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Author(s)	Koike, Naoya; Nunokawa, Mamoru; Saitoh, Hitoshi
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ON MEROMORPHICALLY MULTIVALENT FUNCTIONS

Naoya Koike(小池 尚也, 群馬大学)
Mamoru Nunokawa(布川 護, 群馬大学)
Hitoshi Saitoh(斎藤 齐, 群馬高専)

Abstract. The purpose of this paper is to derive some properties of certain meromorphically multivalent functions in annulus.

1. Introduction

Let Σ_p be the class of functions of the form

$$(1.1) \quad f(z) = 1/z^p + a_0/z^{p-1} + \dots + a_{k+p-1}z^k + \dots,$$

which are analytic in the annulus $D = \{Z : |z| < 1\}$, where $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. For $f(z) \in \Sigma_p$, we define the operator $D^{n+p-1}f(z)$ by

$$(1.2) \quad \begin{aligned} D^{n+p-1}f(z) &= (z^{n+2p-1}f(z)/(n+p-1)!)^{(n+p-1)}/z^p \\ &= 1/z^p + (n+p)a_0/z^{p-1} + (n+p)(n+p+1)a_1/(2!z^{p-2}) + \dots \\ &\quad + (n+p)(n+p+1)\dots(n+k+2p-1)a_{k+p-1}z^k/(k+p)! + \dots, \end{aligned}$$

where n is an integer and $n > -p$.

Recently, Cho and Nunokawa [1] proved that

$$\operatorname{Re}\{z^{p+1}(D^{n+p}f(z))'\} < -\alpha \quad (0 \leq \alpha < p; |z| < 1)$$

$$\text{implies } \operatorname{Re}\{z^{p+1}(D^{n+p-1}f(z))'\} < -\beta \quad (|z| < 1)$$

where

$$\beta = (p + 2\alpha(n+p))/(1+2(n+p)).$$

In the present paper, we show another properties of functions $f(z) \in \Sigma_p$ concerning with the operator $D^{n+p-1} f(z)$.

2. Main results

We need the following lemma due to Jack [2] (or, due to Miller and Mocanu[3]).

Lemma. Let $w(z)$ be non-constant analytic in $U = \{ Z: |z| < 1 \}$ with $w(0)=1$. If $|w(z)|$ attains its maximum value at a point z_0 on the circle $|z|=r<1$, then we have

$$z_0 w'(z_0) = kw(z_0)$$

where k is real and $k \geq 1$.

Theorem 1. If $f(z) \in \Sigma_p$ satisfies

$$(2.1) \quad \operatorname{Re}\{ z^{p+1} (D^{n+p} f(z))' \} > -\alpha \quad (z \in U)$$

for some α ($\alpha > p$), then

$$(2.2) \quad \operatorname{Re}\{ z^{p+1} (D^{n+p-1} f(z))' \} > -\beta \quad (z \in U),$$

where

$$\beta = (p + 2\alpha(n+p))/(1 + 2(n+p)).$$

Proof. Define the function $w(z)$ by

$$(2.3) \quad z^{p+1} (D^{n+p-1} f(z))' = ((p - 2\beta)w(z) - p)/(1+w(z)),$$

$w(z) \neq -1$, with

$$\beta = (p + 2\alpha(n+p))/(1 + 2(n+p)).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Note that

$$(2.4) \quad z(D^{n+p-1} f(z))' = (n+p)D^{n+p} f(z) - (n+2p) D^{n+p-1} f(z).$$

It follows from (2.3) that

$$(2.5) \quad \begin{aligned} (n+p) z^p D^{n+p} f(z) - (n+2p) z^p D^{n+p-1} f(z) \\ = ((p-2\beta)w(z)-p)/(1+w(z)). \end{aligned}$$

Taking the differentiations in both sides of (2.5), we have

$$(2.6) \quad z^{n+1} (D^{n+1} f(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) \\ + 2(p-\beta)zw'(z)/((n+p)(1+w(z))^2).$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| \quad (w(z_0) \neq -1),$$

then, by Lemma, we have

$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), we see that

$$(2.7) \quad \operatorname{Re}\{z_0^{n+1} (D^{n+1} f(z_0))'\} + \alpha \\ = \alpha + \operatorname{Re}\{((p-2\beta)e^{i\theta}-p)/(1+e^{i\theta})\} + 2(p-\beta)k/(n+p) \operatorname{Re}\{e^{i\theta}(1+e^{i\theta})^{-2}\} \\ = \alpha - \beta + (p-\beta)k/((n+p)(1+\cos\theta)) \\ \leq \alpha - \beta + (p-\beta)/2(n+p) = 0$$

for $\alpha > p$ and $\beta = (p+2\alpha(n+p))/(1+2(n+p))$.

This contradicts our condition (2.1). Therefore, $|w(z)| < 1$ for all $z \in U$, or

$$\operatorname{Re}\{z^{n+1} (D^{n+1} f(z))'\} > -\beta \quad (z \in U).$$

Next, we prove

Theorem 2. Let

$$(2.8) \quad F_c(z) = cz^{-c-p} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0)$$

for $f(z) \in \Sigma_p$. If $f(z)$ satisfies

$$(2.9) \quad \operatorname{Re}\{z^{p+1} (D^{n+p-1} f(z))'\} > -\alpha \quad (z \in U)$$

for some α ($\alpha > p$), then

$$(2.10) \quad \operatorname{Re}\{z^{p+1} (D^{n+p-1} F_c(z))'\} > -\beta, \quad (z \in U)$$

where $\beta = (p + 2\alpha c)/(1+2c)$.

Proof. We define the function $w(z)$ by

$$(2.11) \quad z^{p+1} (D^{n+p-1} F_c(z))' = ((p - 2\beta)w(z) - p)/(1 + w(z)) \quad (w(z) \neq -1).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Noting that

$$(2.12) \quad z(D^{n+p-1} F_c(z))' = cD^{n+p-1} f(z) - (c+p) D^{n+p-1} F_c(z),$$

therefore we have

$$(2.13) \quad \begin{aligned} & z^{p+1} (D^{n+p-1} f(z))' \\ &= ((p - 2\beta)w(z) - p)/(1+w(z)) + 2(p-\beta) zw'(z)/c(1+w(z))^2. \end{aligned}$$

Therefore, if we assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1)$$

then Lemma gives us that

$$(2.14) \quad \begin{aligned} \operatorname{Re}\{z_0^{p+1} (D^{n+p-1} f(z_0))'\} + \alpha \\ \leq \alpha - \beta + (p - \beta)/2c \\ = 0 \end{aligned}$$

which contradicts our condition (2.9). This completes the proof of Theorem 2.

References

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Department of Mathematics
University of Gunma
Maebashi, Gunma 371
Japan

Department of Mathematics
University of Gunma
Maebashi, Gunma 371
Japan

Department of Mathematics
Gunma College of Technology
Toriba, Maebashi
Gunma 371
Japan