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Stable ROW-Type Weak Scheme for Stochastic Differential Equations

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Abstract

A stable weak scheme of ROW-type is proposed for stochastic differential equation so as to reserve its stability in the mean square sense and to be of order 2. Desirable restrictions of numerical scheme are investigated to derive a specified one. Through some numerical experiments for the scheme, the stability property and the convergence order are verified.

1 Introduction

Numerical method is one of effective means to obtain the information about stochastic phenomena which analytically unsolvable stochastic differential equations (SDEs) present. Among such methods weak schemes are often used to get approximations of sample characteristics of solutions of SDEs. So far many weak methods have been proposed in the literature (for example, [3, 9, 10]).

On the other hand the importance of stability consideration on numerical schemes for SDEs has been acknowledged similarly as in the case of ordinary differential equations (ODEs) to avoid explosion of numerical solution. Several concepts of analytical stability for SDEs are considered ([1, 5]), while more numerical stability concepts are treated due to many different test equations with additive or multiplicative noise ([6, 8]).

However for SDEs with multiplicative noise we can find a very few weak schemes which have the stability property corressponding to A-stability in ODEs ([2, 8]). Therefore the followings are our goal of the features of a new scheme proposed in the present paper.

1) The scheme is ROW-type of weak order 2.
2) For homogeneous d-dimensional linear SDEs with the diffusion coefficient matrix having only real eigenvalues,
   i) the scheme is asymptotically stable in the mean-square sense with any step-size if the underlying SDEs are asymptotically stable, and
   ii) exactly reserves the instability with sufficiently small step-size if the underlying SDEs are not stable.
3) The scheme is A-stable as for ODEs if the diffusion coefficient matrix vanishes.

We will give numerical experiments for the scheme and show that the scheme works very well with respect to the stability as well as the convergence order.

2 Preliminaries

In this section we will introduce some notations and concepts for later use.

Suppose that

$$dz(t) = \sum_{j=0}^{p} g^j(t, z(t))dw^j(t), \qquad z(t_0) = z_0 \text{ (fixed)}, \qquad t \geq t_0, \tag{2.1}$$

is a d-dimensional SDE which has a unique solution. The sense of the stochastic differential will be specified later. Here $dw^0(t) \equiv dt$ and w^j $(j = 1, \ldots, p)$ stand for mutually independent Wiener processes. Furthermore we assume g^j $(j = 0, 1, \ldots, p)$ are continuous with respect to t, and satisfy

$$g^j(t, \mathbf{o}) = \mathbf{o} \qquad \text{for all} \quad t \geq t_0,$$

so that $z(t) \equiv \mathbf{o}$ is the solution of (2.1) with initial value $z_0 = \mathbf{o}$. We will call this trivial solution as the *equilibrium position*. In the sequel we generally assume $z(t) \in C^d$ for convenience' sake.

The following concepts can be found in [1].

Definition 2.1 (m.s. stability, a.m.s. stability) *The equilibrium position is said to be stable in the mean-square sense (m.s. stable, in short) if, for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$\sup_{t_0 \leq t < \infty} E\left[\|z(t)\|^2\right] \leq \varepsilon \qquad \text{for} \quad \|z_0\| \leq \delta,$$

where $\|z(t)\|$ *stands for the norm* $(z(t)^* z(t))^{1/2}$ *(the mark* * *means the conjugate transposition). Further if*

$$\lim_{t \to \infty} E\left[\|z(t)\|^2\right] = 0 \qquad \text{for all } z_0 \text{ in a neighborhood of } \mathbf{o},$$

the equilibrium position is said to be asymptotically stable in the mean-square sense (a.m.s. stable).

When $E\left[\|z(t)\|^2\right]$ is replaced by $\|E\left[z(t)z(t)^*\right]\| = \left(\sum_{i,j=1}^{d} \left|(E\left[z(t)z(t)^*\right])_{ij}\right|^2\right)^{1/2}$ in Definition 2.1, the (asymptotical) stability in the mean-square sense turns to the (asymptotical) stability of the second moment $E\left[z(t)z(t)^*\right]$.

Let $C_P^l(C^d, C)$ be the totality of l times continuously differentiable C-valued functions on C^d, all of whose partial derivatives of order less than or equal to l have polynomial growth. Let N be a natural number, and for an end time $T > 0$, set the step-size $\Delta t = (T - t_0)/N$ and the step-point $t_n = t_0 + n\Delta t$ $(n = 0, 1, \ldots, N)$. Denote by $z(t_n)$ and z_n the exact and approximate solution of SDE, respectively, at time t_n.

Definition 2.2 (weak order) *([10]) A time discrete approximation* z_N *is said to converge weakly with order* q *at time* t_N *as* $\Delta t \to 0$ *if for each* $G \in C_P^{2(q+1)}(C^d, C)$ *there exists a positive constant* C *independent of* Δt *and satisfying*

$$\mid E\left[G(z(t_N))\right] - E\left[G(z_N)\right] \mid = C\Delta t^q.$$

We will restrict ourselves with autonomous SDEs with $p = 1$ and omit the superscripts with respect to the Wiener processes unless there are ambiguities. When (2.1) is interpreted as an

Itô-type, the following explicit Runge-Kutta scheme proposed by PLATEN (p.485 in [10]) is known.

$$\begin{aligned} z_{n+1} &= z_n + \frac{1}{2}(g^0(\hat{z}_n) + g^0(z_n))\Delta t \\ &+\frac{1}{4}(g^1(z_n^+) + g^1(z_n^-) + 2g^1(z_n))\Delta w_n \\ &+\frac{1}{4}(g^1(z_n^+) - g^1(z_n^-))\{(\Delta w_n)^2 - \Delta t\}\Delta t^{-\frac{1}{2}}, \\ \hat{z}_n &= z_n + g^0(z_n)\Delta t + g^1(z_n)\Delta w_n, \\ z^\pm &= z_n + g^0(z_n)\Delta t \pm g^1(z_n)\sqrt{\Delta t}. \end{aligned} \tag{2.2}$$

Here, Δw_n, which stands for the difference $w(t_{n+1}) - w(t_n)$, is realized by the pseudo-random number whose expectation and covariance are 0 and Δt, respectively. This scheme is proved to be of weak order 2.

The numerical counterpart of the a.m.s. concept is given in [2] as follows.

Definition 2.3 (a.m.s. stability for numerical scheme) *A numerical scheme is said to be asymptotically stable in the mean-square sense (a.m.s. stable) with a step-size Δt if the numerical scheme applied to the SDE, in which the equilibrium position is a.m.s. stable, satisfies the following equation for the step-size.*

$$\lim_{n\to\infty} E\left[\|z_n\|^2\right] = 0.$$

3 Test equations for numerical stability

We will introduce a linear d-dimensional system of SDEs and a linear scalar SDE both with multiplicative noise. These equations devoted to stability analysis have the common necessary and sufficient condition for the a.m.s. stability of their equilibrium positions.

Consider the following homogeneous linear Itô-type SDEs in $\boldsymbol{R}^d$

$$d\boldsymbol{x}(t) = A\boldsymbol{x}(t)dt + B\boldsymbol{x}(t)dw(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \text{ (fixed } \boldsymbol{R}^d\text{-value)}, \qquad t \geq t_0, \tag{3.1}$$

provided that $d \times d$ real matrices A and B are *simultaneously diagonalizable.* The second moment $\boldsymbol{P}(t) = E[\boldsymbol{x}(t)\boldsymbol{x}^T(t)]$ of the solution satisfies the equation ([1])

$$\frac{d\boldsymbol{P}}{dt}(t) = A\boldsymbol{P}(t) + \boldsymbol{P}(t)A^T + B\boldsymbol{P}(t)B^T, \qquad \boldsymbol{P}(t_0) = \boldsymbol{x}_0\boldsymbol{x}_0^T. \tag{3.2}$$

The above assumption implies the diagonalizing matrix T to give the identities

$$\boldsymbol{\Lambda} = T^{-1}AT, \qquad \boldsymbol{\Gamma} = T^{-1}BT,$$

where $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ stand for the diagonal matrices

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1(A) & & 0 \\ & \ddots & \\ 0 & & \lambda_d(A) \end{bmatrix}, \qquad \boldsymbol{\Gamma} = \begin{bmatrix} \lambda_1(B) & & 0 \\ & \ddots & \\ 0 & & \lambda_d(B) \end{bmatrix},$$

and these diagonal components $\lambda_i(A)$ or $\lambda_i(B)$ $(i = 1, \ldots, d)$ denote eigenvalues of A or B, respectively. Of course, we assume that $\lambda_i(A)$, $\lambda_i(B) \in \boldsymbol{C}$ $(i = 1, \ldots, d)$.

Let $\boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t)$. Note that $\boldsymbol{x}(t) \in \boldsymbol{R}^d$, but $\boldsymbol{z}(t) \in \boldsymbol{C}^d, \boldsymbol{T} \in \boldsymbol{C}^{d\times d}$. Putting $\boldsymbol{Q}(t) = E[\boldsymbol{z}(t)\boldsymbol{z}^*(t)]$, we have

$$\boldsymbol{P}(t) = \boldsymbol{T}\boldsymbol{Q}(t)\boldsymbol{T}^*. \tag{3.3}$$

Substitution of (3.3) into (3.2) and simplification yield

$$\frac{d\boldsymbol{Q}}{dt}(t) = \boldsymbol{\Lambda}\boldsymbol{Q}(t) + \boldsymbol{Q}(t)\boldsymbol{\Lambda}^* + \boldsymbol{\Gamma}\boldsymbol{Q}(t)\boldsymbol{\Gamma}^*, \qquad \boldsymbol{Q}(t_0) = (\boldsymbol{T}^{-1}\boldsymbol{x}_0)(\boldsymbol{T}^{-1}\boldsymbol{x}_0)^*.$$

Specifying the components of $\boldsymbol{Q}(t)$ by

$$\boldsymbol{Q}(t) = \begin{bmatrix} q_{11}(t) & \cdots & q_{1d}(t) \\ \vdots & & \vdots \\ q_{d1}(t) & \cdots & q_{dd}(t) \end{bmatrix},$$

we can get

$$\frac{dq_{ii}}{dt}(t) = (2\Re(\lambda_i(\boldsymbol{A})) + |\lambda_i(\boldsymbol{B})|^2) q_{ii}(t), \qquad q_{ii}(t_0) = ((\boldsymbol{T}^{-1}\boldsymbol{x}_0)(\boldsymbol{T}^{-1}\boldsymbol{x}_0)^*)_{ii}.$$

The relationships $\|\boldsymbol{P}(t)\| \leq E[\|\boldsymbol{x}(t)\|^2]$ and $E[\|\boldsymbol{x}(t)\|^2] = \text{trace}\, \boldsymbol{P}(t)$ show that the m.s. stability is equivalent to the stability of the second moment ([1]). It is easy to verify that the fact is also true in the complex case. That is, the stability of $q_{ii}(t)$ is equivalent to the stability of $\boldsymbol{Q}(t)$. Therefore from (3.3) the stability of $q_{ii}(t)$ is equivalent to the m.s. stability in (3.1). This equivalence also holds for the a.m.s. stability.

Thus we can conclude that the a.m.s. stability of the equilibrium position of (3.1) is equivalent to the following inequalities.

$$2\Re(\lambda_i(\boldsymbol{A})) + |\lambda_i(\boldsymbol{B})|^2 < 0, \qquad (i = 1, \ldots, d). \tag{3.4}$$

Denote by j the index for which $2\Re(\lambda_j(\boldsymbol{A})) + |\lambda_j(\boldsymbol{B})|^2 = \max_{1\leq i\leq d}(2\Re(\lambda_i(\boldsymbol{A})) + |\lambda_i(\boldsymbol{B})|^2)$ holds, and further introduce the notations

$$\tilde{\lambda} = \Re(\lambda_j(\boldsymbol{A})) \quad \text{and} \quad \tilde{\sigma} = \pm|\lambda_j(\boldsymbol{B})|, \tag{3.5}$$

which leads an equivalent condition

$$2\tilde{\lambda} + \tilde{\sigma}^2 < 0 \tag{3.6}$$

of (3.4).

Assume that

$$\lambda = \lambda_j(\boldsymbol{A}), \qquad \sigma = \pm\lambda_j(\boldsymbol{B}).$$

The necessary and sufficient condition of the a.m.s. stability for the equilibrium position of scalar Itô-type equation

$$dz(t) = \lambda z(t)dt + \sigma z(t)dw(t), \qquad z(t_0) = z_0, \qquad z(t) \in \boldsymbol{C}. \tag{3.7}$$

turns out to be consistent with (3.6). Therefore the a.m.s. stability of (3.7) is equivalent to that of (3.1).

Equation (3.7) is interpreted to Stratonovich-type equation

$$dz(t) = \hat{\lambda} z(t)dt + \sigma z(t) \circ dw(t), \qquad z(t_0) = z_0, \qquad z(t) \in \boldsymbol{C}, \tag{3.8}$$

where $\hat{\lambda} = \lambda - \dfrac{\sigma^2}{2}$. Then the condition (3.6) can be written in the form

$$\Re(\hat{\lambda}) + \{\Re(\sigma)\}^2 < 0. \tag{3.9}$$

We will investigate numerical stability through these test equations in the following sections.

4 Significance of a.m.s. stability in numerical schemes

Applying Platen's scheme (2.2) to Eq. (3.7), we obtain

$$z_{n+1} = \left\{1 + \lambda\Delta t + \sigma\Delta w_n + \lambda\sigma\Delta t\Delta w_n + \frac{1}{2}\sigma^2\{\Delta w_n^2 - \Delta t\} + \frac{1}{2}\lambda^2\Delta t^2\right\} z_n.$$

Define the amplification factor as

$$P(\Delta t, \Delta w_n) \equiv 1 + \lambda\Delta t + \sigma\Delta w_n + \lambda\sigma\Delta t\Delta w_n + \frac{1}{2}\sigma^2\{\Delta w_n^2 - \Delta t\} + \frac{1}{2}\lambda^2\Delta t^2. \tag{4.1}$$

The expectation of the squared $P(\Delta t, \Delta w_n)$ is given by

$$E[| P(\Delta t, \Delta w_n) |^2] = 1 + (2\tilde{\lambda} + \tilde{\sigma}^2)\Delta t + 2(\tilde{\lambda} + \frac{1}{2}\tilde{\sigma}^2)^2\Delta t^2 \mid \lambda \mid^2 (\tilde{\lambda} + \tilde{\sigma}^2)\Delta t^3 + \frac{1}{4} \mid \lambda \mid^4 \Delta t^4.$$

Therefore, the necessary and sufficient condition for a.m.s. stability of scheme (2.2) is given by

$$(2\tilde{\lambda} + \tilde{\sigma}^2)\Delta t + 2(\tilde{\lambda} + \frac{1}{2}\tilde{\sigma}^2)^2\Delta t^2 + \mid \lambda \mid^2 (\tilde{\lambda} + \tilde{\sigma}^2)\Delta t^3 + \frac{1}{4} \mid \lambda \mid^4 \Delta t^4 < 0.$$

Here introduction of notations $X = \tilde{\lambda}\Delta t$, $Y = \Im(\lambda_j(A))\Delta t$ and $W = \tilde{\sigma}^2\Delta t$ yields

$$2X + W + 2(X + \frac{1}{2}W)^2 + (X^2 + Y^2)(X + W) + \frac{1}{4}(X^2 + Y^2)^2 < 0. \tag{4.2}$$

Denote the left-hand side term by $MS_p(X, Y, W)$. On the other hand the necessary and sufficient condition, in which the equilibrium position of (3.7) is a.m.s. stable, can be expressed with

$$2X + W < 0. \tag{4.3}$$

This means that under the restriction of (4.3), the triplet (X, Y, W) satisfying (4.2) gives the a.m.s. stability criterion of the scheme (2.2) when it is applied to the test equation (3.7).

Second, we will consider the stability with respect to expectation. From (4.1), we have

$$E[P(\Delta t, \Delta w_n)] = 1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2.$$

Therefore the condition, in which the expectation $E[z_n]$ converges to 0 as $n \to \infty$, is

$$\mid 1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2 \mid < 1.$$

We will call it the stability in expectation (e. stability, in short). Define

$$E_p(X, Y) \equiv \mid 1 + (X + \mathrm{i}Y) + \frac{1}{2}(X + \mathrm{i}Y)^2 \mid,$$

then the necessary and sufficient condition, in which the expectation $E[z(t)]$ of the solution of (3.7) converges 0 as $t \to \infty$, can be expressed with $\tilde{\lambda} < 0$, which impies $X < 0$.

Under these circumstances, we will show numerically that the a.m.s. stability is significant. The 2-dimensional equation

$$dx(t) = \begin{bmatrix} 0 & 1 \\ \beta & \gamma \end{bmatrix} x(t)dt + \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} x(t)dw, \qquad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{4.4}$$

serves the numerical test. The parameters are specified as $\alpha = 3$, $\beta = -100$, $\gamma = -25$, $x_1(t_0) = 1$, $x_2(t_0) = 0$. Calculation shows

$$X = \frac{\gamma + \sqrt{\gamma^2 + 4\beta}}{2}\Delta t = -5\Delta t, \qquad Y = 0, \qquad W = \alpha^2 \Delta t = 9\Delta t.$$

Since

$$2X + W = -\Delta t < 0,$$

the equilibrium position of (4.4) is a.m.s. stable.

The region of e. stability $\{(X,Y); E_p(X,Y) < 1\}$ is shown as the shadowed zone in Fig. 4.1. In the Figure points on the X-axis correspond to those for $\Delta t = 2^{-6}, 2^{-5}, \ldots, 2^{-1}$, respectively, in the order closer to the origin. The evaluated $MS_p(X,Y,W)$ are listed in Table 4.1 for every Δt.

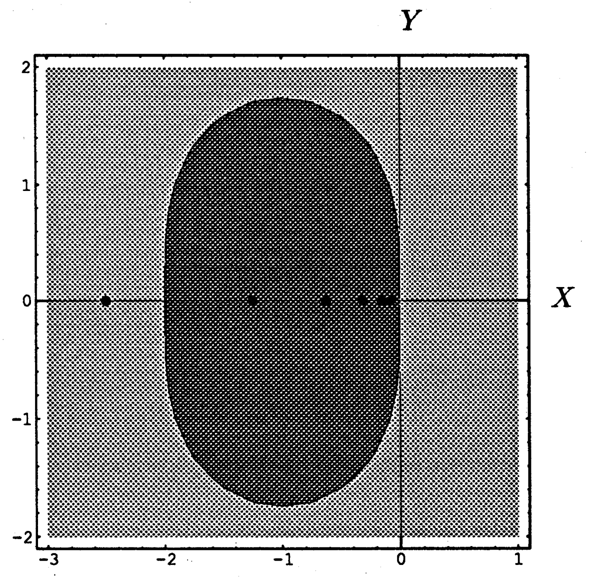


Figure 4.1: The region of $E_p < 1$

$\log_2 \Delta t$	-1	-2	-3	-4	-5	-6
$\|\langle x_N \rangle\|$	8.9×10^{16}	2.0×10^{19}	1.2×10^{6}	0	0	0
$\langle \|x_N\|^2 \rangle$	6.7×10^{34}	3.6×10^{41}	1.6×10^{16}	0	0	0
$MS_p(X,Y,W)$	22	2.0	0.12	-0.034	-0.028	-0.015

Table 4.1: Arithmetic means and second moments at $T = 5$ by scheme (2.2)

Since the two points corresponding to $\Delta t = 2^{-3}$ and 2^{-2} on Fig. 4.1 satisfy the condition $E_p(X,Y) < 1$, one might expect that the arithmetic mean $\langle x_N \rangle$ of their numerical solutions is close to $\mathbf{o}$ as N becomes large. Nevertheless, it explodes numerically as observed in Table 4.1. This phenomenon can be interpreted as follows: Although the condition of e. stability is satisfied, the criterion $MS_p(X,Y,W) < 0$ does not hold. Therefore the arithmetic mean square $\langle \|x_N\|^2 \rangle$ of the numerical solutions explodes for $\Delta t = 2^{-3}$ and 2^{-2}. The arithmetic mean square being large means the existence of the paths that are far from 0. In such a case the arithmetic mean can hardly be close to 0 because the computer simulations have only a finite number of the paths. Consequently even if one wants to know only arithmetic means, one cannot ignore the m.s. stability. This example suggests the significance of a.m.s. stability in numerical schemes.

Finally we note that the set $\{(X, Y, W)\}$ satisfying $MS_p(X, Y, W)$ becomes to a domain in the 3-dim space, which can be called the domain of a.m.s. stability. We show in Fig. 4.2 the cross-sections of the domain of the scheme (2.2) with several planes of Y = const. The region satisfying $2X + W < 0$ and $W \geq 0$ means (3.7) is a.m.s. stable. The region is the part below the straight line $W = 2X$ in Fig. 4.2. The figure tells that the scheme cannot be a.m.s. stable for any X and W when $Y = 2$.

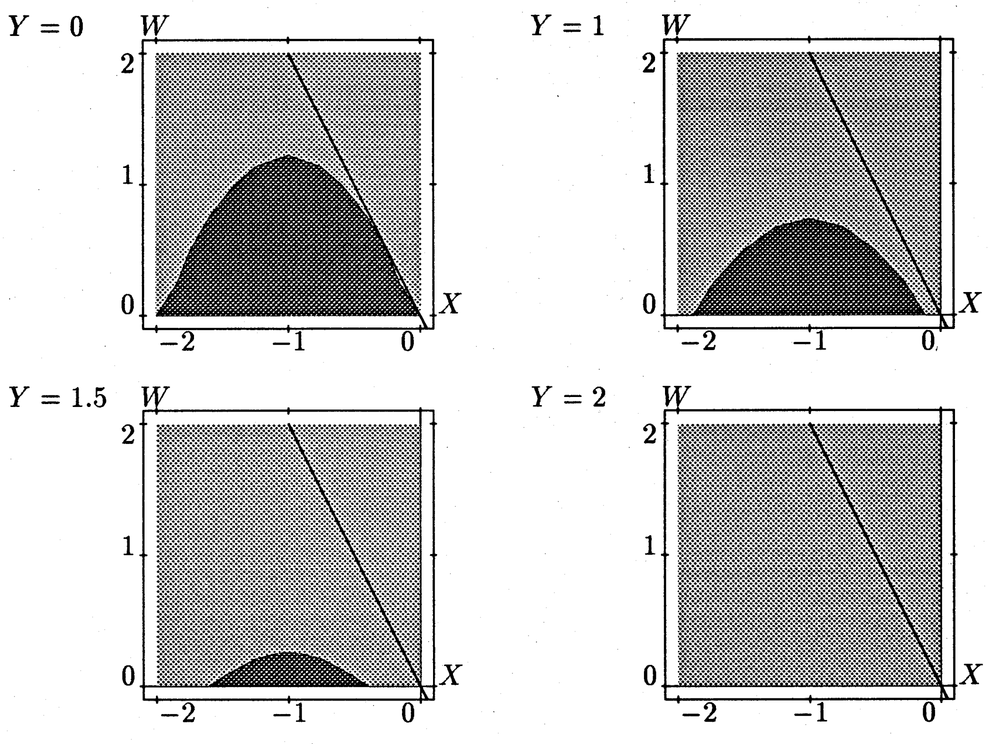


Figure 4.2: The cross-sections of the domain of a.m.s. stability

5 Asymptotically stable scheme in the mean-square sense

5.1 Amplification factor of ROW-type method

Taking the observations in the previous section into account, we shall proceed to obtain a series of numerical schemes possessing better stability property. Here we focus on the ROW-type method of m stages ([12]) given by

$$
\begin{aligned}
\boldsymbol{Y}_i &= \left[I - \gamma \Delta w_n^0 \frac{\partial g^0}{\partial z}(\boldsymbol{z}_n) \right]^{-1} \\
&\times \left[\sum_{j=0}^{1} a_i^j \Delta w_n^j g^j(\boldsymbol{z}_n + \sum_{l=1}^{i-1} \alpha_{il} \boldsymbol{Y}_l) + \sum_{l=1}^{i-1} \left\{ \sum_{j=0}^{1} \gamma_{il}^j \Delta w_n^j \frac{\partial g^j}{\partial z}(\boldsymbol{z}_n) \boldsymbol{Y}_l \right\} \right], \qquad (5.1) \\
\boldsymbol{z}_{n+1} &= \boldsymbol{z}_n + \sum_{j=1}^{m} c_j \boldsymbol{Y}_j
\end{aligned}
$$

for SDE (2.1) interpreted in the Stratonovich sense. Here $\Delta w_n^0 \equiv \Delta t$.

Consider the scalar linear Stratonovich-type SDE

$$
dz(t) = \sum_{j=0}^{1} \hat{f}^j z(t) \circ dw^j(t), \qquad z(t_0) = z_0. \tag{5.2}
$$

Applicaton of (5.1) to (5.2) implies the stage values as

$$
Y_i = \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j [a_i^j z_n + \sum_{l=1}^{i} \mu_{il}^j Y_l], \tag{5.3}
$$

where

$$
\mu_{il}^j = \begin{cases} a_{ii}^j \alpha_{il} + \gamma_{il}^j & (i > l), \\ \gamma & (i = l,\ j = 0), \\ 0 & (i = l,\ j = 1). \end{cases}
$$

Hence introducing the notations

$$
\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_m]^T, \qquad \boldsymbol{a}^j = [a_1^j, a_2^j, \ldots, a_m^j]^T, \qquad \boldsymbol{c} = [c_1, c_2, \ldots, c_m]^T
$$

and

$$
U_j = \begin{bmatrix} \mu_{11}^j & & & 0 \\ \mu_{21}^j & \mu_{22}^j & & \\ \vdots & \vdots & \ddots & \\ \mu_{m1}^j & \mu_{m2}^j & \cdots & \mu_{mm}^j \end{bmatrix},
$$

we obtain

$$
\boldsymbol{Y} = \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j [\boldsymbol{a}^j z_n + U_j \boldsymbol{Y}], \tag{5.4}
$$

$$
-\boldsymbol{c}^T \boldsymbol{Y} + z_{n+1} = z_n. \tag{5.5}
$$

Composition of (5.4) and (5.5) yields

$$
\begin{bmatrix} I - \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j U_j & \boldsymbol{o} \\ -\boldsymbol{c}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{Y} \\ z_{n+1} \end{bmatrix} = z_n \begin{bmatrix} \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j \boldsymbol{a}^j \\ 1 \end{bmatrix} \tag{5.6}
$$

Calculation by Cramer's rule leads to the identify

$$z_{n+1} = \frac{\det\left[I - \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j U_j + \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j \boldsymbol{a}^j \boldsymbol{c}^T\right]}{\det\left[I - \sum_{j=0}^{1} \Delta w_n^j \hat{f}^j U_j\right]} z_n.$$

Put $\Delta W_j = \hat{f}^j \Delta w_n^j$ and define the amplification factor as

$$R(\Delta W_0, \Delta W_1) \equiv \frac{\det\left[I - \sum_{j=0}^{1} \Delta W_j U_j + \sum_{j=0}^{1} \Delta W_j \boldsymbol{a}^j \boldsymbol{c}^T\right]}{\det\left[I - \sum_{j=0}^{1} \Delta W_j U_j\right]}.$$

In what follows, we restrict the method (5.1) in the case of $m = 4$. Then the denominator of $R(\Delta W_0, \Delta W_1)$ is factorized as

$$\det\left[I - \sum_{j=0}^{1} \Delta W_j U_j\right] = D^4, \tag{5.7}$$

where $D = 1 - \gamma \Delta W_0$. On the other hand, by calculation using the condition (Appendix (A)) in which the method (5.1) is of weak order 2, the numerator becomes

$$\begin{aligned} &\det\left[I - \sum_{j=0}^{1} \Delta W_j U_j + \sum_{j=0}^{1} \Delta W_j \boldsymbol{a}^j \boldsymbol{c}^T\right] \\ = \quad & D^4 + \left(\sum_{j=0}^{1} \Delta W_j\right) D^3 + \left\{(\frac{1}{2} - \gamma)\Delta W_0^2 + (1-\gamma)\Delta W_0 \Delta W_1 + \frac{1}{2}\Delta W_1^2\right\} D^2 \qquad (5.8) \\ & + \left\{T_{30}\Delta W_0^3 + T_{21}\Delta W_0^2 \Delta W_1 + (\frac{1}{2} - \gamma)\Delta W_0 \Delta W_1^2 + \frac{1}{6}\Delta W_1^3\right\} D \\ & + T_{40}\Delta W_0^4 + T_{31}\Delta W_0^3 \Delta W_1 + T_{22}\Delta W_0^2 \Delta W_1^2 + T_{13}\Delta W_0 \Delta W_1^3 + \frac{1}{24}\Delta W_1^4. \end{aligned}$$

Here T_{ij} means a polynomial of the formula parameters.

5.2 Desired restrictions on ROW-type method

We shall impose the following four conditions on the ROW-type method. (Refer back to the goal listed in **Introduction.**)

Assuming that $\hat{f}^1 = 0$ and $\Delta W_1 = 0$ hold, we obtain

$$R(\Delta W_0, 0) = 1 + \frac{1}{D}\Delta W_0 + \frac{1}{D^2}(\frac{1}{2} - \gamma)\Delta W_0^2 + \frac{1}{D^3}T_{30}\Delta W_0^3 + \frac{1}{D^4}T_{40}\Delta W_0^4 \tag{5.9}$$

from (5.7) and (5.8). If $\gamma > 0$, $R(\Delta W_0, 0)$ is analytic for $\Re(\Delta W_0) < 0$. Since we assume the equation without stochastic term, the numerical scheme reduces to that for ODEs. Therefore, the necessary and sufficient condition for A-stability of the ODE-case is that for all ΔW_0 satisfying $\Re(\Delta W_0) = 0$ and any $\gamma > 0$ the inequality

$$|R(\Delta W_0, 0)|^2 \leq 1 \tag{5.10}$$

holds ([4]).

On the other hand, we can recognize that in the Fig. 4.2 the whole X-axis never be included in the stability region. This means that Platen's scheme is not A-stable in the ODE-case. In other words, for A-stable methods in the ODE-case, the whole X-axis in the XYW-portrait should be included in the domain of a.m.s. stability. Therefore we impose A-stability in the ODE-case as the first restriction on the ROW-type method.

Assume a combination of the coefficients in the Stratonovich-type SDE (5.2) as $\hat{f}^0 = -\varepsilon$ and $\hat{f}^1 = \varepsilon + i\eta$ for $1 \gg \varepsilon > 0$. The equation turns out to be a.m.s. stable because the condition (3.9) is satisfied. As $\varepsilon \to 0$, we have

$$R(\Delta W_0, \Delta W_1) \to 1 - \frac{1}{2}(\eta \Delta w_n^1)^2 + \frac{1}{24}(\eta \Delta w_n^1)^4 + \{(\eta \Delta w_n^1) - \frac{1}{6}(\eta \Delta w_n^1)^3\}$$

from $\Delta W_0 \to 0$ and $\Delta W_1 \to i\eta \Delta w_n^1$. This yields

$$E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right] \to 1 - \frac{5}{24}\eta^6 \Delta t^3 + \frac{35}{192}\eta^8 \Delta t^4$$

as $\varepsilon \to 0$. Therefore in this case the ROW-scheme (5.1) is impossible to be a.m.s. stable with any Δt . As we have seen above, even if $\hat{f}^0 \in \boldsymbol{R}$, the ROW-schme (5.1) cannot be a.m.s. stable with any Δt provided $\hat{f}^1 \in \boldsymbol{C}$. Thus we restrict (3.8) in the case of $\hat{f}^0 \in \boldsymbol{C}$ and $\hat{f}^1 \in \boldsymbol{R}$, that is, $\hat{f}^0 = \lambda - \frac{\tilde{\sigma}^2}{2}$ and $\hat{f}^1 = \tilde{\sigma}$. (Recall the definition of the quantity $\tilde{\sigma}$ in (3.5).)

We can then represent $E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right]$ as a function with arguments only X, Y and W. We will make the method (5.1) satisfying

$$E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right] < 1 \tag{5.11}$$

under $W \geq 0$ and the condition (4.3). It means that (5.1) is numerically a.m.s. stable with any Δt. When $\gamma > 0$, $E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right]$ is analytic in the left half-plane of X-W for any Y. Therefore (5.11) holds if and only if (5.10) holds and the inequality

$$E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right] \leq 1$$

is satisfied on the extreme line $W = -2X$ of the condition (4.3). Our second restriction is for the above inequality to hold.

The curve which represents the relationship $E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right] = 1$ in the XW-plane is the boundary of the region of a.m.s. stability for (5.1). We will make the scheme (5.1) so that the slope of the boundary curve at the origin $(X, W) = (0, 0)$ is equal to -2. This means that the stability region of (5.1) is consistent with the analytic a.m.s. stability of (3.7) in the neighborhood of the origin. This is our third restriction.

The last restriction is as follows. For $X = 0$ and $W \neq 0$,

$$E\left[\mid R(\Delta W_0, \Delta W_1) \mid^2\right] > 1 \tag{5.12}$$

is desirable because then SDE (3.7) is not stable.

We impose these four restrictions in addition to the conditions for weak order 2 on the ROW-type method (5.1).

5.3 Specification formula parameters

By calculations we find

$$| D |^2 | R(\Delta W_0, 0) |^2 - | D |^2 \qquad (\leq 0 \qquad \text{for } \Re(\Delta W_0) = 0),$$

which is derived from the left-hand side of (5.10), and

$$| D |^2 E \left[| R(\Delta W_0, \Delta W_1) |^2 \right] - | D |^2 \qquad (> 0 \qquad \text{for } X = \Re(\Delta W_0) = 0), \tag{5.13}$$

which is obtained from the left-hand side of (5.12), are polymomials of Δt of degree 8 and their leading coefficients coincide. Denote by l_c the coefficient. For the condition (5.10), l_c should be non-positive. On the other hand, l_c should be non-negative for the condition (5.12). Henceforth we shall take that $l_c = 0$ and the second leading coefficient of (5.13) is positive. Then, for large enough W, (5.12) is satisfied.

Summing up our analyses, we will decide the parameters of the scheme (5.1) so that it possesses four properties described above and is of weak order 2. The order conditions of (5.1) for weak order 2 can be derived according to an analysis previously described in [12] and is listed in Appendix A.

The selection $\gamma = \frac{1}{2}$ and $T_{30} = T_{40} = 0$ suggested from the formulae (A.2), (A.3), (A.4) and (A.5) reduces (5.9) into a simple equation

$$R(\Delta W_0, 0) = 1 + \frac{1}{D} \Delta W_0,$$

which leads the equation

$$| R(\Delta W_0, 0) |^2 - 1 = | \frac{1}{D} |^2 \, 2X.$$

Therefore we can satisfy both (5.10) and $l_c = 0$ when $\gamma = \frac{1}{2}$, because

$$| R(\Delta W_0, 0) |^2 = 1 \qquad \text{if } X = 0.$$

A set of the formula parameter satisfying all of the desired conditions is listed in Appendix B. We call the scheme (5.1) with these parameters by Scheme A. Fig. 5.1 shows the cross-sections (shadowed) of the domain $E\left[| R(\Delta W_0, \Delta W_1) |^2 \right] < 1$ of Scheme A for each Y.

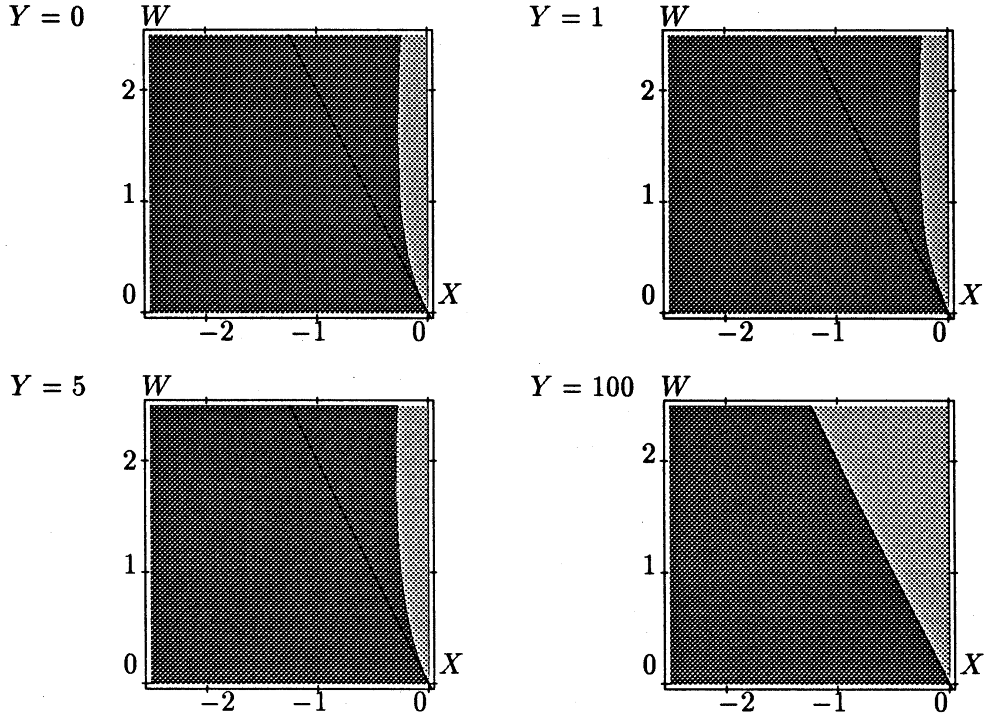


Figure 5.1: The cross-sections of the domain of a.m.s. stability

6 Numerical tests for the new scheme

Solution of the test equation (4.4) with the parameters $\alpha = 3$,$\beta = -100$ and $\gamma = -25$ by Scheme A is listed in Tab. 6.1, which clearly shows that the expectations as well as the mean squares are solved stably.

$\log_2 \Delta t$	-1	-2	-3	-4	-5	-6
$\lVert\langle \boldsymbol{x}_N \rangle\rVert$	0.10	0.0	0.0	0.0	0.0	0.0
$\langle \lVert\boldsymbol{x}_N\rVert^2 \rangle$	0.018	0.0	0.0	0.0	0.0	0.0

Table 6.1: Arithmetic means and second moments by Scheme A

Next we will show a numerical experiment in an oscillatory SDE case. Assume that the parameters $\alpha = 0.5$, $\beta = -37$ and $\gamma = -2$ in the equation (4.4), which imply

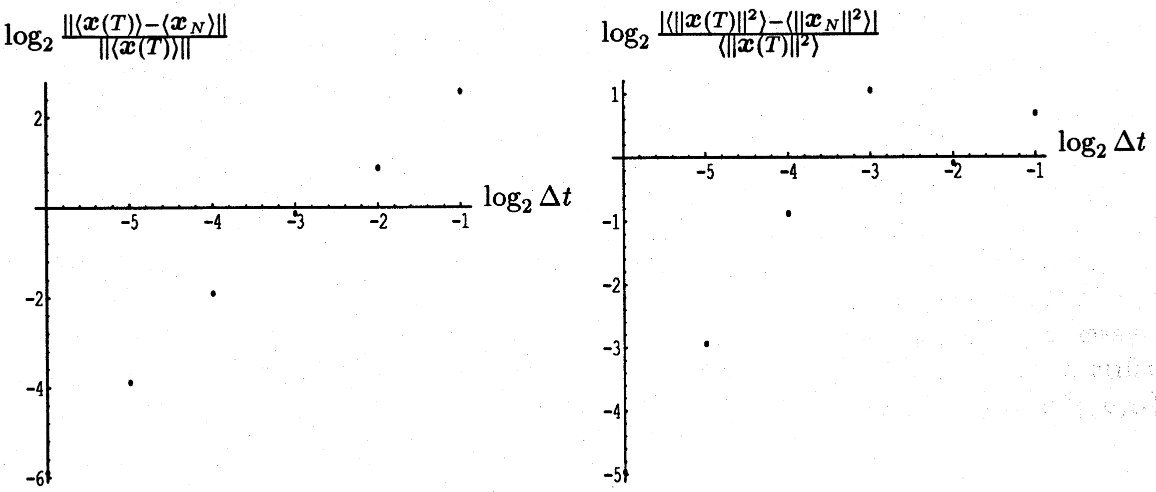
$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ -37 & -2 \end{bmatrix} \boldsymbol{x}(t)dt + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \boldsymbol{x}(t)dw(t), \qquad \boldsymbol{x}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6.1}$$

The eigenvalues of the drift coefficient are $-1 \pm i6$, which yield the inequality

$$2\bar{\lambda} + \bar{\sigma}^2 = 2 \times (-1) + (\frac{1}{2})^2 = -\frac{7}{4} < 0.$$

Thus the equilibrium position of (6.1) is a.m.s. stable.

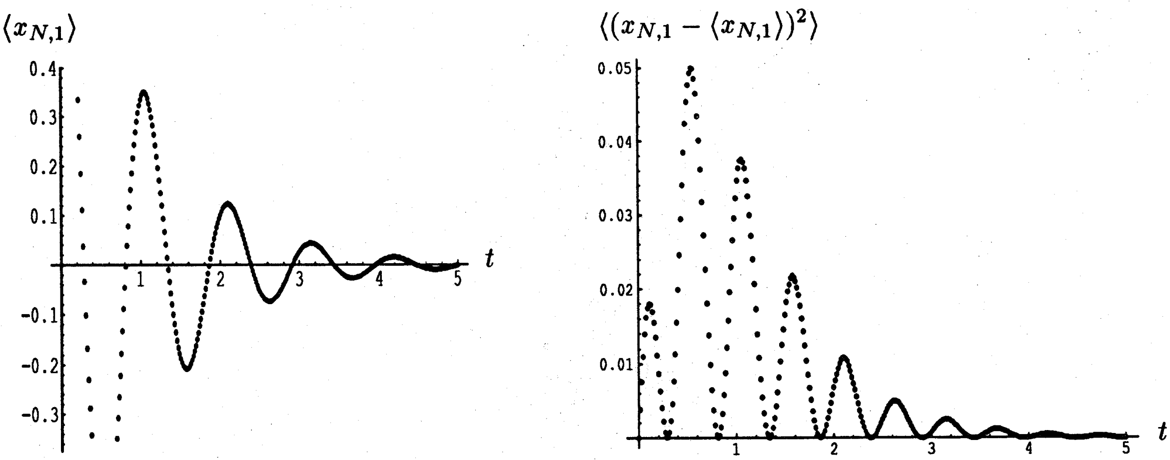
We show in Fig. 6.1 the tendency of the numerical error of Scheme A versus step-size. Here the numerical error is taken as its relative value of the approximate sample characteristics by Scheme A from the exact sample characteristics ([11]). Fig. 6.2 shows the fluctuation of expectation and variance of the first solution element versus time.



The relative error of the expectation

The relative error of the mean square

Figure 6.1: The relative errors of Scheme A at $t = 3$



The fluctuation of the expectation

The fluctuation of the second moment

Figure 6.2: The fluctuation of the expectation and variance of first solution element

7 Concluding remarks

Through the discussion about the numerical stability for SDEs, we raised for numerical schemes several conditions (ref. to them listed in **Introduction**), which were realized with Scheme A of an ROW-type method. Numerical tests confirmed its desirable features.

However, some problems are remained unsolved. First, there are still cases in which the variance of the solution of an SDE must diverge theoretically, but converges numericaly. This phenomenon could be found, for example, when $\alpha = 3$, $\beta = -1/4$ and $\gamma = -3$ in the test equation (4.4) introduced in the paper. This combination of parameters implies that the equation is a.m.s. unstable. Trajectory tracing of the analytical solution with pseudo-random numbers on computer, however, shows convergence. Henceforth the same occurs in the numerical simulation with Scheme A for the equation, too. The reason may be considered that the pseudo-Wiener process does not have paths of big magnitude at the end point of time. The phenomenon suggests that the instability of the numerical scheme may be influenced by pseudo-random numbers. Thus we need to take enough care of the instability of the original SDE even if the numerical result is stable.

Second, we need further study when an eigenvalue of the matrix of diffusion coefficient is a complex number. In the case when the imaginary part vanishes in the stochastic part of the test equation (3.7), it was shown that Scheme A was a.m.s. stable with any Δt if (3.7) was a.m.s. stable (ref. to **5** in the present paper).

However in the case of the complex number in the stochastic part of it, the situation is different. To confirm it, putting $X = \Re(\lambda)\Delta t$ and $Y = \Im\lambda\Delta t$ as Section **4**, denoting by $U = \{\Re(\sigma)\}^2\Delta t$ and $V = \{\Im(\sigma)\}^2\Delta t$, we readily see that the condition (4.3) of a.m.s. stability for (3.7) turns out to

$$X < -\frac{1}{2}U^2 - \frac{1}{2}V^2.$$

We have then four quantities X, Y, U and V to control stability. For $Y = 0$ and $V = 1$, for instance, the region of Scheme A in XU-plane where the condition (5.11) is satisfied is shown in Fig. 7.1 (the shadowed region). The region does not cover the analytical stability region of (3.7) that is below the parabola.

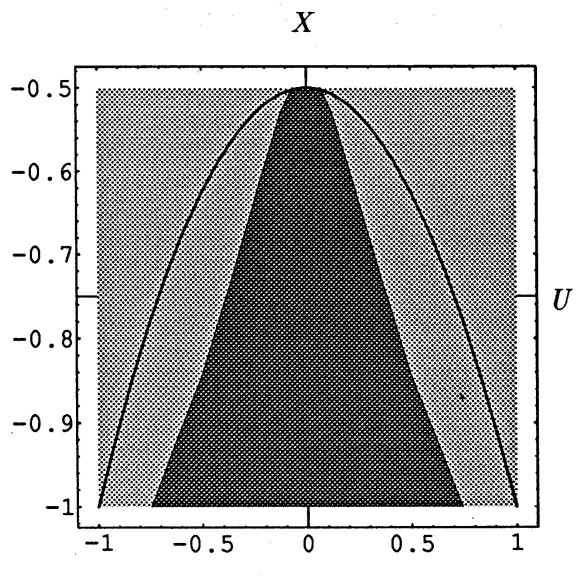


Figure 7.1: The stability region of Scheme A for $Y = 0$ and $V = 1$

Hofmann and Platen [7] try to investigate a way to overcome a similar problem to it, but the schemes they handle with are restricted in lower order weak ones. Refer to [13], too. Therefore a future work still remains to treat higher order weak schemes.

Appendix

A The order conditions for weak order 2

Here we introduce the following notations:

$$\begin{aligned} A_{ij} &= a_i\alpha_{ij} + \gamma^0_{ij}, \qquad B_{ij} = b_i\alpha_{ij} + \gamma^1_{ij}, \\ A_i &= \sum_{j=1}^{i-1} a_j A_{ij}, \qquad B_i = \sum_{j=1}^{i-1} b_j B_{ij}, \\ \bar{B}_i &= \sum_{j=1}^{i-1} b_j \alpha_{ij}. \end{aligned}$$

$$b_1c_1 + b_2c_2 + b_3c_3 + b_4c_4 = 1$$

$$a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4 = 1$$

$$B_2c_2 + B_3c_3 + B_4c_4 = \frac{1}{2}$$

$$a_1B_{21}c_2 + (a_1B_{31} + a_2B_{32})c_3 + (a_1B_{41} + a_2B_{42} + a_3B_{43})c_4 = \frac{1}{2}$$

$$A_{21}b_1c_2 + (A_{31}b_1 + A_{32}b_2)c_3 + (A_{41}b_1 + A_{42}b_2 + A_{43}b_3)c_4 = \frac{1}{2} - \gamma \qquad \text{(A.1)}$$

$$A_2c_2 + A_3c_3 + A_4c_4 = \frac{1}{2} - \gamma \qquad \text{(A.2)}$$

$$\bar{B}_2^2b_2c_2 + \bar{B}_3^2b_3c_3 + \bar{B}_4^2b_4c_4 = \frac{1}{3}$$

$$a_1\bar{B}_2\alpha_{21}b_2c_2 + \bar{B}_3(a_1\alpha_{31} + a_2\alpha_{32})b_3c_3 + \bar{B}_4(a_1\alpha_{41} + a_2\alpha_{42} + a_3\alpha_{43})b_4c_4 = \frac{1}{4}$$

$$\alpha_{21}^2a_2b_1^2c_2 + a_3(\alpha_{31}b_1 + \alpha_{32}b_2)^2c_3 + a_4(\alpha_{41}b_1 + \alpha_{42}b_2 + \alpha_{43}b_3)^2c_4 = \frac{1}{2}$$

$$B_2B_{32}c_3 + (B_2B_{42} + B_3B_{43})c_4 = \frac{1}{6}$$

$$A_{21}b_1B_{32}c_3 + (A_{32}b_2B_{43} + b_1(A_{21}B_{42} + A_{31}B_{43}))c_4 = -\frac{\gamma}{2} \qquad \text{(A.3)}$$

$$a_1B_{21}B_{32}c_3 + (a_2B_{32}B_{43} + a_1(B_{21}B_{42} + B_{31}B_{43}))c_4 = \frac{1}{4}$$

$$A_{32}b_1B_{21}c_3 + (b_1(A_{42}B_{21} + A_{43}B_{31}) + A_{43}b_2B_{32})c_4 = \frac{1}{4} - \frac{\gamma}{2} \qquad \text{(A.4)}$$

$$\bar{B}_2^3b_2c_2 + \bar{B}_3^3b_3c_3 + \bar{B}_4^3b_4c_4 = \frac{1}{4}$$

$$\bar{B}_3\alpha_{32}B_2b_3c_3 + \bar{B}_4(\alpha_{42}B_2 + \alpha_{43}B_3)b_4c_4 = \frac{1}{8}$$

$$B_2B_{32}B_{43}c_4 = \frac{1}{24}$$

$$\bar{B}_2^2b_2B_{32}c_3 + (\bar{B}_2^2b_2B_{42} + \bar{B}_3^2b_3B_{43})c_4 = \frac{1}{12}$$

The above conditons are derived through the rooted tree analysis ([12]).

B The parameter set of Scheme A

$$
\begin{aligned}
a_1 &= -1+\sqrt{3}, \quad a_2 = 7-\frac{5\sqrt{3}}{2}, \quad a_3 = \frac{11}{2}-\frac{5\sqrt{3}}{2}, \quad a_4 = \frac{4}{3}-\frac{1}{\sqrt{3}}, \\
b_1 &= 1, \quad b_2 = \frac{1}{2}, \quad b_3 = \frac{1}{2}, \quad b_4 = 1, \\
c_1 &= \frac{1}{6}, \quad c_2 = \frac{4(-7+4\sqrt{3})}{27}, \quad c_3 = \frac{37-16\sqrt{3}}{27}, \quad c_4 = \frac{2}{3}, \\
\alpha_{21} &= 1, \quad \alpha_{31} = \frac{1979+112(3^{\frac{3}{2}})}{1202}, \quad \alpha_{32} = -\frac{777}{601}-\frac{112(3^{\frac{3}{2}})}{601}, \\
\alpha_{41} &= -\frac{1}{4}, \quad \alpha_{42} = 1, \quad \alpha_{43} = \frac{1}{2}, \quad \gamma = \frac{1}{2}, \\
\gamma_{21}^{0} &= \frac{-29+11\sqrt{3}}{8}, \quad \gamma_{31}^{0} = \frac{-20335+6199\sqrt{3}}{2404}, \quad \gamma_{32}^{0} = \frac{21(167-3^{\frac{5}{2}})}{1202}, \\
\gamma_{41}^{0} &= \frac{70-37\sqrt{3}}{6}, \quad \gamma_{42}^{0} = \frac{-4+\sqrt{3}}{3}, \quad \gamma_{43}^{0} = \frac{-4+\sqrt{3}}{6}. \\
\gamma_{21}^{1} &= \frac{1}{4}, \quad \gamma_{31}^{1} = \frac{73-112(3^{\frac{5}{2}})}{7212}, \quad \gamma_{32}^{1} = \frac{3533+112(3^{\frac{5}{2}})}{3606} \\
\gamma_{41}^{1} &= \frac{71}{216}+\frac{2}{3^{\frac{5}{2}}}, \quad \gamma_{42}^{1} = \frac{-91+16\sqrt{3}}{54}, \quad \gamma_{43}^{1} = -\frac{1}{4}
\end{aligned}
$$

References

[1] L. Arnold. *Stochastic Differential Equations: Theory and Applications.* John Wiley & Sons, New York, 1974.

[2] S.S. Artem'ev. The stability of numerical methods for solving stochastic differential equations. *Bulletin of the Novosibirsk Computing Center*, **2**, 1993.

[3] T.A. Averina and S.S. Artem'ev. A new family of numerical methods for solving stochastic differential equations. *Soviet Math. Dokl.*, **33**:736–738, 1986.

[4] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Systems.* Springer-Verlag, Berlin, 1991.

[5] R.Z. Has'minskiĭ. *Stochastic Stability of Differential Equations.* Sijthoff & Noordhoff International Publishers B.V., Netherlands, 1980.

[6] D.B. Hernandez and R. Spigler. Convergence and stability of implicit runge-kutta method for systems with multiplicative noise. *BIT*, **33**:654–669, 1993.

[7] N. Hofmann and E. Platen. Stability of superimplicit numerical methods for stochastic differential equations. Technical Report SRR 015-94, The Australian National University, 1994.

[8] N. Hofmann and E. Platen. Stability of weak numerical schemes for stochastic differential equations. *Computers Math. Applic.*, **28**:45–57, 1994.

[9] J.R. Klauder and W.P. Petersen. Numerical integration of multiplicative-noise stochastic differential equations. *SIAM J. Numer. Anal.*, **22**:1153–1166, 1985.

[10] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, Berlin, 1992.

[11] Y. Komori, Y. Saito, and T. Mitsui. Some issue in discrete approximate solution for stochastic differential equations. *Computers Math. Applic.*, **28**:269–278, 1994.

[12] Y. Komori, H. Sugiura, and T. Mitsui. Rooted tree analysis of the order conditions of ROW-type scheme for stochastic differential equations. Submitted.

[13] G.N. Milstein, E. Platen, and H. Schurz. Balanced implicit methods for stiff stochastic systems. Technical Report SRR 029-94, The Australian National University, 1994.