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SOME REMARKS ON THE DUGUNDJI EXTENSION THEOREMS

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1. Results that are known or easily proved

Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Let $C(X,Z)$ be the linear space of all continuous mappings from X to Z. A linear transformation $u: C(A,Z)\to C(X,Z)$ is said to be a *Dugundji extender* if u satisfies the following conditions: For each $f\in C(A,Z)$,
(a) $u(f)$ is an extension of f, and
(b) the range of $u(f)$ is contained in the closed convex hull of the range of f.

The study of this area is initiated by Dugundji [2]. He proved that for every closed subspace A of a metrizable space X there exists a Dugundji extender $u: C(A,\mathbb{R})\to C(X,\mathbb{R})$. Michael ([8]) noticed that the Dugundji extender constructed by Dugundji is continous with respect to the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

We shall consider the Dugundji extention theorems on product spaces.

Definition 1.1. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is *D(Z)-embedded* in X if there is a Dugundji extender $u: C(A,Z)\to C(X,Z)$. Furthermore, we say that A is *D-embedded* in X if A is D(Z)-embedded in X for every locally convex linear topological space Z.

Definition 1.2. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is $\pi_{D(Z)}$*-embedded* in X if for every space Y there is a Dugundji extender $u: C(A\times Y,Z)\to C(X\times Y,Z)$. Furthermore, A is said to be π_D*-embedded* in X if A is $\pi_{D(Z)}$-embedded in X for every locally convex linear topological space Z.

Definition 1.3. Let X be a space, A a closed subspace of X and Z a locally convex linear topological space. Then we say that A is *continuously* $\pi_{D(Z)}$*-embedded (resp.* π_D*-embedded)* in X if we can choose the Dugundji extender u as is continuous with respect the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

For a space X and a locally convex linear topological space Z we denote $C_u(X,Z)$ the linear topological space of all continuous mappings from X to Z with the uniform convergence topology, i.e., the sets of the form $V(f)=\{g\in C(X,Z): g(x)-f(x)\in V\}$, where V is a neighborhood of the origin of Z consists a basic neighborhoods of $f\in C_u(X,Z)$. Let $C_{co}(X,Z)$ be the linear topological space of all continuous mappings from X to Z with the compact-open topology.

A mapping $f : X \to Y$ is called a *Z-map* if $f(Z)$ is closed for every zero-set Z of X. Then we have the following.

Theorem 1.1. *Let X and Y be spaces and A a D-embeded subspace of X. Let $p_A : A \times Y \to A$ and $p_Y : A \times Y \to Y$ be the projections. If either of the following conditions is satisfied, then $A \times Y$ is D-embedded in $X \times Y$:*

(1) p_A is a Z-map.

(2) p_Y is a Z-map and there is a continuous Dugundji extender $u : C_u(A, Z) \to C_u(X, Z)$ for every locally convex linear topological space Z.

Theorem 1.2. *([4]) Let X and Y be spaces, A a closed subspace of X and Z a locally convex linear topological space. Suppose that X is locally compact or $X \times Y$ is a k-space. If there exists a continous Dugundji extender $u : C_{co}(A, Z) \to C_{co}(X, Z)$, then $A \times Y$ is D(Z)-embedded in $X \times Y$.*

Remark. In Theorem 1.2, the continuoity of the Dugundji extender u can not be dropped. In fact, let $X = [0, \omega_1] \times [0, \omega] - \{(\omega_1, \omega)\}$ and $A = [0, \omega_1) \times \{\omega\}$ be the closed subspace of X. It is clear that A is D($\mathbb{R}$)-embedded in X. Let $Y = [0, \omega_1]$ be the space with the following topology: For each $y < \omega_1$ y is an isolated point of Y and ω_1 has a neighborhood base of the usual order topology. It follows that $A \times Y$ is not C-embedded in $X \times Y$, and hence $A \times Y$ is not D($\mathbb{R}$)-embedded in $X \times Y$.

In [9] and [10], Stares proved that every closed subspace of spaces satisfying the decreasing (G) is π-embedded and every such space has the Dugundji extension property. Before stating the theorem, we recall the definition of spaces satisfying the decreasing (G) from [1]. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ be a collection of subsets of X, where $\mathcal{W}(x) = \{W(x, n) : n \in \omega\}$ such that $x \in W(x, n)$ for every $x \in X$ and $n \in \omega$. Then we say that $\mathcal{W}$ is decresing if $W(x, n+1) \subset W(x, n)$ for every $n \in \omega$, and $\mathcal{W}$ satisfies (G) if

(G) for each $x \in X$ and each open set U with $x \in U$ there is an open neighborhood $V = V(x, U)$ of x such that $y \in V$ implies $x \in W(y, s) \subset U$ for some $s \in \omega$.

We say that a space X satisfies the decreasing (G) if there is a collection $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfying decreasing (G). We notice that every stratifiable space satisfies the decreasing (G) ([10]). Now, we have the following.

Theorem 1.3. *Let X be a regular space satisfying the decreasing (G) and A a closed subspace of X. Then A is continuously π_D-embedded in X.*

2. Results about GO-spaces

In [7], we proved that for a perfectly normal GO-space X with $E(X)$ is σ-discrete in X, a closed subspace A of X and Z a locally convex linear topological space Z, there is a Dugundji extender u from $C(A, Z)$ to $C(X, Z)$, where $E(X) = \{x \in X : (\leftarrow, x]$ or $[x, \to)$ is open in X $\}$. We extend the theorem above as follows.

Theorem 2.1. *Let X be a perfectly normal GO-space such that $E(X)$ is σ-discrete in X. Then every closed subspace A of X is continuously π_D-embedded in X.*

Proof. Let A be a closed subspace of X. Then $X - A$ is the union of a disjoint family $\mathcal{U}$ of convex components of $X - A$. Since X is perfectly normal, it follows from [3, Theorem 2.4.5] that $\mathcal{U}$ is σ-discrete in X. Let $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}_n$, where $\mathcal{U}_n$ is discrete in X. Similarly, let Int $A = \bigcup \mathcal{V}$, where $\mathcal{V} = \cup_{n=1}^{\infty} \mathcal{V}_n$ is a disjoint and σ-discrete family of convex components of Int A. For each $U \in \mathcal{U}$ we choose $x(U) \in U$. We put $M_{\mathcal{U}} = \{x(U) : U \in \mathcal{U}\}$. For each convex open set C in X, we put

- $l(C) = \max\{a \in A : a < x \text{ for all } x \in C\}$, and
- $r(C) = \min\{a \in A : a > x \text{ for all } x \in C\}$,

if the righthand of the above equations exist.

Then for each n, we put $\mathcal{U}_n^{\ell} = \{U \in \mathcal{U}_n : l(U) \text{ exists}\}$ and $\mathcal{U}_n^{r} = \{U \in \mathcal{U}_n : r(U) \text{ exists}\}$. Similarly, we define $\mathcal{V}_n^{\ell}$ and $\mathcal{V}_n^{r}$. Furtheremore, we put

- $L_n = \{l(U) : U \in \mathcal{U}_n^{\ell}\}$,
- $R_n = \{r(U) : U \in \mathcal{U}_n^{r}\}$,
- $L'_n = \{l(V) : V \in \mathcal{V}_n^{\ell}\}$, and
- $R'_n = \{r(V) : V \in \mathcal{V}_n^{r}\}$.

It is easy to see that all of L_n, R_n, L'_n and R'_n are closed discrete in X. Let $L = \cup_{n=1}^{\infty} L_n$, $R = \cup_{n=1}^{\infty} R_n$, $L' = \cup_{n=1}^{\infty} L'_n$ and $R' = \cup_{n=1}^{\infty} R'_n$. Furthermore, we put

$$B = \{a \in A - (L \cup R) : a \in \overline{\bigcup \mathcal{U}^-(a)}^X \cup \overline{\bigcup \mathcal{U}^+(a)}^X\},$$

where $\mathcal{U}^-(a) = \{U \in \mathcal{U} : x(U) < a\}$ and $\mathcal{U}^+(a) = \{U \in \mathcal{U} : x(U) > a\}$. Let

$$M = M_{\mathcal{U}} \cup L \cup R \cup L' \cup R' \cup (E(X) \cap A) \cup B.$$

Then M is a GO-space and $D = M - B$ is σ-discrete in M. Since $E(M) \subset D$ and D is dense in M, it follows from [3, Theorem 3.1] that M is metrizable. Then there exists a compatible metric ρ on M bounded by 1.

We shall define a mapping $\varphi : X \to 2^A$. Let $x \in X$. If $x \in A$, then we put $\varphi(x) = \{x\}$. Let $x \in X - A$. Then there is $U \in \mathcal{U}_n$ such that $x \in U$.

Case 1. Suppose that $U \in \mathcal{U}_n^{\ell} \cap \mathcal{U}_n^{r}$. If $U = \{x\}$, we put $\varphi(x) = \{\ell(U)\}$. If U contains at least two points, we choose points $s(U)$ and $t(U)$ of U such that $s(U) < t(U)$. We put

$$\varphi(x) = \begin{cases} \{\ell(U)\}, & \text{if } x < s(U), \\ \{\ell(U), r(U)\}, & \text{if } s(U) \le x \le t(U), \\ \{r(U)\}, & \text{if } x > t(U). \end{cases}$$

Case 2. If $U \in \mathcal{U}_n^{\ell}$ and $U \notin \mathcal{U}_n^{r}$, then we put $\varphi(x) = \{\ell(U)\}$.

Case 3. If $U \notin \mathcal{U}_n^{\ell}$ and $U \in \mathcal{U}_n^{r}$, then we put $\varphi(x) = \{r(U)\}$.

Case 4. Finally, we suppose that $U \notin \mathcal{U}_n^{\ell} \cup \mathcal{U}_n^{r}$. Then we put $\varphi(x) = \{a(U)\}$, where $a(U)$ is defined in the proof of Theorem 2.1 in [7]. Then we can see that $\varphi : X \to 2^A$ is upper semicontinuous.

To define an extender $u : C(A \times Y, Z) \to C(X \times Y, Z)$, let $f \in C(A \times Y, Z)$. First, for each n and each $U \in \mathcal{U}_n$ we shall define a continuous function $f_U : U \times Y \to Z$. We consider the following four cases.

Case 1. Suppose that $U \in \mathcal{U}_n^\ell \cap \mathcal{U}_n^r$. If $U = \{x\}$, we define $f_U(x,y) = f(l(U),y)$ for each $y \in Y$. If U contains at least two points, we define

$$f_U(x,y) = \begin{cases} f(l(U),y), & \text{if } x < s(U), \\ (1-\psi_U)(x) \cdot f(l(U),y) + \psi_U(x) \cdot f(r(U),y), & \text{if } s(U) \leq x \leq t(U), \\ f(r(U),y), & \text{if } x > t(U), \end{cases}$$

for each $(x,y) \in U \times Y$, where $\psi_U : X \to I$ is a continuous mapping such that $(\leftarrow, l(U)] \subset \psi_U^{-1}(0)$ and $[r(U), \to) \subset \psi_U^{-1}(1)$.

Case 2. If $U \in \mathcal{U}_n^\ell$ and $U \notin \mathcal{U}_n^r$, then we put $f_U(x,y) = f(l(U),y)$ for each $(x,y) \in U \times Y$.

Case 3. If $U \notin \mathcal{U}_n^\ell$ and $U \in \mathcal{U}_n^r$, then we put $f_U(x,y) = f(r(U),y)$ for each $(x,y) \in U \times Y$.

Case 4. If $U \notin \mathcal{U}_n^\ell \cup \mathcal{U}_n^r$, $f_U(x,y) = f(a(U),y)$ for each $(x,y) \in U \times Y$.

We define a function $u(f) : X \times Y \to Z$ as follows:

$$u(f)(x,y) = \begin{cases} f(x,y), & \text{if } x \in A, \\ f_U(x,y), & \text{if } x \in U \text{ for some } U \in \mathcal{U}. \end{cases}$$

In a similar fashion to [7], we can see that $u(f)$ is a continuous extension of f and the range of $u(f)$ is contained in the closed convex hull of the range of f.

By use of the upper semicontinuity of φ, we can show that the extender u above is continuous with respect to the point convergence topology, compact-open topology and uniform convergence topology (cf. [8]).

In a similar fashion as the proof of Theorem 2.1, we obtain the following (in fact, the proof of this case is more simple than Theorem 2.1).

Theorem 2.2. *Let X be a GO-space, A a closed subspace of X and $X - A = \bigcup \mathcal{U}$, where $\mathcal{U}$ is a disjoint family of convex components of $X - A$. If $\mathcal{U}' = \{U \in \mathcal{U} : U$ has neither $l(U)$ nor $r(U)\}$ is discrete in X, then A is continuously π_D-embedded in X.*

Corollary 2.1. *Let X be a locally compact GO-space. Then every closed subspace A of X is continuously π_D-embedded in X.*

Corollary 2.2. *Every closed subspace of the Sorgenfrey line $\mathbb{S}$ is continuously π_D-embedded.*

Corollary 2.3. *Let X be a GO-space such that the underlining ordered set is well-ordered. Then every closed subspace A of X is continuously π_D-embedded.*

Now, we have the following corollaries.

Corollary 2.4. *Let $X_i (i = 1, 2, \cdots n)$ be perfectly normal GO-spaces with $E(X_i)$ σ-discrete in X_i and A_i are closed subsets in X_i. Then, $\prod_{i=1}^n A_i$ is D-embedded in $\prod_{i=1}^n X_i$.*

Corollary 2.5. *Let κ be an ordinal and $A_i (i = 1, 2, \cdots, n)$ are closed subsets of κ. Then $\prod_{i=1}^n A_i$ is D-embedded in κ^n.*

Remark. In [5], Heath and Lutzer proved that for every closed subspace A of a GO-space X there is a simultaneous extender $u : C^*(A) \to C^*(X)$. However, Heath, Lutzer and Zenor [6] proved that there is no Dugundji extender $u : C^*(\mathbb{Q}) \to C^*(\mathbb{M})$ which is continuous when both function spaces are equipped with the compact-open topology nor the pointwise convergence topology, where $\mathbb{M}$ is the Michael line and $\mathbb{Q}$ is the subspace of $\mathbb{M}$ consisting of all rationals.

References

1. P. J. Collins and A. W. Roscoe, Criteria for metrisability, Proc. Amer. Math. Soc. 90 (1984), 631–640.
2. J. Dugundji, An extension of Tietze's theorem, Pacific J. Math. 1 (1951), 353–367.
3. M. J. Faber, Metrizability in generalized ordered spaces, Math. Center Tracts 53 (1974), 1–120.
4. S. Fujii, π-embedding and Dugundji extension theorem, Master Thesis, University of Tsukuba (1984).
5. R. W. Heath and D. J. Lutzer, Dugundji extension theorems for linearly ordered spaces, Pacific J. Math. 55 (1974), 419–425.
6. R. W. Heath, D. J. Lutzer and P. L. Zenor, On continuous extenders, in : H. M. Starrakas and K. R. Allen, eds., Studies in Topology (Academic Press, New York, 1975), 203–213.
7. Y. Hattori, π-embeddings and Dugundji extension theorems for generalized ordered spaces, preprint.
8. E. A. Michael, Some extension theorems for continuous functions, Pacific J. Math. 3 (1953), 789–806.
9. I. S. Stares, Extension of functions and generalized metric spaces, D. Phil. Thesis, Oxford University, Oxford (1994).
10. I. S. Stares, Concerning the Dugundji extension property, Topology Appl. 63 (1995), 165–172.