Title	Algebraic semantics for predicate logics and their completeness(Non-Classical Logics and Their Kripke Semantics)
Author(s)	Ono, Hiroakira
Citation	数理解析研究所講究録 (1995), 927: 88-103
Issue Date	1995-11
URL	http://hdl.handle.net/2433/59913
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
927巻 1995年 88-103

Algebraic semantics for predicate logics and their completeness

小野寛晰
Hiroakira Ono
School of Information Science, JAIST
Tatsunokuchi, Ishikawa, 923-12, Japan
ono@jaist.ac.jp

1 Introduction

Algebraic semantics for nonclassical propositional logics provides us a powerful tool in studying logical properties which are common among many logics. In fact, it has been producing a lot of interesting general results by the help of universal algebraic methods. On the other hand, it seems that there have been little progress in the study of algebraic semantics for nonclassical predicate logics. For some special logics like the intuitionistic logic, we can show the completeness with respect to algebraic semantics. But, at present there seems to be no general way of proving completeness for a broad class of predicate logics. What is worse, there are uncountably many intermediate predicate logics which are incomplete with respect to algebraic semantics (see [10]).

Recently, the author proved in [11] the completeness of some of basic substructural predicate logics with respect to algebraic semantics, by using the Dedekind-MacNeille (DM) completion. Since existing Kripke-type semantics for these predicate logics are rather unsatisfactory, it would be proper now to consider seriously the possibility and the limitation of algebraic semantics once again. In the following, we will show how the above method can be extended to other logics, where difficulties arise and when incompleteness occurs.

To explain how the completeness can be proved, we will take the intuitionistic predicate logic *Int* for example. Obviously, Heyting algebras will be taken to define algebraic semantics for *Int*. But we must consider here how to interpret quantifiers in our semantics. Usually, we will assume moreover that these Heyting algebras are *complete* as lattices and will interpret universal and existential quantifiers by (possibly infinite) meets and joins in them. Thus, we will take any pair $\langle \mathbf{A}, V \rangle$ of a complete

Heyting algebra **A** and a non-empty set V, which determines the *domain*, for an algebraic structure for *Int*. Then, our goal is to show that *Int* is complete with respect to the class of all algebraic structures for *Int*.

Now suppose that a formula α is not provable in *Int*. Let **A** be the Lindenbaum algebra of *Int* and f be the *canonical mapping* from the set of formulas to **A**. Clearly, f can be regarded as a valuation on an algebraic structure determined by **A** with a countable set V. Then, it is easy to see that $f(\alpha)$ is not equal to the greatest element 1 of **A**. But this doesn't complete our proof, since the Lindembaum algebra **A** is not complete. What remains is to embed **A** into a complete Heyting algebra **B**. Moreover, this embedding h from **A** to **B** must preserve *every existing infinite meet and join* in **A**, in order to make the composite $h \circ f$ a valuation on an algebraic structure $\langle \mathbf{B}, V \rangle$. This is the essential point in completeness proofs.

There exist several standard methods for completion, *i.e.* methods of obtaining a complete algebra from the original algebra. Any complete algebra thus obtained is called a *completion* of the original algebra. For instance, for any Heyting algebra A, we can take the complete Heyting algebra consisting of all *complete ideals* of A (see e.g. [15]), or the complete Heyting algebra obtained by the DM-completion of A. (For general information on DM-completion, see [9].) The DM-completion method in its extended form works well also for algebras connected with basic substructural logics. This is what we showed in the paper [11]. We notice that Rasiowa, who proved the completeness of *Int* with respect to algebraic semantics for the first time in [13], employed the DM-completion. So, our results in [11] may be regarded as an extension of her result.

On the other hand, the DM-completion doesn't work well for logics in which $\forall x(\alpha(x) \vee \beta) \supset (\forall x \alpha(x) \vee \beta)$ holds, where x doesn't occur free in β. In algebraic terms, this formula represents the following infinite distributive law; $\bigcap_i (a_i \cup b) = (\bigcap_i a_i) \cup b$. In order to get completions of these algebras, it seems that we need a quite different idea. In fact, to prove the completeness of the classical predicate logic, Rasiowa and Sikorski introduced a way of completion of Boolean algebras by using the notion of *Q-filters* and proving so-called Rasiowa-Sikorski Lemma in [14]. This method is extended also for the intermediate predicate logic obtained from *Int* by adding the above formula as the axiom ([5], [12], [16]) and for ω^+-valued predicate logic ([7], [8]).

The purpose of the present paper is two-fold. First, we will show completeness theorems for some predicate logics with respect to algebraic semantics. In fact, we will apply the DM-completion method to some substructural predicate logics with modality and with distributive law. For the first case, we will generalize the method in [11] with certain modifications by Bucalo in [1]. For the second case, we will introduce a way of dealing with distributive law, as the DM-completion of a given distributive lattice is not always distributive. Of course, this is still not enough to prove the completeness of logics with infinite distributive law mentioned in the above, like the relevant logic *RQ*.

Then, we will discuss an inherent weakness of algebraic semantics, from which incompleteness results come. This weakness will come from essential differences between *instantiations* in logic and those in algebra. It will be shown that most of algebraic incompleteness results obtained so far will be caused by them.

2 Dedekind-MacNeille completion

Throughout this paper, we will assume the familiarity with the notations and the terminologies in [11]. First, we will discuss the Dedekind-MacNeille completion of algebras related to substructural logics.

Definition 1 *A structure* $\mathbf{A} = \langle A, \rightarrow, \cup, \cap, *, 1, 0, \top, \bot \rangle$ *is an FL-algebra if*
(1) $\langle A, \cup, \cap, \top, \bot \rangle$ *is a lattice with the least element* $\bot$ *and the greatest element* $\top$ *satisfying* $\top = \bot \rightarrow \bot$,
(2) $\langle A, *, 1 \rangle$ *is a monoid with the identity 1,*
(3) $z * (x \cup y) * w = (z * x * w) \cup (z * y * w)$, *for every* $x, y, z, w \in A$,
(4) $x * y \leq z$ *iff* $x \leq y \rightarrow z$, *for every* $x, y, z \in A$,
(5) 0 is an element of A.

Obviously, $\rightarrow, \cup, \cap$ and $*$ determine the interpretation of logical connectives $\supset, \vee, \wedge$ and the fusion (or, the multiplicative conjunction), respectively, in a given FL-algebra. When an FL-algebra $\mathbf{A}$ satisfies $x * y = y * x$ for every $x, y \in A$, it is called an FL_e-algebra. It can be easily verified that a Heyting algebra is just an FL_e-algebra in which $\cap = *, 0 = \bot$ and $1 = \top$.

We say that an FL-algebra $\mathbf{A}$ is *complete* if $\langle A, \cup, \cap \rangle$ is complete as a lattice, which moreover satisfies $y * (\bigcup_i x_i) * z = \bigcup_i (y * x_i * z)$ for every $x_i, y, z \in A$. In [11], we have introduced the Dedekind-MacNeille completion of any FL-algebra. For the sake of simplicity, we will give here a definition of the Dedekind-MacNeille completion of FL_e-algebras. As for the details, see [11].

Suppose that an FL_e-algebra $\mathbf{A}$ is given. For each subset U, V of A, we will define

$$U \cdot V = \{x * y; x \in U \text{ and } y \in V\},$$
$$U \Rightarrow V = \{z \in A; \text{for any } x \in U, z * x \in V\}.$$

Next, for each subset U of A, define

$$U^{\rightarrow} = \{x \in A; u \leq x \text{ for any } u \in U\},$$
$$U^{\leftarrow} = \{x \in A; x \leq u \text{ for any } u \in U\}.$$

That is, $U^{\rightarrow}$ and $U^{\leftarrow}$ are the set of all upper bounds of U and the set of all lower bounds of U, respectively. Now define an operation C on the power set $\wp(A)$ of A by $C(U) = (U^{\rightarrow})^{\leftarrow}$ for any subset U of A. It is easy to see that the operation C is a closure operation, *i.e.*, it satisfies the following three conditions; (1) $U \subseteq C(U)$, (2) $C(C(U)) \subseteq C(U)$ and (3) $U \subseteq V$ implies $C(U) \subseteq C(V)$. Moreover, it satisfies also that (4) $C(U) \cdot C(V) \subseteq C(U \cdot V)$. We say that U is *DM-closed*, or simply *closed* if $C(U) = U$ holds. We define $\tilde{A}$ to be the set of all DM-closed subsets of A. For any $a \in A$, let I_a be the principal ideal generated by a, *i.e.*, $I_a = \{x \in A; x \leq a\}$. Then, I_a belongs to $\tilde{A}$.

It can be easily seen that if both U and V are in $\tilde{A}$ then $U \cap V$ and $U \Rightarrow V$ are also in $\tilde{A}$. But this doesn't hold always for $\cup$ and $\cdot$. So, we define $U \star V = C(U \cdot V)$ and $U \sqcup V = C(U \cup V)$. Then, we can show the following.

Theorem 1 *Let* $\mathbf{A}$ *be an* FL_e*-algebra. Define* $\tilde{\mathbf{A}} = \langle \tilde{A}, \Rightarrow, \sqcup, \cap, \star, C(\{1\}), C(\{0\}), A, C(\emptyset)\rangle$. *Then,* $\tilde{\mathbf{A}}$ *is a complete* FL_e*-algebra. Moreover, the mapping* h *from* A *to* $\tilde{A}$ *defined by* $h(a) = I_a$ *for each* $a \in A$, *is an embedding which preserves all existing infinite meets and joins in* A. *Moreover, when* $\mathbf{A}$ *is complete* h *is an isomorphism.*

By a slight modification of the definition of $\tilde{\mathbf{A}}$, we can show the similar result for any FL-algebra. The complete FL-algebra $\tilde{\mathbf{A}}$ thus obtained is called the *Dedekind-MacNeille completion* (abbreviated to the DM-completion) of $\mathbf{A}$.

As a consequence of the above result, we can derive the completeness theorem of basic substructural predicate logics with respect to algebraic semantics determined by some classes of complete FL-algebras corresponding to these logics. To show this, it is enough to take the Lindenbaum algebra of a given logic for $\mathbf{A}$. As for the details, see [11]. See also the arguments developed in [2], in which *completion operators* and *completion algebras* are introduced for Heyting algebras.

In [15], the completeness theorem of the intuitionistic predicate logic Int was proved by using the complete Heyting algebra consisting of all *complete ideals* of the Lindembaum algebra of Int. Here, a nonempty subset W of a given lattice L is a *complete ideal* if

(1) if $x \in W$ and $y \leq x$ then $y \in W$,

(2) if $x, y \in W$ then $x \cup y \in W$,

(3) if $X \subseteq W$ and $\bigcap X$ exists then $\bigcap X \in W$.

It can be shown that (1) for any FL_e-algebra $\mathbf{A}$, if a subset U of A is closed then it is a complete ideal and (2) the converse holds when $\mathbf{A}$ is a Heyting algebra. Therefore, the DM-completion of a given Heyting algebra is nothing else but the completion obtained by collecting all of its complete ideals.

On the other hand, it is well-known that the DM-completion of a given distributive lattice is not necessarily distributive. So, we need to clarify the reason why the DM-completion works well for Heyting algebras. To see this, we will check carefully the proof of the fact that $\tilde{\mathbf{A}}$ is an FL_e-algebra when $\mathbf{A}$ is an FL_e-algebra. The point is the distributivity of $\star$ over $\sqcup$ in $\tilde{\mathbf{A}}$, which corresponds to the condition (3) of the Definition 1. When $\mathbf{A}$ is a Heyting algebra, $\star = \cap$ holds in $\tilde{\mathbf{A}}$ and therefore the usual distributive law follows from this.

In the present case, as it is obvious that $\star$ is commutative, it suffices to show the following distributive law:

$$U \star (V \sqcup W) = (U \star V) \sqcup (U \star W) \tag{1}$$

for every $U, V, W \in \tilde{A}$. By the monotonicity and the properties of closure operation C, we have only to show that

$$U \star (V \sqcup W) \subseteq (U \star V) \sqcup (U \star W).$$

Then this can be derived from the property (4) of the closure operation C, *i.e.*, $C(U) \cdot C(V) \subseteq C(U \cdot V)$. Now, let us examine the proof of this inclusion. (As for FL-algebras, the proof will be more complicated. See the proof of Lemma 4.3 for the detail.) Suppose that $a \in C(U) \cdot C(V)$. Then there exist $u \in C(U)$ and $v \in C(V)$ such that $a = u * v$. We will show that $a \in C(U \cdot V)$, *i.e.*,

for any w, if $z \leq w$ holds for any $z \in U \cdot V$ then $a \leq w$.

So, suppose that $z \leq w$ for any $z \in U \cdot V$. Let y be an arbitrary element of V. For any $x \in U$, $x * y \in U \cdot V$ and therefore $x * y \leq w$. Then, $x \leq y \to w$. Now, $x \leq y \to w$ holds for any $x \in U$. Since $u \in C(U)$, $u \leq y \to w$. Hence $y * u = u * y \leq w$ and thus $y \leq u \to w$ for any $y \in V$. Again, since $v \in C(V)$, $v \leq u \to w$. Thus, $a = u * v \leq w$.

The above proof shows that the distributive law (1) holds for $\tilde{\mathbf{A}}$ as long as the semigroup $\langle A, * \rangle$, which is a reduct of $\mathbf{A}$, is *residuated*, *i.e.*, for every $x, y \in A$, the *residual* $x \to y$ of y by x with respect to $*$ exists. In particular, in the case of Heyting algebras, algebras obtained by the DM-completion are also distributive because of the existence of $\rightarrow$ which is the residual operation with respect to the meet $\cap$.

These facts will be used in the proof of the completeness theorem for a relevant predicate logic in Section 3.

3 Completeness theorems for modal and relevant predicate logics

In this section, we will show how the Dedekind-MacNeille completion works in proving the completeness theorem for some modal and relevant predicate logics. We will show this only for some particular logics, intending to convey our basic idea, but the method developed here will be easily extended to many other logics.

In [11], we have shown how the embedding theorems for substructural logics obtained by the DM-completion can be extended to those for substructural logics with exponentials. The basic idea is to embed the non-modal reduct of a given algebra $\mathbf{A}$ into a complete algebra $\mathbf{B}$ by a mapping h using the DM-completion, and then to introduce exponentials on $\mathbf{B}$ in such a way that the mapping h preserves also exponentials. Bucalo [1] modified the way of constructing exponentials and proved the completeness theorem for modal subsystems of the intuitionistic linear predicate logic with exponentials. (Precisely speaking, she hasn't mentioned these completeness results explicitly in her paper. But this is obvious from Lemma 3.7 in [1].)

This suggests us a certain applicability of our method in this modified form to many substructural modal predicate logics. To show this, let us consider the modal logic $K.FL_e$ over the logic FL_e with the modality $\Box$. The logic $K.FL_e$ is obtained from the sequent calculus FL_e by adding the following rule of inference for $\Box$, which is usually used to introduce the classical modal logic K. Here, $\Box\Gamma$ denotes the sequence of formulas $\Box\gamma_1, \ldots, \Box\gamma_m$ when Γ is $\gamma_1, \ldots, \gamma_m$:

$$\frac{\Gamma \rightarrow \alpha}{\Box\Gamma \rightarrow \Box\alpha}$$

Corresponding to this logic, we will define *modal* FL_e-algebras as follows.

Definition 2 *A pair $\langle \mathbf{A}, \mu \rangle$ of an FL_e-algebra $\mathbf{A}$ and an operation μ on A is a modal FL_e-algebra if*
(1) $x \leq y$ implies $\mu(x) \leq \mu(y)$ for each $x, y \in A$,
(2) $\mu(x) * \mu(y) \leq \mu(x * y)$ for each $x \in A$,
(3) $1 \leq \mu(1)$.

As usual, $\Box$ in modal formulas can be interpreted by the operation μ in a modal FL_e-algebra $\langle \mathbf{A}, \mu \rangle$. Let $\tilde{\mathbf{A}}$ be the DM-completion of $\mathbf{A}$ and h be the embedding from $\mathbf{A}$ to $\tilde{\mathbf{A}}$ introduced in Theorem 1. Following [1], define an operation $\tilde{\mu}$ on $\tilde{A}$ by

$$\tilde{\mu}(a) = \bigcup\{h(\mu(x)); x \in A, h(x) \leq a\} \tag{2}$$

for each $a \in \tilde{A}$. Then, we can show easily that $\langle \tilde{\mathbf{A}}, \tilde{\mu} \rangle$ is also a modal FL_e-algebra and that $h(\mu(x)) = \tilde{\mu}(h(x))$ for each $x \in A$. Thus, by using the argument mentioned in the previous section, we can show the following.

Theorem 2 *The substructural modal predicate logic $K.FL_e$ is complete with respect to the class of complete modal FL_e-algebras.*

Similar results will hold also for some extensions of FL_e with additional modal axioms, as long as the algebraic counterparts of modal axioms will be transferred from μ to $\tilde{\mu}$. That is, if an algebraic property corresponding to a given modal axiom holds for μ then so does for $\tilde{\mu}$. For example, take axioms $\Box A \supset A$ and $\Box A \supset \Box\Box A$. Then, it is easily verified that if $\mu(x) \leq x$ holds for any $x \in A$ then $\tilde{\mu}(a) \leq a$ holds for any $a \in \tilde{A}$ and that if $\mu(x) \leq \mu(\mu(x))$ holds for any $x \in A$ then $\tilde{\mu}(a) \leq \tilde{\mu}(\tilde{\mu}(a))$ holds for any $a \in \tilde{A}$. Therefore, we can derive the completeness theorem for modal logics which are extensions of $K.FL_e$ having either or both of these modal axioms, and also some structural rules. In particular, we can get an alternative proof of the completeness theorem of classical modal predicate logic $S4$, by combining this with results in [11] (cf. [13]).

Although the above transferring property will hold only for limited cases, there are some rooms where the DM-completion still works. The similar idea will work also for modal logics with $\Diamond$ as a logical connective. (Note that in weaker systems, $\Diamond$ cannot be treated as the dual of $\Box$.) In this case, if σ is an algebraic operation corresponding to $\Diamond$ in the original algebra, the operation $\tilde{\sigma}$ on its DM-completion will be defined by

$$\tilde{\sigma}(a) = \bigcap\{h(\sigma(x)); x \in A, a \leq h(x)\}. \tag{3}$$

Next, we will discuss the completeness theorem for relevant predicate logics. The main obstacle here lies in the fact that though the distributive law holds between the disjunction and the (additive) conjunction, the implication $\supset$ is the residual not with respect to the additive conjunction, but to the fusion. Therefore, it may happen that the DM-completion of a given relevant algebra is not distributive, as we have remarked in Section 2.

To overcome this difficulty, we will add a new logical connective $\sqsupset$ to the original system so that the algebraic operation $\rightarrow^*$ corresponding to $\sqsupset$ will be the residual with respect to $\cap$. In the following, we will show how this idea will work.

Here we will take the relevant predicate logic DFL_e as an example. The logic DFL_e is defined as a sequent system obtained from the predicate calculus FL_e by adding the following two kinds of sequents as the intial sequents;

$$\alpha \wedge (\beta \vee \gamma) \rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \tag{4}$$

$$\exists x\alpha(x) \wedge \beta \rightarrow \exists x(\alpha(x) \wedge \beta) \tag{5}$$

An FL_e-algebra **A** is a *distributive* FL_e-algebra (or, a DFL_e-algebra) if it is a distributive lattice. A DFL_e-algebra is *complete*, if it is complete as a lattice and also satisfies the following infinite distributive law;

$$\bigcup_i a_i \cap b = \bigcup_i (a_i \cap b) \tag{6}$$

In the rest of this section, we will show the following theorem.

Theorem 3 *The relevant predicate logic DFL_e is complete with respect to the class of complete DFL_e-algebras.*

It is almost trivial to show the soundness. On the other hand, as we have mentioned already, the DM-completion of a given DFL_e-algebra may not be distributive. Therefore, we will introduce DFL_e-algebras with the new operation $\rightarrow^*$ satisfying the following;

for every a, b, c, $a \cap b \leq c$ if and only if $a \leq b \rightarrow^* c$.

Let us call these algebras, DFL_e^+-algebras. It is easy to see that in any DFL_e^+-algebra, if the join $\bigcup_i a_i$ exists then $\bigcup_i (a_i \cap b)$ exists also for any b for which the above equation 6 holds. By using the argument stated in Section 2, the DM-completion $\tilde{\mathbf{A}}$ of a given DFL_e^+-algebra **A** is also distributive. For any subset U, V of A, define

$U \Rightarrow^* V = \{z \in A;$ for any $x \in U, z \cap x \in V\}$.

Then it can be shown that if both U and V are closed then $U \Rightarrow^* V$ is also closed. Moreover, for any closed U and V, the following holds;

$U \cap W \subset V$ if and only if $W \subset U \Rightarrow^* V$.

Thus, the equation 6 holds by using the above argument, and hence $\tilde{\mathbf{A}}$ becomes a complete DFL_e^+-algebra.

To show the completeness of DFL_e, suppose that a given formula α is not provable in DFL_e, or more precisely, the sequent $\rightarrow \alpha$ is not provable in DFL_e. Let $\mathbf{A}$ be the Lindenbaum algebra of DFL_e and f be the canonical mapping from the set of formulas to A. Then, $\mathbf{A}$ is a DFL_e- algebra in which the inequality $1 \leq f(\alpha)$ doesn't hold. But, as $\mathbf{A}$ is not necessarily a DFL_e^+-algebra, we cannot apply the DM-completion to it.

So, we will introduce an extension of DFL_e with the logical connective $\sqsupset$, which corresponds to the algebraic operation $\rightarrow^*$. First, we will introduce a (cut-free) sequent calculus D_0FL_e which is equivalent to DFL_e. To do this, we will follow Dunn's idea developed in [3] and use *intensional* and *extensional* sequences. But here we will borrow the notations from Slaney [19], since our system D_0FL_e can be defined simply by eliminating *I-weakening* from his LL_{DBCK} and by adding rules for quantifiers. (To keep the consistency of our terminologies, we need some literal translations of logical symbols in [19], *i.e.*, symbols $:, \rightarrow$ and & in [19] will be replaced by $\rightarrow, \supset$ and $*$ (for fusion), respectively. Moreover, in usual sequent systems for substructural logics, the commas in the left side of sequents means the intensional combination, *i.e.*, they can be interpreted as the fusions. By this reason, in our formulation of D_0FL_e presented below, *we will use the commas for intensional combination and the symbol* | *for extensional one.* Without any difficulty, we can also extend D_0FL_e to the one with the constants which the language of the original FL_e contains, but we will omit the details.)

Now, we will give here the definition of D_0FL_e. (For more information, consult [3] and [19].) First, we will define *antecedents* inductively as follows;
(1) any formula is an antecedent,
(2) empty expression is an antecedent,
(3) if each of $X_1, \ldots, X_n$ is an antecedent then the expressions of the form $X_1, \ldots, X_n$ and $X_1 | \ldots | X_n$ are also antecedents.

In the above (3), the first expression is called an intensional combination, which means roughly the combination of $X_1, \ldots, X_n$ by the fusion, and the second expression an extensional combination, which means the combination by the additive conjunction. Similarly to [19], we will use expressions like $\Gamma(X)$ to denote an antecedent in which X occurs as a sub-antecedent. Also, for a given $\Gamma(X)$, the expression $\Gamma(Y)$ for an antecedent Y denotes the antecedent obtained from $\Gamma(X)$ by replacing the indicated occurrence of X by Y.

Sequents of D_0FL_e are expressions of the form $\Gamma \to \alpha$, where Γ is an antecedent and α is a formula. The initial sequents of D_0FL_e consist of sequents of the form $\alpha \to \alpha$. The rules of inference of D_0FL_e are given as follows:

$$\frac{X \to \alpha \quad \Gamma(\alpha) \to \delta}{\Gamma(X) \to \delta}\ (cut)$$

$$\frac{\Gamma(X) \to \delta}{\Gamma(X|Y) \to \delta} \qquad \frac{\Gamma(X|X) \to \delta}{\Gamma(X) \to \delta}$$

$$\frac{\Gamma(X|Y) \to \delta}{\Gamma(Y|X) \to \delta} \qquad \frac{\Gamma(X,Y) \to \delta}{\Gamma(Y,X) \to \delta}$$

$$\frac{\Gamma(\alpha) \to \delta \quad \Gamma(\beta) \to \delta}{\Gamma(\alpha \vee \beta) \to \delta}$$

$$\frac{\Gamma \to \alpha}{\Gamma \to \alpha \vee \beta} \qquad \frac{\Gamma \to \beta}{\Gamma \to \alpha \vee \beta}$$

$$\frac{\Gamma(\alpha|\beta) \to \delta}{\Gamma(\alpha \wedge \beta) \to \delta} \qquad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma|\Delta \to \alpha \wedge \beta}$$

$$\frac{\Gamma(\alpha,\beta) \to \delta}{\Gamma(\alpha * \beta) \to \delta} \qquad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma, \Delta \to \alpha * \beta}$$

$$\frac{\Gamma \to \alpha \quad \Delta(\beta) \to \delta}{\Delta(\alpha \supset \beta, \Gamma) \to \delta} \qquad \frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}$$

$$\frac{\Gamma(\alpha(x)) \to \delta}{\Gamma(\exists z \alpha(z)) \to \delta}\ (\exists \to) \qquad \frac{\Gamma \to \alpha(y)}{\Gamma \to \exists z \alpha(z)}$$

$$\frac{\Gamma(\alpha(y)) \to \delta}{\Gamma(\forall z \alpha(z)) \to \delta} \qquad \frac{\Gamma \to \alpha(x)}{\Gamma \to \forall z \alpha(z)}\ (\to \forall)$$

As usual, the variable condition must be satisfied when we use $(\exists \to)$ and $(\to \forall)$.

Since any sequent of DFL_e is of the form $\alpha_1, \ldots, \alpha_k \to \beta$, it can be regarded also as a sequent of D_0FL_e by identifying commas appearing in it with commas in the definition of antecedents. Then, we can show the following, similarly to [19].

Theorem 4 *For any sequent S of DFL_e, S is provable in DFL_e if and only if it is provable in D_0FL_e.*

Next, we will introduce an extension DFL_e^+ of DFL_e, which has the new operation $\sqsupset$. For this operation, it has the initial sequent $\alpha \wedge (\alpha \sqsupset \beta) \rightarrow \beta$ and the following rule of inference;

$$\frac{\alpha \wedge \gamma \rightarrow \beta}{\gamma \rightarrow \alpha \sqsupset \beta}$$

Corresponding to DFL_e^+, we will introduce also an extension D_1FL_e of D_0FL_e, which is obtained from the latter by adding the following rules of inference for $\sqsupset$;

$$\frac{\Gamma \rightarrow \alpha \quad \Delta(\beta) \rightarrow \delta}{\Delta(\alpha \sqsupset \beta|\Gamma) \rightarrow \delta} \qquad \frac{\Gamma|\alpha \rightarrow \beta}{\Gamma \rightarrow \alpha \sqsupset \beta}$$

Similarly to Theorem 4, we can show the following.

Theorem 5 *For any sequent S of DFL_e^+, S is provable in DFL_e^+ if and only if it is provable in D_1FL_e.*

Moreover, we can show the following.

Theorem 6 *The cut elimination theorem holds for D_1FL_e.*

By using Theorems 6, 4 and 5, we have the following.

Theorem 7 *DFL_e^+ is a conservative extension of DFL_e.*

Now, we will give a proof of Theorem 3. Suppose that a formula α is not provable in DFL_e. Then, by Theorem 7, neither is it provable in DFL_e^+. Let $\mathbf{B}$ be the Lindenbaum algebra of DFL_e^+ and f be the canonical mapping from the set of formulas to B, for which $1 \leq f(\alpha)$ doesn't hold. It is easy to see that $\mathbf{B}$ is a DFL_e^+-algebra. Now let $\tilde{\mathbf{B}}$ be the DM-completion of $\mathbf{B}$. Then, $\tilde{\mathbf{B}}$ is a complete DFL_e^+-algebra and *a fortiori* a complete DFL_e-algebra. Let h be the embedding from $\mathbf{B}$ to $\tilde{\mathbf{B}}$ which preserves all exisiting meets and joins in B. Then, the composite $h \circ f$ is a valuation on $\tilde{\mathbf{B}}$ for which $1 \leq (h \circ f)(\alpha)$ doesn't hold. This completes the proof of Theorem 3.

This method will work when the extension by adding $\sqsupset$ is conservative over the original system. On the other hand, it will not work well for relevant logics like RQ in which $\forall x(\alpha(x) \vee \beta) \supset (\forall x\alpha(x) \vee \beta)$ holds. For, the DM-completion doesn't work at all for them, as we mentioned in Section 1.

4 Algebraic incompleteness

In the previous section, we have shown that the Dedekind-MacNeille completion works well in proving the completeness theorems for substructural predicate logics. In this section, we will show an inherent weakness of algebraic semantics, from which many incompleteness results come. In the following, we will discuss mainly the incompleteness phenomena among intermediate predicate logics, *i.e.*, logics between the classical logic and the intuitionistic.

In the following, we will say an intermediate predicate logic L is *algebraically incomplete* if there is no class of algebraic structures (in the sense of Section 1) with complete Heyting algebras such that L is complete with respect to it. The algebraic incompleteness was first pointed out by the present author in [10] (see also [12]). In fact, the following result was shown in it (Theorem 2.4).

Theorem 8 *There are uncountably many intermediate predicate logics which are algebraically incomplete.*

After that, several results on algebraic incompleteness have been shown (*e.g.* [6], [17] and [20]). In the following, we will try to make it clear that there exists an essential difference between *instantiations* in logic and in algebra, from which most of algebraic incompleteness results obtained up to now follow.

To show this, as an example let us take the logic LF obtained from the intuitionistic predicate logic Int by adding the following axiom F: $\exists x \forall y (P(x) \supset P(y))$. It is easy to see that the propositional fragment of this logic is equal to the intuitionistic propositional logic. (As for the propositional fragment of a given logic, see Section 5 of [10], for instance.) On the other hand, the above axiom can be expressed in algebraic terms as: $\bigcup_i \bigcap_j (a_i \supset a_j)$. As a special case, if these indeces i and j run over the finite set $\{1, 2\}$, this becomes

$$((a_1 \supset a_1) \cap (a_1 \supset a_2)) \cup ((a_2 \supset a_1) \cup (a_2 \supset a_2))$$

which is equivalent to $(a_1 \supset a_2) \cup (a_2 \supset a_1)$. On the other hand, this term is not always equal to 1 in any Heyting algebra. Here, we can see a certain descrepancies.

Now let us state the above argument in a more formal way. Define formulas N_1 and Lin as follows;

$$N_1 \equiv (\exists x P(x) \supset \forall x P(x))$$
$$Lin \equiv (p \supset q) \wedge (q \supset p)$$

Then, by using the above argument we can show first that

> for any algebraic strucuture $\langle \mathbf{A}, V \rangle$ with a complete Heyting algebra $\mathbf{A}$, if the formula F is valid in $\langle \mathbf{A}, V \rangle$ then the formula $Lin \vee N_1$ is also valid in it.

In proving this, we use the fact that when the cardinality of the set V is greater than 1, $f(N_1) = 0$ for some valuation f, where 0 is the least element of a given Heyting algebra. On the other hand, we can show that

> the formula $Lin \vee N_1$ is not provable in the logic LF.

From these two facts, we can easily derive the incompleteness of LF. So, it remains to give a proof of the second statement. For this purpose, we will use Kripke-sheaf semantics introduced in [17]. (Here, we will not give the definition of Kripke-sheaf semantics. As for the details, consult [17].) We will take the following Kripke sheaf $\mathbf{S} = \langle \langle D, \rho \rangle, \langle W, \leq \rangle, \pi \rangle$ such that

1. $D = \{u_0, u_1, v, w\}$ with the binary relation ρ such that $x\rho y$ if and only if (1) $x = y$ or (2) $x = u_i$ (for $i = 0, 1$) and either $y = v$ or $y = w$,
2. $W = \{a, b, c\}$ with the binary relation $\leq$ such that $x \leq y$ if and only if (1) $x = y$ or (2) $x = a$,
3. $\pi(u_0) = \pi(u_1) = a, \pi(v) = b$ and $\pi(w) = c$.

Roughly speaking, u_0 and u_1 are elements in the world a, which is smaller than both b and c, and both of them become v (and w) in the world b (and c, respectively). Then, it can be shown that the formula F is valid in this Kripke-sheaf $\mathbf{S}$ but the formula $Lin \vee N_1$ is not. Similarly, we can show the following. (See also [6] and [17].)

Theorem 9 *Let L be any intermediate predicate logic which is obtained from Int by adding one of the following axioms;* $\exists x \forall y(P(x) \supset P(y))$, $\exists x \forall y(P(y) \supset P(x))$, $\exists x(P(x) \supset \forall y P(y))$, $\exists x(\exists y P(y) \supset P(x))$, $\neg\neg\exists x P(x) \supset \exists x \neg\neg P(x)$. *Then, L is algebraically incomplete.*

Our present method can cover also other incompleteness results mentioned in [6]. In these examples discussed above, the incompleteness is caused by the fact that an instantiation of the algebraic expression for a given axiom produces a stronger principle, when we take a finite set for the index set. To the contrary, sometimes it happens that an instantiation gives a weaker principle, and this also causes the incompleteness. To explain this, let D be the formula $\forall x(\alpha(x) \vee \beta) \supset (\forall x \alpha(x) \vee \beta)$, in which the variable x doesn't occur free in β. Then, its algebraic form will be (equivalent to) $\bigcap_i(a_i \cup b) = (\bigcap_i a_i) \cup b$. If the indeces i and j run over a finite set, this equality expresses the finite distributivity and therefore is deduced from the usual distributive law. Now for each positive integer m define the formula $R_m(x)$ and N_m as follows, where each $P_j(x)$ is a predicate symbol;

$$R_1(x) \equiv P_1(x),$$
$$R_{m+1}(x) \equiv ((\textstyle\bigwedge_{i=1}^m \neg P_i(x)) \wedge P_{m+1}(x)),$$
$$N_m \equiv (\textstyle\bigwedge_{i=1}^m \exists x R_i(x) \supset \forall x(\bigvee_{i=1}^m R_i(x))).$$

It can be easily checked that the formula N_m is valid in an algebraic structure $\langle \mathbf{A}, V\rangle$ if and only if the cardinality of the set V is not greater than m. Now, by the above argument and the fact that any Heyting algebra is distributive, we have that

> for any algebraic strucuture $\langle \mathbf{A}, V\rangle$ with a complete Heyting algebra $\mathbf{A}$, if the formula $D \vee N_m$ is valid in $\langle \mathbf{A}, V\rangle$ then the formula D is also valid in it.

On the other hand, we can show that D is not provable in the logic obtained from *Int* by adding $D \vee N_m$ as the axiom. Thus this logic is algebraically incomplete. In fact, this is the essence of the proof of Theorem 8.

As we have discussed in the above, there seems to be a big difference between instantiations in logic and in algebra. In other words, though usually we interpret quantitiers as infinite joins and meets in algebraic semantics, *i.e.*, we deal with quantifiers like infinite disjunctions and conjunctions, this will be not always appropriate. So, one of the most important questions in the study of algebraic semantics for predicate logics would be how to give an appropriate interpretation of quantifiers, in other words, how to extend algebraic semantics.

References

[1] A. Bucalo, Modalities in linear logic weaker than the exponential "of course": algebraic and relational semantics, *Journal of Logic, Language and Information 3*, (1994) pp.211-232.

[2] A.G. Dragalin, *Mathematical intuitionism, Introduction to proof theory*, Nauka, Moscow, 1979 (English translation: American Mathematical Society, Rhode Island, 1987).

[3] J.M. Dunn, Consecution formulation of positive R with co-tenability and t, in *Entailment: The logic of relevance and necessity, vol.1*, eds. by A.R. Anderson and N.D. Belnap, Princeton Univ. Press, pp. 381-391, 1975.

[4] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford, 1963.

[5] S. Görnemann, A logic stronger than intuitionism, *Journal of Symbolic Logic 36*, (1971) pp.249-261.

[6] Y. Komori, Some results on the super-intuitionistic predicate logics, *Reports on Mathematical Logic 15*, (1983) pp.13-31.

[7] L. Maksimova and D. Vakarelov, Representation theorems for generalized Post algebras of order ω^+, *Bulletin de l'Academie Polonaise des Sciences, Sér. Sci. Math. Astronom. Phys. 22*, (1975) pp.757-764.

[8] L. Maksimova and D. Vakarelov, Semantics for ω^+-valued predicate calculi, *Bulletin de l'Academie Polonaise des Sciences, Sér. Sci. Math. Astronom. Phys. 22*, (1975) pp.765-771.

[9] R.N. McKenzie, G.F. McNulty and W.F. Tailor, *Algebras, lattices, varieties* vol.1, Wadsworth & Brooks/Cole, California, 1987.

[10] H. Ono, A study of intermediate predicate logics, *Publications of RIMS, Kyoto Univ. 8*, (1973) pp.619-649.

[11] H. Ono, Semantics for substructural logics, in *Substructural logics*, eds. by K. Došen and P. Schroeder-Heister, Oxford Univ. Press, pp.259-291, 1993.

[12] H. Ono and C. Rauszer, On an algebraic and Kripke semantics for intermediate logics, in *Universal algebra and applications*, PWN - Polish Scientific Publishers, Warszawa, pp.431-438, 1982.

[13] H. Rasiowa, Algebraic treatment of the functional calculi of Heyting and Lewis, *Fundamenta Matematicae 38*, (1951) pp.99-126.

[14] H. Rasiowa and R. Sikorski, A proof of the completeness theorem of Gödel, *Fundamenta Mathematicae 37*, (1950) pp.193-200.

[15] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, PWN - Polish Scientific Publishers, Warszawa, 1963.

[16] C. Rauszer and B. Sabalski, Notes on the Rasiowa-Sikorski lemma, *Studia Logica 34*, (1975) pp.225-228.

[17] V.B. Shehtman and D.P. Skvortsov, Semantics of non-classical first order predicate logics, in *Mathematical logic*, ed. by P.P. Petkov, Plenum Press, pp.105-116, 1990.

[18] D.P. Skvortsov and V.B. Shehtman, Maximal Kripke-type semantics for modal and superintuitionistic predicate logics, *Annals of Pure and Applied Logic 63*, (1993) pp.69-101.

[19] J. Slaney, Solution to a problem of Ono and Komori, *Journal of Philosophical Logic 18*, (1989) pp.103-111.

[20] N.-Y. Suzuki, Some results on the Kripke sheaf semantics for super-intuitionistic predicate logics, *Studia Logica 52*, (1993) pp.73-94.