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Asymptotic Stability versus Exponential Stability in Linear Volterra Difference Equations of Convolution Type

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§1. INTRODUCTION.

Volterra difference equations of convolution type have been investigated by the first author in [3,4,5]. In particular, the resolvent matrix was defined and used to establish a variation of constants formula. These results constitute the discrete analogue of the theory of Volterra integrodifferential equations [1,6,7,8]. A question was raised by Corduneanu and Lakshmikantham [1] of whether or not uniform asymptotic stability implies exponential stability in linear Volterra integrodifferential equations. In [8] Murakami answered this question negatively. In this paper we will extend Murakami's result to Volterra difference equations. It will be shown that if the zero solution is uniformly asymptotically stable, then it is exponentially stable if and only if the kernel decays exponentially (Theorem 5).

Consider the linear Volterra difference system of convolution type

$$x(n+1)=Ax(n)+\sum_{j=0}^{n}B(n-j)x(j), \tag{L}$$

where A is a $k\times k$ constant matrix and $B(n)\in l^1(Z^+)$ is a $k\times k$ matrix-valued function defined on the set of nonnegative integers Z^+.

The resolvent matrix $R(n)$ of (L) is defined as the unique solution of the matrix equation

$$R(n+1)=AR(n)+\sum_{j=0}^{n}B(n-j)R(j),\quad R(0)=I,\ n\in Z^+. \tag{R}$$

Let $y(n)$ denote the solution of the equation

$$y(n+1) = Ay(n) + \sum_{j=0}^{n} B(n-j)y(j) + g(n). \tag{1}$$

Then by the variation of constants formula [4,5], we obtain

$$y(n) = R(n)y(0) + \sum_{j=0}^{n-1} R(n-j-1)g(j). \tag{2}$$

For any $s \in Z^+$ and initial function $\varphi : [0,s] \mapsto R^k$, there is only one solution $x(n,s,\varphi) \equiv x(n)$ which satisfies Equation (L) on $[s,\infty)$ and $x(n) = \varphi(n)$ on $[0,s]$. Here all our intervals are discrete, e.g. $[0,s] = \{0,1,2,\ldots s\}$.

Although our stability definitions are standard, we will state them here for the convenience of the reader. If $\varphi : [0,s] \mapsto R^k$, then $\|\varphi\|_{[0,s]} = \sup\{|\varphi(j)| : j \in [0,s]\}$.

Definition 1. *The zero solution of (L) is said to be:*
(i) uniformly stable (US) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $s \in Z^+$ and φ is an initial function on $[0,s]$ with $\|\varphi\|_{[0,s]} < \delta$ then $|x(n,s,\varphi)| < \varepsilon$ for all $n \geq s$;
(ii) uniformly attractive (UA) if there exists $\mu > 0$ such that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in Z^+$ such that if $s \in Z^+$ and φ is an initial function on $[0,s]$ with $\|\varphi\|_{0,s]} < \mu$ then $|x(n,s,\varphi)| < \varepsilon$ for all $n \geq s+N$;
(iii) uniformly asymptotically stable (UAS) if it is both US and UA;
(iv) exponentially stable (ES) if there exist positive constants K and η with $\eta \in (0,1)$ such that

$$|x(n,s,\varphi)| \leq K\eta^{n-s}\|\varphi\|_{[0,s]}, \quad n \geq s \geq 0$$

for any initial function φ on $[0,s]$.

§2. UNIFORM ASYMPTOTIC STABILITY.

In this section we will establish some necessary and sufficient conditions for UAS. One of the main tools used here is the Z-transform method [2,3,4,5]. Recall that the Z-

transform $\tilde{x}(z)$ of a sequence $x(n)$ is defined as

$$\tilde{x}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

Let

$$h(n) := \sum_{r=0}^{\infty} \left| \sum_{j=0}^{n-1} R(n-j-1)B(j+r+1) \right|. \tag{3}$$

The main result in this section now follows.

Theorem 2. *For equation (L) the following statements are equivalent.*

(I) $\det(zI - A - \tilde{B}(z)) \neq 0$ for $|z| \geq 1$.

(II) $R(n) \in l^1(Z^+)$.

(III) *The zero solution of (L) is UAS.*

(IV) *Both $R(n)$ and $h(n)$ of (3) tend to zero as $n \to \infty$.*

Proof. (I)$\Longrightarrow$ (II): Define the matrix function $\hat{B}(n)$ by letting $\hat{B}(r) = B(r)$ for all $r \neq 0$ and $\hat{B}(0) = B(0) + A$. Then Equation (R) may be now written in the form

$$R(n+1) = \hat{B}(n) + \sum_{j=1}^{n} \hat{B}(n-j)R(j). \tag{4}$$

By the discrete Gronwall's inequality [5], Equation (4) yields

$$|R(n)| \leq (1+\alpha)^n = \beta^n$$

where $\alpha = \sum_{n=0}^{\infty} |\hat{B}(n)|$. Hence

$$\begin{aligned} \tilde{R}(z) &= z(zI - A - \tilde{B}(z))^{-1} \\ &= (I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z))^{-1}, \quad |z| > \beta > 1. \end{aligned} \tag{5}$$

For sufficiently large γ,

$$\inf_{|z|>\gamma} |\det(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z))| \geq \frac{1}{2}.$$

Furthermore, also on the compact annulus $1 \leq |z| \leq \gamma$, $\inf \det(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)) \neq 0$. Hence it follows that

$$\inf |\det(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z))| > 0 \text{ for all } |z| \geq 1. \tag{6}$$

Applying a theorem due to Wiener (cf. [2, p.251]), we conclude by (6) that there exists an $H(n) \in l^1(Z^+)$ such that

$$\tilde{H}(z)(I - \frac{1}{z}A - \frac{1}{z}\tilde{B}(z)) = I \text{ for } |z| \geq 1.$$

Then it follows from (5) that $\tilde{H}(z) = \tilde{R}(z)$ for $|z| > \beta$, and hence $R(n) \equiv H(n) \in l^1(Z^+)$ as required.

(II)$\Longrightarrow$ (III): Assume that $R(n) \in l^1(Z^+)$. Then from Equation (L),

$$\begin{aligned} x(n+\tau+1, \tau, \varphi) &= Ax(n+\tau, \tau, \varphi) + \sum_{j=0}^{n+\tau} B(n+\tau-j)x(j, \tau, \varphi) \\ &= Ax(n+\tau, \tau, \varphi) + \sum_{j=0}^{n} B(n-j)x(j+\tau, \tau, \varphi) \\ &\quad + \sum_{j=1}^{\tau} B(n+j)\varphi(\tau-j). \end{aligned}$$

It follows by the variation of constants formula that

$$x(n+\tau, \tau, \varphi) = R(n)\varphi(\tau) + \sum_{j=0}^{n-1} R(n-j-1) \sum_{s=1}^{\tau} B(j+s)\varphi(\tau-s)$$

or

$$|x(n+\tau, \tau, \varphi)| \leq \|\varphi\|_{[0,\tau]}[|R(n)| + \sum_{j=0}^{n-1} |R(n-j-1)| \sum_{s=j+1}^{\infty} |B(s)|]. \tag{7}$$

In Equation (7), the first term $|R(n)| \to 0$ as $n \to \infty$; the second term also tends to zero as $n \to \infty$ since it is the convolution of an l^1 function with one which tends to zero as $n \to \infty$. Therefore the quantity

$$|R(n)| + \sum_{j=0}^{n-1} |R(n-j-1)| \sum_{s=j+1}^{\infty} |B(s)|$$

is bounded and tends to zero as $n \to \infty$, and hence the zero solution of (L) is UAS.

(III)$\Longrightarrow$ (IV): Assume that the zero solution of (L) is UAS. Then $|x(n+\tau,\tau,\varphi)| \to 0$ as $n \to \infty$. If $\tau = 0$, then $|x(n,0,\varphi(0))| = |R(n)\varphi(0)| \to 0$ as $n \to \infty$. Consequently, $|R(n)| \to 0$ as $n \to \infty$. Now

$$\begin{aligned} x(n+\tau+1,\tau,\varphi) &= Ax(n+\tau,\tau,\varphi) + \sum_{j=0}^{n+\tau} B(n+\tau-j)x(j,\tau,\varphi) \\ &= Ax(n+\tau,\tau,\varphi) + \sum_{r=0}^{n} B(n-r)x(r+\tau,\tau,\varphi) + \sum_{j=0}^{\tau-1} B(n+\tau-j))\varphi(j). \end{aligned}$$

By the variation of constant formula we have

$$x(n+\tau,\tau,\varphi) = R(n)\varphi(\tau) + \sum_{j=0}^{n-1} R(n-j-1)\sum_{r=0}^{\tau-1} B(j+\tau-r)\varphi(r)$$

or

$$|x(n+\tau,\tau,\varphi) - R(n)\varphi(\tau)| = |\sum_{r=0}^{\tau-1}(\sum_{j=0}^{n-1} R(n-j-1)B(j+r+1))\varphi(\tau-r-1)|.$$

Since $x(n+\tau,\tau,\varphi) \to 0$ and $R(n)\varphi(\tau) \to 0$ uniformly for $\tau \geq 0$ as $n \to \infty$, it follows that

$$\sup_{\tau\geq 0} |\sum_{r=0}^{\tau-1}(\sum_{j=0}^{n-1} R(n-j-1)B(j+r+1))\varphi(\tau-r-1)| \to 0 \text{ as } n \to \infty.$$

Consequently, $\sup_{\tau\geq 0}\sum_{r=0}^{\tau-1} |\sum_{j=0}^{n-1} R(r-j-1)B(j+r+1)| \to 0$ as $n \to \infty$. Therefore, the limit $\lim_{\tau\to\infty}\sum_{r=0}^{\tau-1} |\sum_{j=0}^{n-1} R(n-j-1)B(j+r+1)| = h(n)$ exists for $n \in Z^+$, and it satisfies the relation that $h(n) \to 0$ as $n \to \infty$.

(IV)$\Longrightarrow$ (I): Assume that the condition (IV) holds. Claim that $\det[zI - A - \tilde{B}(z)] \neq 0$ for all $|z| \geq 1$. If this claim is false, then there exist a complex number z_0 with $|z_0| \geq 1$ and a unit vector $y_0 \in R^k$ such that

$$(z_0I - A - \tilde{B}(z_0))y_0 = 0.$$

Hence

$$z_0 y_0 = A y_0 + \sum_{n=0}^{\infty} B(n) y_0 z_0^{-n}. \tag{8}$$

Set $y(n) = z_0^n y_0$. Then $y(n)$ is bounded for $n \in (-\infty, 0]$, and

$$|y(n)| \geq |y_0| = 1, \quad \text{for } n \in Z^+. \tag{9}$$

Now using Equation (8) we get

$$\begin{aligned} y(n+1) &= z_0^{n+1} y_0 \\ &= z_0^n [A y_0 + \sum_{j=0}^{\infty} B(j) y_0 z_0^{-j}]. \end{aligned}$$

Thus

$$y(n+1) = Ay(n) + \sum_{j=0}^{n} B(j) y(n-j) + \sum_{j=n+1}^{\infty} B(j) y(n-j).$$

By the variation of constants formula, it follows that

$$\begin{aligned} y(n) &= R(n) y_0 + \sum_{j=0}^{n-1} R(n-j-1) \left(\sum_{s=j+1}^{\infty} B(s) y(j-s) \right) \\ &= R(n) y_0 + \sum_{r=0}^{\infty} \left(\sum_{j=0}^{n-1} R(n-j-1) B(j+r+1) \right) y(-r-1). \end{aligned}$$

Since $R(n) \to 0$ and $h(n) \to 0$ as $n \to \infty$, it follows that $y(n) \to 0$ as $n \to \infty$, which contradicts (9). The proof of the claim is now complete.

Remark. In both Volterra integrodifferential equations and Volterra difference equations, it is widely believed that the resolvent matrix $R(n)$ of Equation (E) possesses the same properties of a fundamental matrix of an ordinary difference or differential equation. In particular, it is assumed that

$$R(n-s) R(s) = R(n). \tag{10}$$

The false statement (10) leads to the false claim that UAS implies ES for Equation (L). The next lemma shows that (10) is possessed only by ordinary difference equations.

Lemma 3. *$R(n-s)R(s) = R(n)$ for all $n \geq s \geq 0$ if and only if $B(n) = 0$ for all $n = 1, 2, \ldots$.*

Proof. Sufficiency is trivial.
Necessity. Suppose that $R(n-s)R(s) = R(n)$ for all $n \geq s \geq 0$. Then from Equation (R) we have

$$R(n-s+1) = AR(n-s) + \sum_{j=0}^{n-s} B(n-s-j)R(j).$$

Multiply both sides by R(s) and change the indexing to obtain

$$R(n-s+1)R(s) = AR(n-s)R(s) + \sum_{j=s}^{n} B(n-j)R(j-s)R(s)$$

or

$$R(n+1) = AR(n) + \sum_{j=s}^{n} B(n-j)R(j). \tag{11}$$

Subtracting (11) from (R) gives

$$\sum_{j=0}^{s-1} B(n-j)R(j) = 0 \quad \text{for all } n \geq s \geq 0. \tag{12}$$

Letting $s = 1$ in (12) yields $B(n) = 0$ for all $n = 1, 2, 3, \ldots$. Consequently, Equation (L) must have the form $x(n+1) = [A + B(0)]x(n)$.

§3. EXPONENTIAL STABILITY.

A matrix-valued function $C(n)$ on Z^+ is said to *decay exponentially* whenever it satisfies $|C(n)| \leq M\nu^n$ $(n \in Z^+)$ for some $M > 0$ and $\nu \in (0, 1)$.

Theorem 4. *Suppose $|R(n)| \to 0$ as $n \to \infty$. Then $R(n)$ decays exponentially if and only if $B(n)$ is so.*

Proof of "only if " part. We assume that $|R(n)| \leq M\nu^n$ $(n \in Z^+)$ for some $M > 0$ and $\nu \in (0, 1)$. Then $\tilde{R}(z) = \sum_{n=0}^{\infty} R(n)z^{-n}$ absolutely converges on $|z| > \nu$. Moreover,

$\tilde{B}(z) = \sum_{n=0}^{\infty} B(n)z^{-n}$ absolutely converges on $|z| \geq 1$. Taking the Z-transform of Equation (R), we obtain

$$(zI - A - \tilde{B}(z))\tilde{R}(z) = zI, \quad |z| \geq 1.$$

Hence $\tilde{R}(z)$ is nonsingular and $zI - A - \tilde{B}(z) = z\tilde{R}(z)^{-1}$ for all z with $|z| \geq 1$. Since $\tilde{R}(z)$ is continuous on $|z| > \nu$, the fact that $\tilde{R}(z)$ is nonsingular for z with $|z| = 1$ implies $\inf\{|\det \tilde{R}(z)| : |z| = 1\} > 0$. Then $\inf\{|\det \tilde{R}(z)| : 1 - 2\delta \leq |z| \leq 1\} > 0$ for some $\delta \in (0, (1-\nu)/2)$, and hence $\tilde{R}(z)$ is nonsingular for z with $|z| \geq 1 - 2\delta$. Since $\tilde{R}(z)$ is analytic on the annulus $|z| > 1 - 2\delta$, so is $\tilde{R}(z)^{-1}$. Consider a function $F(z)$ defined by

$$F(z) = zI - A - z\tilde{R}(z)^{-1}, \qquad |z| > 1 - 2\delta,$$

which is analytic on the annulus $|z| > 1 - 2\delta$, and let

$$F(z) = \sum_{n=-\infty}^{\infty} a(n)z^n \qquad (1 - 2\delta < |z| < \infty)$$

be Laurent's expansion of $F(z)$. Since $F(z) = \tilde{B}(z)$ on $|z| \geq 1$, we obtain $\sup_{|z|\geq 1} |F(z)| = \sup_{|z|\geq 1} |\sum_{n=0}^{\infty} B(n)z^{-n}| \leq \sum_{n=0}^{\infty} |B(n)| =: M_1 < \infty$. Then

$$\begin{aligned} |a(n)| &= |\frac{1}{2\pi i}\int_{|z|=L} \frac{F(z)}{z^{n+1}}dz| \\ &\leq \frac{1}{2\pi}\frac{M_1}{L^{n+1}}2\pi L = \frac{M_1}{L^n} \end{aligned}$$

for all $L \geq 1$. Let $L \to \infty$ in the above to get $|a(n)| = 0$ for $n \geq 1$. Consequently,

$$F(z) = \sum_{n=0}^{\infty} a(-n)z^{-n}, \qquad 1 - 2\delta < |z| < \infty.$$

In particulr, we get

$$\sum_{n=0}^{\infty} |a(-n)|(1-\delta)^{-n} < \infty. \tag{13}$$

Since $\sum_{n=0}^{\infty} B(n)z^{-n} = \tilde{B}(z) = F(z) = \sum_{n=0}^{\infty} a(-n)z^{-n}$ on $|z| \geq 1$, it follows from the uniqueness of Laurent's expansion that $B(n) = a(-n)$ for all $n \in Z^+$. Thus we obtain

$\sum_{n=0}^{\infty}|B(n)|(1-\delta)^{-n} < \infty$ by (13), and hence $\sup_{n\geq 0}|B(n)|(1-\delta)^{-n} =: M_2 < \infty$. Then $|B(n)| \leq M_2(1-\delta)^n$ for all $n \in Z^+$, which shows that $B(n)$ decays exponentially.

Proof of "if "part. Suppose $|B(n)| \leq M_3\nu_1^n$ $(n \in Z^+)$ for some $M_3 > 0$ and $\nu_1 \in (0,1)$. Then $\tilde{B}(z) = \sum_{n=0}^{\infty} B(n)z^{-n}$ absolutely converges on $|z| > \nu_1$. On the other hand, $\sup_{n\geq 0}|R(n)|d^{-n}$ $(=: c) < \infty$ for some constant $d \geq 1$, and hence the series $\tilde{R}(z) = \sum_{n=0}^{\infty} R(n)z^{-n}$ absolutely converges on $|z| > d$. Taking the Z-transform of Equation (R), we obtain

$$(zI - A - \tilde{B}(z))\tilde{R}(z) = zI, \qquad |z| > d.$$

Thus $zI - A - \tilde{B}(z)$ is nonsingular and $\tilde{R}(z) = z(zI - A - \tilde{B}(z))^{-1}$ for all z with $|z| > d$. We assert that

$$\det[zI - A - \tilde{B}(z)] \neq 0, \qquad |z| \geq 1. \tag{14}$$

Indeed, since $|B(n)| \leq M_3\nu_1^n$ for $n \in Z^+$, the function $h(n)$ of (3) satisfies

$$\begin{aligned} h(n) &\leq M_3\sum_{r=0}^{\infty}\sum_{j=0}^{n-1}|R(n-j-1)|\nu_1^{j+r+1} \\ &\leq M_3/(1-\nu_1)\sum_{j=0}^{n-1}|R(n-j-1)|\nu_1^{j+1}. \end{aligned}$$

Then $h(n) \to 0$ as $n \to \infty$, because of $0 < \nu_1 < 1$ and $|R(n)| \to 0$ as $n \to \infty$. Hence the assertion (14) follows from Theorem 2.

Now, since $zI - A - \tilde{B}(z)$ is continuous on $|z| > \nu_1$, by (14) one can choose a constant $\delta_1 \in (0, (1-\nu_1)/2)$ so small that $\inf\{|\det(zI - A - \tilde{B}(z))| : 1 - 2\delta_1 \leq |z| \leq 1\} > 0$. Then $zI - A - \tilde{B}(z)$ is nonsingular for z with $|z| > 1 - 2\delta_1$, and $(zI - A - \tilde{B}(z))^{-1}$ is analytic on $|z| > 1 - 2\delta_1$. Consider a function $G(z) = z(zI - A - \tilde{B}(z))^{-1}$ on $|z| > 1 - 2\delta_1$, and let $G(z) = \sum_{n=-\infty}^{\infty} b(n)z^n$ $(1 - 2\delta_1 < |z| < \infty)$ be Laurent's expansion of $G(z)$. Since $\tilde{R}(z) = G(z)$ on $|z| > d$, we obtain

$$\sup\{|G(z)| : |z| \geq 2d\} = \sup\{\sum_{n=0}^{\infty}|R(n)z^{-n}| : |z| \geq 2d\}$$

$$\leq \sum_{n=0}^{\infty} |R(n)|(2d)^{-n}$$
$$\leq c\sum_{n=0}^{\infty} 2^{-n} = 2c.$$

By the same reasoning as for $F(z)$, one can get $b(n) = 0$ for all $n \geq 1$, and $\sum_{n=0}^{\infty} |b(-n)|(1-\delta_1)^{-n} < \infty$. On the other hand, since $\sum_{n=0}^{\infty} R(n)z^{-n} = \tilde{R}(z) = G(z) = \sum_{n=0}^{\infty} b(-n)z^{-n}$ on $|z| > d$, it follows from the uniqueness of Laurent's expansion that $R(n) = b(-n)$ for all $n \in Z^+$. Consequently, $\sum_{n=0}^{\infty} |R(n)|(1-\delta_1)^{-n} < \infty$, and hence $\sup_{n\geq 0} |R(n)|(1-\delta_1)^{-n} < \infty$. This implies that $R(n)$ decays exponentially.

Theorem 5. *Suppose that the zero solution of (L) is UAS. Then the zero solution of (L) is ES if and only if $B(n)$ decays exponentially.*

Proof. The "only if " part follows from Theorem 4, immediately. We will establish the "if " part. Assume that $|B(n)| \leq M_1\nu_1^n$ $(n \in Z^+)$ for some $M_1 > 0$ and $\nu_1 \in (0,1)$. Since $R(n)$ decays exponentially by Theorem 4, there exist some $M > 0$ and $\nu \in (\nu_1, 1)$ such that $|R(n)| \leq M\nu^n$ for all $n \in Z^+$. Let any $\tau \in Z^+$ and any initial function φ on $[0, \tau]$ be given. By (7), we get

$$\begin{aligned} |x(n+\tau, \tau, \varphi)| &\leq \|\varphi\|_{[0,\tau]}[M\nu^n + \sum_{j=0}^{n-1} M\nu^{n-j-1} \sum_{s=j+1}^{\infty} M_1\nu_1^s] \\ &\leq M\|\varphi\|_{[0,\tau]}[1 + \frac{M_1\nu_1}{(1-\nu_1)(\nu-\nu_1)}]\nu^n, \end{aligned}$$

which shows the exponential stability of the zero solution of (L).

Example 6. Consider the scalar difference equation

$$x(n+1) = \frac{1}{4}x(n) + \sum_{j=0}^{n} \frac{1}{(2(n-j)+1)(2(n-j)+3)}x(j).$$

Here $A = 1/4$, $B(n) = 1/[(2n+1)(2n+3)]$ which is in $l^1(Z^+)$ since $\sum_{n=0}^{\infty} B(n) = \frac{1}{2}$. Since $A + \sum_{n=0}^{\infty} B(n) < 1$, it follows from Corollary 2.4 in [3] that the zero solution is UAS. By Theorem 5 it follows that the zero solution is not ES.

Example 7. Consider the scalar difference equation

$$\begin{aligned} x(n+1) &= ax(n) + \sum_{j=0}^{n}(-1/2)^{n-j}x(j) \\ &= (a+1)x(n) + \sum_{j=1}^{n}(-1/2)^{n-j}x(j). \end{aligned}$$

Here $A = a$ and $B(n) = (-1/2)^n$ which deacays exponentially. The equation $z - A - \tilde{B}(z) = 0$ $(|z| \geq 1)$ is equivalent to the equation $2z^2 - (2a+1)z - a = 0$ $(|z| \geq 1)$, which has no roots if and only if $-2 < a < 1/3$. Therefore, by Theorems 1 and 5, the zero solution of the above equation is ES if and only if $-2 < a < 1/3$. We note that the zero solution of the equation $x(n+1) = (a+1)x(n)$ which has no delay terms is unstable if $0 < a < 1/3$.

REFERENCES

[1] C. Corduneanu and V. Lakshmikantham, Equations with Unbounded Delays, a Survey, *Nonlinear Anal. TMA* **4** (1980), 831-877.

[2] C.A. Desoer and M. Vidyassager, *Feedback Systems: Input-Output Properties,* Academic Press, New York, 1975.

[3] S. Elaydi, Stability of Volterra Difference Equations of Convolution Type, *Dynamical Systems (eds. Liao et al)*, World Scientific, Singapore, 1993, pp. 66-73.

[4] S. Elaydi, Periodicity and Stability of Linear Volterra Difference Systems, *J. Math. Anal. Appl.* **181** (1994), 483-492.

[5] S. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag 1995 (to appear).

[6] Y. Hino and S. Murakami, Stability Properties of Linear Volterra Equations, *J. Differential Equations* **89** (1991), 121-137.

[7] Y. Hino and S. Murakami, Total Stability and Uniform Asymptotic Stability for

Linear Volterra Equations, *J. London Math. Soc.* **(2) 43** (1991), 305-312.

[8] S. Murakami, Exponential Asymptotic Stability for Scalar Linear Volterra Equations, *Differential and Integral Equations* **4** (1991), 519-525.