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# SHIFT MAPS AND ATTRACTORS

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## 1 Introduction.

All spaces considered here are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space. A *continuum* is connected, nondegenerate compactum. Let $R$ be the real line and $R^n$ the Euclidean $n$-dimensional space. Let $S$ be the unit circle in the plane $R^2$. For a manifold $M$, $\partial M$ denotes the manifold boundary. Let $F: Y \to Y$ be a homeomorphism of a space $Y$ (onto itself) with metric $d$ and let $\Lambda$ be a compact subset of $Y$. Then $\Lambda$ is said to be an *attractor* of $F$ provided that there exists an open neighborhood of $U$ of $\Lambda$ in $Y$ such that

$$F(\mathrm{Cl}(U)) \subset U \text{ and } \Lambda = \cap_{n\geq 0} F^n(U).$$

Note that $F(\Lambda) = \Lambda$. Moreover, if for each $y \in Y$ $\lim_{n\to\infty} d(F^n(y), \Lambda) = 0$, then we say that $\Lambda$ is a *global attractor* of $F$, where $d(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$ for sets $A, B$. Let $f: X \to X$ and $g: Y \to Y$ be maps. Then $f$ is *topologically conjugate to* $g$ if there is a homeomorphism $\phi: X \to Y$ such that $\phi \cdot f = g \cdot \phi$.

The notion of shift maps is very convenient for dynamical systems. Let $\mathbf{X} = \{X_n, p_{i,i+1} | i = 1, 2, \ldots\}$ be an inverse sequence of compacta $X_i$ and maps $p_{i,i+1}: X_{i+1} \to X_i (i = 1, 2, \ldots)$ and let

$$\mathrm{invlim}\, \mathbf{X} = \{(x_i)_{i=1}^{\infty} | x_i \in X_i, p_{i,i+1}(x_{i+1}) = x_i \text{ for each } i\} \subset \prod_{i=1}^{\infty} X_i.$$

Then invlim$\mathbf{X}$ is a topological space as a subspace of the product space $\prod_{i=1}^{\infty} X_i$. Then invlim$\mathbf{X}$ is a compactum. Let $f : X \to X$ be a map of a compactum $X$. Consider the following special inverse limit space:

$$(X, f) = \{(x_i)_{i=1}^{\infty} | \, x_i \in X \text{ and } f(x_{i+1}) = x_i \text{ for each } i \geq 1\}.$$

Define a map $\tilde{f} : (X, f) \to (X, f)$ by $\tilde{f}(x_1, x_2, \ldots,) = (f(x_1), x_1, \ldots,)$. Then $\tilde{f}$ is a homeomorphism and it is called the *shift map* of $f$. A map $f : X \to Y$ of compacta is a *near homeomorphism* if $f$ can be approximated arbitrarily closely by homeomorphisms from $X$ onto $Y$.

Isbell proved that if $X = \text{invlim}\{X_i, p_{i,i+1}\}$ where each $X_i$ is a compactum which can be embedded into $R^n$ ($n$ fixed), then $X$ can be embedded into $R^{2n}$. Barge and Martin proved that if $f : I \to I$ is any map of the unit interval $I = [0, 1]$, then there is a homeomorphism $F : R^2 \to R^2$ such that $(I, f)$ is contained in $R^2$, $F$ is an extension of the shift map $\tilde{f} : (I, f) \to (I, f)$, and $(I, f)$ is a global attractor of $F$.

## 2 Shift maps of compact polyhedra in $R^n$.

In this section, we obtain the following theorem which is a generalization of Barge-Martin's theorem, and which is related to Isbell's theorem.

**Theorem 2.1** *If $P$ is a compact polyhedron in $R^n$ and $f : P \to P$ is any map, then there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that $(P, f)$ is contained in $R^{2n}$, $F$ is an extension of the shift map $\tilde{f} : (P, f) \to (P, f)$ of $f$, and $(P, f)$ is an attractor of $F$. Moreover, if $P$ is collapsible, then $F$ can be chosen so that $(P, f)$ is a global attractor of $F$.*

To prove the above theorem, we need the following lemma which was proved by Brown.

**Lemma 2.2** *Let $\mathbf{X} = \text{invlim}\{X_i, p_{i,i+1}\}$ be an inverse sequence of compacta $X_i$. If each $p_{i,i+1} : X_{i+1} \to X_i$ is a near homeomorphism, then* invlim$\mathbf{X}$ *is homeomorphic to $X_i$ for each $i$.*

By using the above lemma, we can easily obtain the following.

**Lemma 2.3** *Suppose that $X$ is a compact subset of a compactum $Y$ and $f : X \to X$ is a map of $X$. If there is an extension $h : Y \to Y$ of $f$ such that $h$ is a near homeomorphism and there is a neighborhood $N$ of $X$ in $Y$ such that $h(N) \subset X$, then there is a homeomorphism $F : Y \to Y$ such that $F$ is topologically conjugate to $\tilde{h} : (Y, h) \to (Y, h)$, $(X, f)$ is contained in $Y$, $F$ is an extension of the shift map $\tilde{f} : (X, f) \to (X, f)$ of $f$, and $(X, f)$ is an attractor of $F$.*

# 3 Shift maps of the unit circle S.

In this section, for the special case $P = S$ we obtain the following.

**Theorem 3.1** *Let $f : S \to S$ be any map of the unit circle $S$, then there is a homeomorphism $F : R^3 \to R^3$ such that $(S, f)$ is contained in $R^3$, $F$ is an extension of the shift map $\tilde{f} : (S, f) \to (S, f)$, and $(S, f)$ is an attractor of $F$.*

**Corollary 3.2** *Let $f : S \to S$ be a map of the unit circle $S$ with $|\deg(f)| \geq 1$, then there is a homeomorphism $F : S^3 \to S^3$ of the 3-sphere $S^3$ such that $(S, f) \subset S^3$, $F$ is an extension of $\tilde{f}$, $(S, f)$ is an attractor of $F$ and if $X$ is the attractor of $F^{-1}$, then $F^{-1}|X : X \to X$ is topologically conjugate to the shift map $\tilde{g} : (S, g) \to (S, g)$, where $g : S \to S$ is the natural covering map with $\deg(g) = \deg(f)$.*

Note that there is a finite graph $G$ which is naturally embedded into $R^3$ and a homeomorphism $f : G \to G$ such that there is no near homeomorphism $F : R^3 \to R^3$ which in an extension of $f$. Naturally, we have the following problem.

**Problem 3.3** *If $f : G \to G$ is a map of any finite graph $G$, does there exist a homeomorphism $F : R^3 \to R^3$ such that $(G, f) \in R^3$, $F$ is an extension of the shift map $\tilde{f} : (G, f) \to (G, f)$, and $(G, f)$ is an attractor of $F$ ?*

# 4 Everywhere chaotic homeomorphisms in the sense of Li-Yorke on manifolds and k-dimensional Menger manifold.

In this section, we deal with everywhere chaotic homeomorphisms in the sense of Li-Yorke. By using the notions of attractor and shift map, we can show that every manifold and k-dimensional Menger manifold admit such chaotic homeomorphisms.

A map $f : X \to X$ is *sensitive* if there is $\tau > 0$ such that for each $x \in X$ and each neighborhood $U$ of $x$ in $X$, there is a point $y \in U$ and a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > \tau$. A map $f : X \to X$ is *accessible* if for any nonempty open sets $U, V$ of $X$ and each $\epsilon > 0$, there are two points $x \in U$, $y \in V$ and a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$.

Let $f : Y \to Y$ be a map and $\tau > 0$. A subset $S$ of $Y$ is called a *$\tau$-scrambled set* of $f$ if the next three conditions are satisfied: For each $x, y \in S$ with $x \neq y$,

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and
3. $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > \tau$ for any periodic point $p$ of $f$.

If there is an uncountable $\tau$-scrambled set $S$ of $f$, then we say that $f$ is *$\tau$-chaotic (in the sense of Li-Yorke)* on $S$. A map $f : Y \to Y$ is *everywhere chaotic* if there is $\tau > 0$ such that $f$ is $\tau$-chaotic on almost all Cantor sets in $Y$, i.e., for any closed subset $A$ of $Y$ and $\epsilon > 0$, there is an Cantor set $C$ in $Y$ such that $d_H(A, C) < \epsilon$ and $f$ is $\tau$-chaotic on $C$, where $d_H$ denotes the Hausdorff metric.

Then we have the following characterization of everywhere chaotic homeomorphism.

**Theorem 4.1** *Let $f : X \to X$ be a map of a compactum $X$. Then $f$ is everywhere chaotic if and only if $f$ is sensitive and accessible.*

Then we have the following theorem.

**Theorem 4.2** *Every compact n-manifold ($n \geq 2$) admits an everywhere chaotic homeomorphism.*

For the case of Menger manifolds, we obtain the following theorem.

**Theorem 4.3** *If $P$ is a compact connected polyhedron with $\dim P \leq k$, then there is a $k$-dimensional compact Menger manifold $M^k$ such that $M^k$ is $(k-1)$-homotopy equivalent to $P$ satisfying the following property; if a map $f : P \to P$ is $(k-1)$-homotopic to $id_P$, then there is a Z-set $P'$ such that $P'$ is homeomorphic to $P$, $(P, f)$ is contained in $M^k - P'$ and there is a homeomorphism $F : M^k \to M^k$ such that $F|P' = id_{P'}$, $(P, f)$ is a global attractor of $F|M^k - P'$ and $F$ is an extension of $\tilde{f} : (P, f) \to (P, f)$. In particular, if $P$ is $(k-1)$-connected compact polyhedron with $\dim P \leq k$, then for any map $f : P \to P$, there is a homeomorphism $F : \mu^k \to \mu^k$ of the $k$-dimensional Menger compactum $\mu^k$ such that $(P, f)$ is contained in $\mu^k - \{*\}$ $(* \in \mu^k)$, $F(*) = *$, $(P, f)$ is a global attractor of $F|\mu^k - \{*\}$, and $F$ is an extension of $\tilde{f}$.*

**Corollary 4.4** *Let $f : G \to G$ be any map of a compact connected graph $G$. Then there is a homeomorphism $F : \mu^1 \to \mu^1$ of the Menger curve $\mu^1$ such that $(G, f) \subset \mu^1 - \{*\}, F(*) = *, (G, f)$ is a global attractor of $F|\mu^1 - \{*\}$ and $F$ is an extension of $f$.*

By using the above theorem, we obtain the following.

**Theorem 4.5** *Every compact Menger manifold admits an everywhere chaotic homeomorphism. In particular, every compact Menger manifold admits a sensitive homeomorphism.*

**Remark 4.6** *There is a Z-set $X$ in $\mu^k$ $(k \geq 1)$ such that for any homeomorphism $h : X \to X$, there is no homeomorphism $F : \mu^k \to \mu^k$ so that $F$ is an extension of $h$ and $X$ is an attractor of $F$.*

For the case of chaos of Devaney, the following problem remains open.

**Problem 4.7** *Do compact Menger manifolds admit chaotic homeomorphisms in the sense of Devaney ?*