KURENAI

Kyoto University Research Information Repository

KYOTO UNIVERSITY

Title	Semidefinite Programming Relaxation for Nonconvex Quadratic Programs(Discrete and Continuous Structures in Optimization)
Author(s)	Fujie, Tetsuya; Kojima, Masakazu
Citation	(1996), 945: 57-67
Issue Date	1996-04
URL	http://hdl.handle.net/2433/60225
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

57

Semidefinite Programming Relaxation for Nonconvex Quadratic Programs

東京工業大学　藤江 哲也　(Tetsuya Fujie)

東京工業大学　小島 政和　(Masakazu Kojima)

Abstract.   Any quadratic inequality in the n-dimensional Euclidean space can be relaxed into a linear matrix inequality in $(1+n) \times (1+n)$ symmetric matrices. Based on this principle, we extend the Lovász-Schrijver SDP (semidefinite programming) relaxation developed for a 0-1 integer program to a general nonconvex QP (quadratic program), and present some fundamental characterization of the SDP relaxation including its equivalence to a relaxation using convex-quadratic valid inequalities for the feasible region of the QP.

Key words.   Semidefinite Program, Relaxation Method, Interior-Point Method, Linear Matrix Inequality, Nonconvex Quadratic Program

1   Introduction.

We use the symbols $\mathcal{S}(m)$ for the set of $m \times m$ symmetric matrices, and $\mathcal{S}(m)_+$ (or $\mathcal{S}(m)_{++}$) for the cone consisting of $m \times m$ symmetric positive semidefinite (or positive definite, respectively) matrices. We are concerned with a canonical form QP:

$$\text{Minimize } \boldsymbol{c}^T \boldsymbol{y} \text{ subject to } \boldsymbol{y} \in \mathcal{F}. \tag{1}$$

Here

$$\left.\begin{aligned}
\mathcal{F} &\equiv \left\{ \boldsymbol{y} \in R^{1+n} : y_0 = 1,\ \boldsymbol{y}^T \boldsymbol{P}_k \boldsymbol{y} \le 0\ (k=1,2,\dots,m) \right\}, \\
\boldsymbol{c} &\equiv \begin{pmatrix} \gamma \\ \boldsymbol{d} \end{pmatrix} \in R^{1+n},\ \boldsymbol{y} \equiv \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \in R^{1+n}, \\
\boldsymbol{P}_k &\equiv \begin{pmatrix} \pi_k & \boldsymbol{q}_k^T/2 \\ \boldsymbol{q}_k/2 & \boldsymbol{Q}_k \end{pmatrix} \in \mathcal{S}(1+n)\ (k=1,2,\dots,m), \\
\pi_k &\in R,\ \boldsymbol{q}_k \in R^n,\ \boldsymbol{Q}_k \in \mathcal{S}(n)\ (k=1,2,\dots,m).
\end{aligned}\right\} \tag{2}$$

Note that the feasible region $\mathcal{F}$ is contained in the n-dimensional hyperplane $H \equiv \{\boldsymbol{y} \in R^{1+n} : y_0 = 1\}$, and that the function $H \ni \boldsymbol{y} \to \boldsymbol{y}^T \boldsymbol{P}_k \boldsymbol{y} \in R$ involved in the inequality constraint is convex (or linear) if and only if $\boldsymbol{Q}_k \in \mathcal{S}(n)_+$ (or $\boldsymbol{Q}_k = \boldsymbol{O}$, respectively). $\boldsymbol{Q}_k$ can be indefinite, so that the feasible region $\mathcal{F}$ of the QP (1) is a nonconvex subset of the hyperplane H in general. The canonical form QP (1) covers various mathematical programs such as 0-1 IPs, general nonconvex QPs and linear complementarity problems.

SDP(Semidefinite Program) has been considered as a powerful tool for relaxation of many combinatorial optimization problems, since it yields a tight bound [1, 2, 5, 6, 8, 13, 17] and it is efficiently solvable by interior methods [1, 2, 3, 4, 7, 9, 12, 16]. SDP relaxation is originally proposed by Lovász[10] for stable set problems.

Among many literatures related on SDP relaxation, this paper was motivated by Alizadeh [1] where an elementary outline of the SDP relaxation method proposed by Lovász-Schrijver [11] for 0-1 IPs (integer programs) was presented. The aim of this paper is to present a general method for constructing an SDP which serves as a relaxation of the QP (1) and some fundamental properties on the SDP relaxation. Our SDP relaxation method may be regarded as a straightforward extension of the Lovász-Schrijver SDP relaxation method [11] for 0-1 IPs to the QP (1). It is also characterized in terms of

- a dual of Shor's relaxation method [14] (see also [15]) for general nonconvex QPs (this will be discussed in Section 5), and

- a relaxation using *convex-quadratic valid inequalities* for the feasible region $\mathcal{F}$.

Let

$$\boldsymbol{P} \equiv \begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{S}(1+n), \ \pi \in R, \ \boldsymbol{q} \in R^n, \ \boldsymbol{Q} \in \mathcal{S}(n).$$

We say that an inequality $\boldsymbol{y}^T \boldsymbol{P} \boldsymbol{y} \leq 0$ is a convex-quadratic (or linear) valid inequality for $\mathcal{F}$ if

$$\boldsymbol{Q} \in \mathcal{S}(n)_+ \ (\text{or } \boldsymbol{Q} = \boldsymbol{O}, \text{ respectively}) \ \text{ and } \ \boldsymbol{y}^T \boldsymbol{P} \boldsymbol{y} \leq 0 \ \text{ for every } \ \boldsymbol{y} \in \mathcal{F}.$$

Then co $\mathcal{F}$, the convex hull of $\mathcal{F}$ is completely determined by all the convex-quadratic valid inequalities for $\mathcal{F}$;

$$\text{co } \mathcal{F} = \bigcap_{\boldsymbol{P} \in \mathcal{V}} \{\boldsymbol{y} \in R^{1+n} \ : \ y_0 = 1, \ \boldsymbol{y}^T \boldsymbol{P} \boldsymbol{y} \leq 0\},$$

where $\mathcal{V}$ denotes the set of all matrices $\boldsymbol{P} \in \mathcal{S}(1+n)$ that induce convex-quadratic valid inequalities for $\mathcal{F}$. (The identity above is well-known when $\mathcal{V}$ is the set of all matrices $\boldsymbol{P} \in \mathcal{S}(1+n)$ that induce linear valid inequalities for $\mathcal{F}$). The discussion above leads us to a relaxation of the QP (1) using all convex-quadratic valid inequalities for $\mathcal{F}$ that we can generate as a nonnegative combination of the quadratic inequalities of the QP (1):

$$\text{Minimize } \boldsymbol{c}^T \boldsymbol{y} \text{ subject to } \boldsymbol{y} \in \widetilde{\mathcal{F}}, \tag{3}$$

where

$$\widetilde{T} \equiv \{t \in R^m \ : \ t \geq 0, \ \sum_{k=1}^{m} t_k \mathbf{Q}_k \in \mathcal{S}(n)_+\},$$

$$f_t(\mathbf{y}) \equiv \mathbf{y}^T \left(\sum_{k=1}^{m} t_k \mathbf{P}_k\right) \mathbf{y} \ \text{ for every } \mathbf{y} \in R^{1+n} \ (t \in \widetilde{T}),$$

$$\widetilde{\mathcal{F}} \equiv \left\{\mathbf{y} \in R^{1+n} \ : \ y_0 = 1 \ \text{ and } f_t(\mathbf{y}) \leq 0 \ (t \in \widetilde{T})\right\},$$

$$= \left\{\mathbf{y} \in R^{1+n} \ : \ \begin{array}{l} y_0 = 1 \ \text{ and } \mathbf{y}^T \left(\sum_{k=1}^{m} t_k \mathbf{P}_k\right) \mathbf{y} \leq 0 \\ \text{for every } t \geq 0 \ \text{ such that } \sum_{k=1}^{m} t_k \mathbf{Q}_k \in \mathcal{S}(n)_+ \end{array}\right\}. \qquad (4)$$

Although the derivation of the relaxation (3) of the QP (1) is simple and straightforward, it seems difficult to implement the relaxation (3) on computer because the set $\widetilde{T}$ over which the index vector t of the convex-quadratic inequality $f_t(\mathbf{y}) \leq 0$ changes is a continuum, non-polyhedral and convex subset of R^m in general. Under a moderate assumption (Condition 2.2), the main theorem (Theorem 2.3) establishes the equivalence between the SDP relaxation and the relaxation (3) using convex-quadratic valid inequalities. Thus the SDP relaxation may be regarded as an implementable version of the relaxation (3).

We give the main theorem without proof in Section 2. Section 3 states a basic principle which makes it possible for us to extend the Lovász-Schrijver SDP relaxation method for IPs to nonconvex QPs. In Section 4, we present Shor's relaxation method [14], and show some duality relation among the SDP relaxation, Shor's relaxation and the relaxation (3) using convex-quadratic valid inequalities. The discussions in Sections 3 and 4 are not only necessary to prove the main theorem, but also helpful to the readers' deep understanding of the SDP relaxation. Section 5 is devoted to a proof of the main theorem.

2 Main Theorem.

For every $\mathbf{A} \in \mathcal{S}(m)$ and $\mathbf{B} \in \mathcal{S}(m)$, $\mathbf{A} \bullet \mathbf{B}$ denotes their inner product, i.e., $\mathbf{A} \bullet \mathbf{B} \equiv \text{Tr } \mathbf{A}^T \mathbf{B}$ (the trace of $\mathbf{A}^T \mathbf{B}$). It should be noted that any linear function $g : \mathcal{S}(m) \to R$ can be written as $g(\mathbf{Y}) = \mathbf{A} \bullet \mathbf{Y}$ for some $\mathbf{A} \in \mathcal{S}(m)$. Define

$$\mathbf{C} \equiv \begin{pmatrix} \gamma & \mathbf{d}^T/2 \\ \mathbf{d}/2 & \mathbf{O} \end{pmatrix} \in \mathcal{S}(1+n),$$

$$\widehat{\mathcal{G}} \equiv \{\mathbf{Y} \in \mathcal{S}(1+n)_+ \ : \ Y_{00} = 1, \ \mathbf{P}_k \bullet \mathbf{Y} \leq 0 \ (k = 1, 2, \ldots, m)\},$$

$$\widehat{\mathcal{F}} \equiv \{\mathbf{Y} \mathbf{e}_0 \ : \ \mathbf{Y} \in \widehat{\mathcal{G}}\}, \ \mathbf{e}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \in R^{1+n}. \qquad (5)$$

Obviously, $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{F}}$ are convex subsets of $\mathcal{S}(1+n)$ and R^{1+n}, respectively. We now introduce the SDP which will serve as a relaxation of the QP (1):

$$\text{Minimize} \quad \boldsymbol{C} \bullet \boldsymbol{Y} \quad \text{subject to} \quad \boldsymbol{Y} \in \widehat{\mathcal{G}}; \tag{6}$$

We can rewrite the SDP as a convex minimization problem in the Euclidean space:

$$\text{Minimize} \quad \boldsymbol{c}^T \boldsymbol{y} \quad \text{subject to} \quad \boldsymbol{y} \in \widehat{\mathcal{F}}. \tag{7}$$

The two problems (6) and (7) above are equivalent in the sense that:

Lemma 2.1

1. $\boldsymbol{y}$ is a feasible solution of the problem (7) if and only if $\boldsymbol{y} = \boldsymbol{Y}\boldsymbol{e}_0$ for some feasible solution $\boldsymbol{Y}$ of the problem (6).

2. $\boldsymbol{y}$ is a minimum solution of the problem (7) if and only if $\boldsymbol{y} = \boldsymbol{Y}\boldsymbol{e}_0$ for some minimum solution $\boldsymbol{Y}$ of the problem (6).

3. $\inf\{\boldsymbol{C} \bullet \boldsymbol{Y} \; : \; \boldsymbol{Y} \in \widehat{\mathcal{G}}\} = \inf\{\boldsymbol{c}^T \boldsymbol{y} \; : \; \boldsymbol{y} \in \widehat{\mathcal{F}}\}.$ ∎

We will be mainly concerned with the convex minimization problem (7) instead of the SDP (6). If we restrict ourselves to QPs derived from 0-1 IPs, our construction of the problem (7) is a special case of the Lovász-Schrijver [11] relaxation method. We impose the following condition on the feasible region $\widehat{\mathcal{G}}$ of the SDP (6) in the main theorem below.

Condition 2.2 There is an interior point $\boldsymbol{Y}$ of the feasible region $\widehat{\mathcal{G}}$ of the SDP (6), a $\boldsymbol{Y} \in \mathcal{S}(1+n)_{++}$ satisfying $Y_{00} = 1$ and $\boldsymbol{P}_k \bullet \boldsymbol{Y} < 0$ $(k = 1, 2, \ldots, m)$. ∎

Now we consider the convex minimization problem (3) introduced in the Introduction as a relaxation of the QP (1) using convex-quadratic valid inequalities for $\mathcal{F}$. If all the extreme points and all the extreme rays of $\widetilde{\mathcal{F}}$ are contained in $\mathcal{F}$ then $\widetilde{\mathcal{F}}$ coincides with co $\mathcal{F}$, the convex hull of $\mathcal{F}$ and the problem (3) gives the best convex relaxation of the QP (1). But $\widetilde{\mathcal{F}} \neq$ co $\mathcal{F}$ in general. We focus our attention to a subset of extreme points of $\widetilde{\mathcal{F}}$ which are shown to be contained in $\mathcal{F}$ ((iii) of Theorem 2.3). We say that a point $\boldsymbol{y} \in \widetilde{\mathcal{F}}$ is *a strictly convex boundary point of* $\widetilde{\mathcal{F}}$ if there exists a $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)^T \geq \boldsymbol{0}$ such that

$$\boldsymbol{y}^T \left(\sum_{k=1}^m t_k \boldsymbol{P}_k \right) \boldsymbol{y} = 0 \quad \text{and} \quad \sum_{k=1}^m t_k \boldsymbol{Q}_k \in \mathcal{S}(n)_{++}. \tag{8}$$

It should be noted that the definition of a strictly convex boundary point depends on the algebraic representation of $\mathcal{F}$. That is, a strictly convex boundary point $\boldsymbol{y}$ of $\widetilde{\mathcal{F}}$ of the representation (4) is not necessarily a strictly convex boundary point of $\widetilde{\mathcal{F}}$ of a distinct representation. See section 6.

Now we are ready to state:

Theorem 2.3 (main theorem)

1. $\mathcal{F} \subseteq \widehat{\mathcal{F}} \subseteq \widetilde{\mathcal{F}}$.

2. Suppose that the feasible region $\widehat{\mathcal{G}}$ satisfies Condition 2.2. Then

$$\inf\{c^T y \ : \ y \in \widehat{\mathcal{F}}\} = \inf\{c^T y \ : \ y \in \widetilde{\mathcal{F}}\} \tag{9}$$

for every $c \in R^{1+n}$, and $\widetilde{\mathcal{F}} = cl\,\widehat{\mathcal{F}}$, the closure of $\widehat{\mathcal{F}}$.

3. Every strictly convex boundary point y of $\widetilde{\mathcal{F}}$ belongs to $\mathcal{F}$. ∎

Proof of the theorem is given in Section 5.

3   A Single Quadratic Inequality.

The most important principle behind the SDP relaxation is: Any quadratic inequality in the n-dimensional Euclidean space can be relaxed into a linear matrix inequality in $(1 + n) \times (1 + n)$ symmetric matrices. We will associate each vector $y = (1, y_1, y_2, \ldots, y_n)^T$ in R^{1+n} with a $(1 + n) \times (1 + n)$ symmetric matrix

$$Y = yy^T = \begin{pmatrix} 1 & y_1 & y_2 & \cdots & y_n \\ y_1 & y_1 y_1 & y_1 y_2 & \cdots & y_1 y_n \\ y_2 & y_2 y_1 & y_2 y_2 & \cdots & y_2 y_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_n & y_n y_1 & y_n y_2 & \cdots & y_n y_n \end{pmatrix} \in \mathcal{S}(1 + n). \tag{10}$$

The matrix $Y \in \mathcal{S}(1 + n)$ contains all the constant, linear and quadratic "atomic" terms, *i.e.*, the nonzero constant term 1, the n linear terms $y_1, y_2, \ldots, y_n$ and the n^2 quadratic terms $y_1 y_1, y_1 y_2, \ldots, y_n y_n$ in its elements, so that we can represent any function consisting of linear and quadratic forms of $y_1, y_2, \ldots, y_n$ in terms of a linear combination of those terms, *i.e.*, a linear function $P \bullet Y$ of Y for some $P \in \mathcal{S}(1 + n)$.

By the construction, for

$$P \equiv \begin{pmatrix} \pi & q^T/2 \\ q/2 & Q \end{pmatrix} \in \mathcal{S}(1 + n), \ q \in R^n, Q \in \mathcal{S}(n), \tag{11}$$

we see that

$$y^T P y = P \bullet Y \tag{12}$$

whenever

$$y = Y e_0, \ y_0 = 1 \ \text{and} \ Y = yy^T. \tag{13}$$

On the other hand, we know that an $(1 + n) \times (1 + n)$ matrix Y satisfies (13) for some $y = (y_0, y_1, \ldots, y_n)^T \in R^{1+n}$ if and only if

$$y = Y e_0, \ Y_{00} = 1, \ Y \in \mathcal{S}(1 + n)_+ \ \text{and rank } Y = 1.$$

Hence

$$y \in R^{1+n}, \ y^T P y \le 0 \ \text{and} \ y_0 = 1$$

if and only if

$$y = Y e_0 \in R^{1+n}, \ P \bullet Y \le 0, \ Y_{00} = 1, \ Y \in \mathcal{S}(1+n)_+ \ \text{and rank} \ Y = 1.$$

Dropping the last rank condition rank $Y = 1$, we obtain:

Lemma 3.1 *Let* $P \in \mathcal{S}(1+n)$. *If*

$$y^T P y \le 0 \quad and \ y_0 = 1 \tag{14}$$

then

$$y = Y e_0 \in R^{1+n}, \ P \bullet Y \le 0, \ Y_{00} = 1 \quad and \quad Y \in \mathcal{S}(1+n)_+ \tag{15}$$

for some $Y \in \mathcal{S}(1+n)$. ∎

(Relaxation by dropping the rank condition as mentioned above has been utilized in many papers [1, 2, 5, 7, 8, 10, 13, 17], etc.).

Lemma 3.2 *Let* P *be a* $(1+n) \times (1+n)$ *symmetric matrix of the form (11). Suppose that a* $(1+n) \times (1+n)$ *matrix* Y *and* $y \in R^{1+n}$ *satisfy the relation (15). Let*

$$y = \begin{pmatrix} 1 \\ x \end{pmatrix} \ and \ Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Then

$$y^T P y = P \bullet Y - Q \bullet (X - xx^T) \le -Q \bullet (X - xx^T).$$

If in addition the $n \times n$ *matrix* Q *is positive semi-definite then* y *satisfies the relation (14).*

Proof: By the definitions of the matrices P and Y, we have that

$$\begin{aligned}
y^T P y &= \pi + q^T x + x^T Q x \\
&= \pi + q^T x + Q \bullet X - Q \bullet (X - xx^T) \\
&= P \bullet Y - Q \bullet (X - xx^T) \\
&\le -Q \bullet (X - xx^T).
\end{aligned}$$

Thus we have shown the first assertion. It follows from $Y \in \mathcal{S}(1+n)_+$ and $Y_{00} = 1$ that $X - xx^T \in \mathcal{S}(n)_+$. Hence if $Q \in \mathcal{S}(n)_+$ then $Q \bullet (X - xx^T) \ge 0$; hence $y^T P y \le 0$. ∎

4 Duality.

Applying Shor's relaxation method [14] to the QP (1), we obtain an SDP

$$\text{Maximize } t_0 \text{ subject to } \boldsymbol{t} \in T^d, \tag{16}$$

where

$$T^d \equiv \left\{ \boldsymbol{t} = (t_0, t_1, \ldots, t_m)^T \ : \ \begin{array}{l} \boldsymbol{C} - t_0 \boldsymbol{e}_0 \boldsymbol{e}_0^T + \sum_{i=1}^m t_i \boldsymbol{P}_i \in \mathcal{S}(1+n)_+, \\ t_i \geq 0 \ (i = 1, 2, \ldots, m) \end{array} \right\}.$$

Between the two problems (16) and (1), the following relation holds.

Lemma 4.1 ([14], see also [17]) *If* $\boldsymbol{t} = (t_0, t_1, \ldots, t_m)^T \in R^{1+m}$ *is a feasible solution of the SDP (16) and* $\boldsymbol{y} \in R^{1+n}$ *a feasible solution of the QP (1), then their objective values* t_0 *and* $\boldsymbol{c}^T \boldsymbol{y}$ *satisfies the inequality* $t_0 \leq \boldsymbol{c}^T \boldsymbol{y}$; $\sup\{t_0 \ : \ \boldsymbol{t} \in T^d\} \leq \inf\{\boldsymbol{c}^T \boldsymbol{y} \ : \ \boldsymbol{y} \in \mathcal{F}\}$.

Proof: Assume that $\boldsymbol{t} = (t_0, t_1, \ldots, t_m)^T \in T^d$ and $\boldsymbol{y} \in \mathcal{F}$. Then

$$0 \leq \boldsymbol{y}^T \left(\boldsymbol{C} - t_0 \boldsymbol{e}_0 \boldsymbol{e}_0^T + \sum_{i=1}^m t_i \boldsymbol{P}_i \right) \boldsymbol{y} = \boldsymbol{c}^T \boldsymbol{y} - t_0 + \sum_{i=1}^m t_i \boldsymbol{y}^T \boldsymbol{P}_i \boldsymbol{y} \leq \boldsymbol{c}^T \boldsymbol{y} - t_0.$$

(This proof is essentially due to [17]). ∎

The SDP (16) is corresponding to the Lagrangian dual of the QP (1). See the papers [13, 14, 15] for details.

It is easily verified that the SDPs (6) and (16) are dual to each other. Hence, from the duality theorem (see, for example, Theorem 4.2.1 of [12]) and Lemma 2.1, we obtain:

Lemma 4.2 (Duality between (3) and (16))

1. *If* $\boldsymbol{t} = (t_0, t_1, \ldots, t_m)^T \in R^{1+m}$ *is a feasible solution of the SDP (16) and* $\boldsymbol{y} \in R^{1+n}$ *a feasible solution of the problem (3), their objective values* t_0 *and* $\boldsymbol{c}^T \boldsymbol{y}$ *satisfy* $t_0 \leq \boldsymbol{c}^T \boldsymbol{y}$; $\sup\{t_0 \ : \ \boldsymbol{t} \in T^d\} \leq \inf\{\boldsymbol{c}^T \boldsymbol{y} \ : \ \boldsymbol{y} \in \widehat{\mathcal{F}}\}$.

2. *Suppose that Condition 2.2 holds and that* $-\infty < \hat{g} \equiv \inf\{\boldsymbol{c}^T \boldsymbol{y} \ : \ \boldsymbol{y} \in \widehat{\mathcal{F}}\}$. *Then the SDP (16) has a maximum solution* $\boldsymbol{t}^* \in R^{1+m}$ *with the maximum objective value* $t_0^* = \hat{g}$. ∎

The lemma below establishes a weak duality relation between the convex minimization problem (3) and the SDP (16).

Lemma 4.3 *If* $\boldsymbol{t} = (t_0, t_1, \ldots, t_m) \in R^{1+m}$ *is a feasible solution of the SDP (16) and* $\boldsymbol{y} \in R^{1+n}$ *a feasible solution of the problem (3), their objective values* t_0 *and* $\boldsymbol{c}^T \boldsymbol{y}$ *satisfy* $t_0 \leq \boldsymbol{c}^T \boldsymbol{y}$;
$\sup\{t_0 \ : \ \boldsymbol{t} \in T^d\} \leq \inf\{\boldsymbol{c}^T \boldsymbol{y} \ : \ \boldsymbol{y} \in \widetilde{\mathcal{F}}\}$.

Proof: Suppose that $t \in T^d$ and $y \in R^{1+n} \in \tilde{\mathcal{F}}$. Let

$$Z \equiv \begin{pmatrix} \zeta & w^T \\ w & Q \end{pmatrix} = C - t_0 e_0 e_0^T + \sum_{k=1}^m t_k P_k,$$

where $\zeta \in R$, $w \in R^n$ and $Q \in \mathcal{S}(n)$. We see by the definitions of the matrices C, $e_0 e_0^T$, $P_k \in \mathcal{S}(1+n)$ $(k = 1, 2, \ldots, m)$ that $Q = \sum_{k=1}^m t_k Q_k$. On the other hand, it follows from $Z \in \mathcal{S}(1+n)_+$ that $Q = \sum_{k=1}^m t_k Q_k \in \mathcal{S}(n)_+$. Hence we obtain from $y \in \tilde{\mathcal{F}}$ that $y^T \left(\sum_{k=1}^m t_k P_k \right) y \leq 0$. Consequently,

$$0 \leq y^T Z y = y^T C y - t_0 y^T e_0 e_0^T y + y^T \left(\sum_{k=1}^m t_k P_k \right) y \leq c^T y - t_0.$$

∎

5　Proof of the Main Theorem.

(i) The first inclusion relation $\mathcal{F} \subseteq \hat{\mathcal{F}}$ follows from Lemma 3.1. To prove the second inclusion relation $\hat{\mathcal{F}} \subseteq \tilde{\mathcal{F}}$, assume that $y \in \hat{\mathcal{F}}$. Then there exists a $Y \in \hat{\mathcal{G}}$ such that $y = Y e_0$; specifically Y satisfies $0 \geq P_k \bullet Y$ $(k = 1, 2, \ldots, m)$. Hence

$$\left(\sum_{k=1}^m t_k P_k \right) \bullet Y \leq 0 \text{ for every } t = (t_1, t_2, \ldots, t_m)^T \geq 0.$$

By Lemma 3.2, we see that

$$y^T \left(\sum_{k=1}^m t_k P_k \right) y \leq 0 \text{ whenever } \sum_{k=1}^m t_k Q_k \in \mathcal{S}(n)_+.$$

This implies $y \in \tilde{\mathcal{F}}$. Thus we have shown that $\hat{\mathcal{F}} \subseteq \tilde{\mathcal{F}}$.

(ii) Since $\hat{\mathcal{F}} \subseteq \tilde{\mathcal{F}}$, we know that

$$\inf\{c^T y : y \in \tilde{\mathcal{F}}\} \leq \inf\{c^T y : y \in \hat{\mathcal{F}}\} \tag{17}$$

for every $c \in R^{1+n}$. Let $c \in R^{1+n}$ be fixed arbitrarily. If $\inf\{c^T y : y \in \hat{\mathcal{F}}\} = -\infty$ then $\inf\{c^T y : y \in \tilde{\mathcal{F}}\} = -\infty$ by (17). Hence we obtain the equality (9). So assume that $\hat{g} \equiv \inf\{c^T y : y \in \hat{\mathcal{F}}\} > -\infty$. By Lemma 4.2, there exists a maximum solution $t^* = (t_0^*, t_1^*, \ldots, t_m^*)^T \in R^{1+m}$ of the SDP (16) with the objective value $t_0^* = \hat{g}$. We also see by Lemma 4.3 that $t_0^* \leq \inf\{c^T y : y \in \tilde{\mathcal{F}}\}$. Therefore

$$\inf\{c^T y : y \in \tilde{\mathcal{F}}\} \leq \inf\{c^T y : y \in \hat{\mathcal{F}}\} = \hat{g} = t_0^* \leq \inf\{c^T y : y \in \tilde{\mathcal{F}}\}.$$

Thus we have shown the equality (9). By the construction, $\tilde{\mathcal{F}}$ is a closed convex subset of R^{1+n} and $\hat{\mathcal{F}}$ is a convex subset of R^{1+n}. Hence the identity (9) for every $c \in R^{1+n}$

implies that $\tilde{\mathcal{F}} = \mathrm{cl}\ \hat{\mathcal{F}}$.

(iii) Assume on the contrary that $y \notin \mathcal{F}$ for some strictly convex boundary point y of $\tilde{\mathcal{F}}$. It follows from $y \notin \mathcal{F}$ that $y^T P_j y > 0$ for some $j \in \{1, 2, \ldots, m\}$. Since y is a strictly convex boundary point of $\tilde{\mathcal{F}}$, there exists some $t = (t_1, t_2, \ldots, t_m)^T \geq 0$ for which

$$y_0 = 1, \quad y^T \left(\sum_{k=1}^{m} t_k P_k \right) y = 0 \ \text{ and } \ \sum_{k=1}^{m} t_k Q_k \in \mathcal{S}(n)_{++}$$

holds. Hence if $\epsilon > 0$ is sufficiently small, we obtain

$$y_0 = 1, \quad y^T \left(\sum_{k=1}^{m} t_k P_k + \epsilon P_j \right) y > 0 \ \text{ and } \ \sum_{k=1}^{m} t_k Q_k + \epsilon Q_j \in \mathcal{S}(n)_{++},$$

which is a contradiction to the assumption that $y \in \tilde{\mathcal{F}}$. This completes the proof of the main theorem.

6 Concluding Discussion.

The effectiveness of the SDP relaxation for a nonconvex QP (or a 0-1 IP) depends on the representation of its feasible region using linear and/or quadratic inequalities. Suppose that the feasible region $\mathcal{F}$ of the canonical form QP (1) is bounded and involves some linear inequality constraints

$$y^T P_k y \equiv \pi_k + q_k^T x \leq 0 \ (k \in K),$$

where $y = \begin{pmatrix} 1 \\ x \end{pmatrix}$, and $K \subseteq \{1, 2, \ldots, m\}$. Let S denote the polyhedral region determined by these linear inequalities;

$$S = \left\{ y = \begin{pmatrix} 1 \\ x \end{pmatrix} \ : \ \pi_k + q_k^T x \leq 0 \ (k \in K) \right\}.$$

We want to cut off all the vertices of S that do not lie in $\mathcal{F}$ when we apply the SDP relaxation. As we will see below, this is always possible if we replace those linear inequality constraints by convex-quadratic inequality constraints

$$y^T P_k' y \equiv (\pi_k + q_k^T x)(\pi_k' + q_k^T x) \leq 0 \ (k \in K),$$

where

$$P_k' \equiv \begin{pmatrix} \pi_k' \pi_k & (\pi_k' + \pi_k) q_k^T / 2 \\ (\pi_k' + \pi_k) q_k / 2 & Q_k' \end{pmatrix} \in \mathcal{S}(1+n), \ Q_k' \equiv q_k q_k^T \in \mathcal{S}(n)_+ \ (k \in K),$$

and π_k' $(k \in K)$ are sufficiently large numbers such that

$$\pi_k + q_k^T x \leq 0 \leq \pi_k' + q_k^T x \ (k \in K) \ \text{ for every } y = \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathcal{F}. \tag{18}$$

Let

$$\mathcal{F}' = \left\{ y \in R^{1+n} \; : \; \begin{array}{l} y_0 = 1, \\ y^T P_i y \leq 0 \; (i \in \{1, 2, \ldots, m\} \backslash K), \\ y^T P'_k y \leq 0 \; (k \in K) \end{array} \right\}$$

The condition (18) above on π'_k ($k \in K$) ensures that $\mathcal{F}' = \mathcal{F}$. Suppose that $y = \begin{pmatrix} 1 \\ x \end{pmatrix}$ is a vertex of S. Then there exists a subset K' of K such that

$$y^T P_k y = \pi_k + q_k^T x = 0 \; (k \in K') \quad \text{and} \quad \{q_k \in R^n \; : \; k \in K'\} \quad \text{forms a basis of } R^n.$$

Hence

$$\sum_{k \in K'} y^T P'_k y = 0 \quad \text{and} \quad \sum_{k \in K'} Q'_k \in \mathcal{S}(n)_{++}.$$

If in addition $y \in \tilde{\mathcal{F}}'$ then y is a strictly convex boundary point of $\tilde{\mathcal{F}}'$; hence $y \in \mathcal{F}'$ by (iii) of Theorem 2.3. Therefore we can conclude that every vertex y of S belongs to $\tilde{\mathcal{F}}'$ if and only if $y \in \mathcal{F}'$.

References

[1] W. F. Alizadeh, "Interior point methods in semidefinite programming with application to combinatorial optimization," *SIAM Journal on Optimization* **5** (1995) 13–51.

[2] W. F. Alizadeh, J. -P. A. Haeberly and M. L. Overton, "Primal-dual interior-point methods for semidefinite programming," 1994.

[3] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, (SIAM, Philadelphia, 1994).

[4] R. M. Freund, "Complexity of an algorithm for finding an approximate solution of a semidefinite program with no regularity assumption," Technical report OR 302-94, Operations Research Center, MIT, 1994.

[5] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *Journal of Assoc. Comput. Mach.* to appear. . A preliminary version appeared in *Proceedings of the 26th Annual ACM Symposium on Theory of Computing* (1994) 422–431.

[6] C. Helmberg, S. Poljak, F. Rendl and H. Wolkowicz, "Combining semidefinite and polyhedral relaxation for integer programs," Lecture note in Computer Science 538 (1995) 124–134.

[7] C. Helmberg, F. Rendl, R. J. Vanderbei and H. Wolkowicz, "An interior-point method for semidefinite programming," SIAM Journal on Optimization, to appear.

[8] D. E. Knuth, "The sandwich theorem," *Electronic Journal of Combinatorics* 1 (1995) 1–48.

[9] M. Kojima, S. Shindoh and S. Hara, "Interior-point methods for the monotone semidefinite linear complementarity problems," Research Report #282, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152, Japan, April 1994, Revised April 1995.

[10] L. Lovász, "On the Shannon capacity of a graph," *IEEE Transactions on Information Theory* 25 (1979) 1–7.

[11] L. Lovász and A. Schrijver, "Cones of matrices and set functions and 0-1 optimization," *SIAM Journal on Optimization* 1 (1991) 166–190.

[12] Ju. E. Nesterov and A. S. Nemirovskii, *Interior Point Polynomial Methods in Convex Programming: Theory and Applications* (SIAM, Philadelphia, 1993).

[13] S. Poljak, F. Rendl and H. Wolkowicz, "A recipe for semidefinite relaxation for (0,1)-quadratic programming," *Journal of Global Optimization* 7 (1995) 51–73.

[14] N. Z. Shor, "Quadratic optimization problems," *Soviet Journal of Computer and Systems Sciences* 25 (1987) 1–11.

[15] N. Z. Shor, "Dual quadratic estimates in polynomial and boolean programming," *Annals of Operations Research* 25 (1990) 163–168.

[16] L. Vandenberghe and S. Boyd, "A primal-dual potential reduction method for problems involving matrix inequalities," *Mathematical Programming* 69 (1995) 205–236.

[17] L. Vandenberghe and S. Boyd, " Semidefinite Programming," Informations Systems Laboratory, Stanford University, 1994.