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MANIFOLDS WITH FINITE FUNDAMENTAL GROUP AS CODIMENSION-2 FIBRATORS

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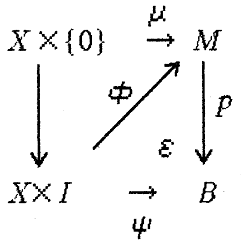
1. Introduction

D. Coram and P. Duvall in [CD1] introduced approximate fibrations as a generalization of both Hurewicz fibrations and cell-like maps. A proper map $p: M \to B$ between locally compact ANRs is called an *approximate fibration* if its map has the following approximate homotopy lifting property : Given an open cover ε of B, an arbitrary space X, and two maps $\mu : X \to M$ and $\Psi : X \times I \to B$ such that $\Psi_0 = p\,\mu$, there exists a map $\Phi : X \times I \to M$ such that $\Phi_0 = \mu$ and $p\,\Phi$ is ε-close to Ψ.



The following results of D. Coram and P. Duvall motived our work.

Fact. 1.1 [CD1]. *Let $p: M \to B$ be an approximate fibration. Then*
(i) *a path $\omega: I \to B$ induces a shape equivalence from $p^{-1}(\omega(0))$ to $p^{-1}(\omega(1))$:*
(ii) *each fiber is a pointed fundamental ANR (a pointed FANR).*

An FANR is the shape analog of an ANR. It is known that a space S is a pointed FANR if and only if S has the shape of a CW complex. The term is defined along with the term " shape " in [MS]. Our purpose in this paper is to examine a converse to the part (ii), that is, given a compact ANR N and an upper semicontinuous decomposition G of a locally compact ANR M satisfying that each element of G is shape equivalent to N, is the decomposition map $p: M \to M/G$ an approximate fibration ? Because of the following fact, we would like to focus attention on that both M and N are manifolds.

Fact. 1.2 [DH, Theorem 3.1.]. *Let $p: M \to B$ be an approximate fibration of a connected m-manifold (without boundary) onto an ANR B. Then B is a k-dimensional generalized manifold; moreover, if M is orientable, then the fiber of p has the shape of a Poincare duality space of formal dimension $m - k$.*

A closed connected manifold N^n is called a *codimension-2 fibrator* (respectively, *codimension-2 orientable fibrator*) if whenever G is an upper semicontinuous decomposition of an arbitrary (respectively, orientable) $(n+2)$-manifold M satisfying that each element of G is shape equivalent to N, then the decomposition map $p: M \to B$ is an approximate fibration.

Main Question 1.3. *What is a manifold which is a codimension-2 fibrator or codimension-2* *orientable fibrator*.

In [D1], Daverman showed that all simply connected manifolds, closed surfaces with nonzero Euler characteristic, and real projective n-spaces (n > 1) are codimension-2 fibrators. Nonfibrators in codimension-2 include all closed manifolds admitting a fixed point free cyclic action having a orbit space homotopy equivalent to itself. Therefore S^1, the torus and the Klein bottle are nonfibrators. For example, there exists a map p from S^3 to S^2 that every fiber of p has the shape of S^1 and p is not an approximate fibration. See [CD3].

Since all simply connected manifolds and real projective n-spaces ($n > 1$) are codimension-2 fibrators, it is quite natural to ask the following :

Question 1.4[D1, Question 6.3]. *Is every closed n-manifold with finite fundamental group a codimension-2 fibrator ?*

Question 1.5[D3, Question 8.2]. *Are Lens spaces codimension-2 fibrators ?*

2. Closed hopfian manifolds and continuity sets

Throughout this paper, all spaces are locally compact, metrizable ANRs, and all manifolds are finite dimensional, connected, and boundaryless. Whenever we allow boundary, the object will be called a manifold with boundary. Homology is computed with integer coefficients unless another coefficient module is mentioned.

By the *degree* of a map between closed n-manifolds, we mean more precisely the absolute degree ; namely the nonnegative integer determining the induced homomorphism's effect on nth (integral or Z_2) homology, up to sign.

Recall that a group H is *hopfian* if every epimorphism $H \to H$ is an isomorphism and that a manifold M is *aspherical* if $\pi_i(M)$ is the trivial group for all i > 1. Call a closed manifold N *hopfian* if it is orientable and every degree one map $N \to N$ is a homotopy equivalence.

Theorem 2.1[D4. Theorem 2.2]. *A closed, orientable n-manifold N is a hopfian manifold if any of the following conditions holds*:

(1) $n \leqq 4$ *and* $\pi_1(N)$ *is hopfian ;*

(2) $\pi_1(N)$ *is hopfian and its integral group ring,* $Z\pi_1(N)$, *is Noetherian ;*

(3) $\pi_1(N)$ *contains a nilpotent subgroup of finite index ; or*

(4) $\pi_i(N)$ *is trivial,* $1 < i < n - 1$, *and* $\pi_1(N)$ *is hopfian.*

All closed, orientable n-maniflods N with $\pi_1(N)$ finite and asperical n-manifold N' with $\pi_1(N')$ hopfian are hopfian manifolds.

Given a orientable closed n-manifold N^n, an upper semicontinuous decomposition G of a orientable manifold M^m is *N-like* if each element of G is shape equivalent to N. For simplicity or familiarity, we shall assume that each element of an N^n-like upper semicontinuous decomposition G is an ANR having the homotopy type of N^n ; experts can easily modify the proofs to treat the more general situation. See [D1].

Let N^n be a closed n-manifold, M^{n+2} be an $(n+2)$-manifold and G be an N^n-like upper semicontinuous decomposition of M. For each $g \in G$ there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$. We will define the *continuity set* $C \subset M/G$ of the decomposition map $p : M \to$

M/G:

$C = \{ p(g) \in M/G$: there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$ such that $R_g \mid g' : g' \to g$ is a degree one map for all $g' \in G$ with $g' \subset U_g \}$.

D. Coram and P.Duvall in [CD3] have shown that C is a dense, open subset of M/G. The following result comes from [CD2]. See [D4, Theorem 2.1].

Proposition 2.2. *Suppose N^n is a closed orientable hopfian n-manifold and M^{n+2} is an (n+2)-manifold. Then the N-like decomposition map $p : M \to B$ is an approximate fibration over its continuity set C.*

The following results of the decomposition spaces is found in [D1].

Theorem 2.3. *Let N^n be a closed n-manifold, M^{n+2} be an (n+2)-manifold and G be an N^n-like upper semicontinuous decomposition of M. If both N and M are orientable, then the decomposition space $B = M/G$ is a 2- manifold and $D = B \setminus C$ is locally finite in M/G, where C represents the continuity set of the decomposition map $p : M \to B$. If either N or M is nonorientable, B is a 2- manifold with boundary (possibly empty) and $D' = (\mathrm{Int}\ B) \setminus C'$ is locally finite in B, where C' represents the mod 2 continuity set.*

3. Closed hopfian manifolds with finite first homology group

The setting throughout this section is that N^n is a closed hopfian n-manifold, G is an N-like upper semicontinuous decomposition of an orientable $(n+2)$-manifold M^{n+2}, the decomposition space M/G is denoted by B and the induced quotient map is denoted by $p : M \to B$.

Proposition 3.1 [Im2, Lemma 3.2]. *Let g be an element of G satisfying $g \neq g_0$ and $g \subset U_{g_0}$. If the restriction $R_{g_0} \mid g : g \to g_0$ of $R_{g_0} : U_{g_0} \to g_0$ induces an H_1-isomorphism $(R_{g_0} \mid g)_* : H_1(g) \to H_1(g_0)$, then $R_{g_0} \mid g' : g' \to g_0$ is a degree one map for all $g' \in G$ with $g' \subset U_{g_0}$.*

It is obtained from the above propositions and Theorem 2.3 that every closed hopfian manifold N satisfying $H_1(N) = 0$ is a codimension-2 orientable fibrator. One may notice that every simply connected closed manifold is a codimension-2 fibrator. For further details, see Theorem 2.3 and Corollary 2.4 of [D1]. The proof of Theorem 2.5 in [C] leads to the following conclusion.

Theorem 3.2. *Every closed hopfian manifold N^n satisfying $H_1(N)$ is finite is a codimension-2 orientable fibrator.*

Corollary 3.3 [cf. D2]. *Let N^n be a closed orientable n-manifold. We consider the following two conditions:*

(a) *$\pi_1(N)$ is finite;*

(b) *$\pi_1(N)$ is hopfian, $H_1(N)$ is finite and N is aspherical.*

Then N is a codimension-2 orientable fibrator.

Let S^n, P^n and $L(p,q)$ denote the n-sphere, real projective n-space and Lens space of type (p,q), respectively.

Corollary 3.4. *For m, n > 1, any finite product of* S^m, P^{2n-1} *and* $L(p,q)$ *is a codimension-2 orientable fibrator.*

4. Products of codimension-2 orientable fibrators

In [Im1], Im proved that any finite product of closed, orientable surfaces with negative Euler characteristic is a codimension-2 orientable fibrator. It is quite natural to ask the following :

Question 4.1. Is a product of codimension-2 orientable fibrators a codimension-2 orientable fibrator ?

In this section, we will state that a product of most known codimension-2 orientable fibrators is a codimension-2 orientable fibrator, that is, a product of any closed manifold with finite fundamental group and any closed orientable surface F with nonzero Euler characteristic is a codimension-2 orientable fibrator.

Definition 4.2. A group G is said to be *residually finite* if for each $g \in G \setminus \{1\}$, there exist a finite group H and a homomorphism $F : G \to H$ with $F(g) \neq 1$.

Hempel [H2] showed that for each surface F, $\pi_1(F)$ is residually finite.

Lemma 4.3 [H2, Lemma 15. 17]. *If G is a finitely generated, residually finite group, then G is hopfian.*

It is clear that every finite group is residually finite and hopfian.

Corollary 4.4. *Any finite product of finitely generated, residually finite groups is hopfian.*

Lemma 4.5. *If* N^n *is a closed orientable m-manifold with finite fundamental group and if F be a closed orientable surface with negative Euler characteristic* $\chi(F)$, $N \times F$ *is a hopfian* $(n+2)$*-manifold.*

As in the proof of [Im 2, Theorem 3.4], the following theorem can be proved.

Theorem 4.6. *Suppose a closed orientable n-manifold* N^n *satisfies that* $\pi_1(N)$ *is finite and F be a closed orientable surface with negative Euler characteristic. Then an (n+2)-manifold* $N \times F$ *is a codimension-2 orientable fibrator.*

5. Applications

When $p : M \to B$ is an approximate fibration, for then, just as with Hurewicz fibrations, there is a homotopy exact sequence

$$\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}(b)) \to \pi_i(M) \to \pi_i(B) \to \cdots$$

Theorem 5.1. *Let N^n be a closed hopfian n-manifold with finite first homology group, G be an N-like upper semicontinuous decomposition of an orientable $(n+2)$-manifold M^{n+2} and $p : M \to B = M/G$ be the decomposition map. If $\pi_1(M) = 1 = \pi_2(M)$, the inclusion map $p^{-1}(b) \to M$ is a homotopy equivalence for each $b \in B$.*

Proof. From Theorem 3.2, $p : M \to B$ is an approximate fibration. Thus there is a homotopy exact sequence

$$\cdots \to \pi_2(M) \to \pi_2(B) \to \pi_1(p^{-1}(b)) \to \pi_1(M) \to \pi_1(B) \to 1.$$

It follows from $\pi_1(M) = 1 = \pi_2(M)$ that $\pi_1(B) = 1$ and $\pi_2(B) \approx \pi_1(p^{-1}(b))$. We see $H_2(B) \approx \pi_2(B) \approx \pi_1(p^{-1}(b)) \approx H_1(p^{-1}(b))$, therefore, $H_2(B) \approx \pi_2(B)$ is a finite abelian group. Since B is a 2-manifold, $H_2(B)$ is either 0 or Z, thus $H_2(B) \approx \pi_2(B) \approx 0$. Thus B is homeomorphic to E^2. We recognize from the above homotopy exact sequence that the inclusion map $p^{-1}(b) \to M$ is a homotopy equivalence. □

Corollary 5.2. *Let N^n be a closed hopfian n-manifold with finite first homology group and an $(n+2)$-manifold M^{n+2} satisfying $\pi_1(M) = 1 = \pi_2(M)$. If N is not homotopy equivalent to M, then there is no N-like upper semicontinuous decomposition of M. Particularly, there is no N-like upper semicontinuous decomposition of E^{n+2} or S^{n+2}.*

We call a closed n-manifold N a *codimension k PL fibrator* if, for all orientable PL $(n+k)$-manifold M, B is a polyhedron, and PL maps $p : M \to B$ satisfying that each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N, p is an approximate fibration. The following result stems from [D5, Theorem 2.10].

Theorem 5.3. *If the closed orientable n-manifold N has a closed (k - 1)-connected universal cover, then N is a codimension-k PL fibrator.*

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