−248	28	16576512	55	−13136687601	80	1597178431536
4	492	31	−39493539	56	16220381536	83	−2691907586232
7	−4119	32	52255768	59	−30197680312	84	3196800943968
8	7256	35	−117966288	60	37017882624	87	−5321761716339
11	−33512	36	153541020	63	−67515206970	88	6294842638512
12	53008	39	−331534572	64	82226601996	91	−10359073015248
15	−192513	40	425691312	67	−147197952744	92	12207820353536
16	287244	43	−884736744	68	178211037024	95	−19874477925452
19	−885480	44	1122626864	71	−313645814923	96	23340149127216
20	1262512	47	−2257837845	72	377674773768	99	−37616060991672
23	−3493982	48	2835861520	75	−654403831496	100	44031499225500

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