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Note on certain analytic functions

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Abstract

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{A}$, a subclass $\mathcal{B}_k(\alpha,\beta,\gamma)$ of $\mathcal{A}$ is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be a member of the subclass $\mathcal{B}_k(\alpha,\beta,\gamma)$ of $\mathcal{A}$ if it satisfies

$$\mathrm{Re}\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\} > \gamma \qquad (k \in \mathbb{N} = \{1,2,3,\ldots\};\ z \in \mathbb{U}) \tag{1.2}$$

for some $a_j \in \mathbb{R}$ $(j = 2,3,4,\ldots,k), \alpha \in \mathbb{R}, \beta \in \mathbb{R}$ $(\beta \neq 0)$, and $\gamma \in \mathbb{R}$ $(0 \leqq \gamma < k!\alpha a_k\,;\, a_1 = 1)$. We consider some properties for functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$.

Remark 1. $\mathcal{B}_k(\alpha,\beta,\gamma)$ is convex.

Because, for $f(z) \in \mathcal{B}_k(\alpha,\beta,\gamma)$ and $g(z) \in \mathcal{B}_k(\alpha,\beta,\gamma)$, we define

$$F(z) = (1-t)f(z) + tg(z) \qquad (0 \leqq t \leqq 1).$$

Then

$$\begin{aligned}
&\mathrm{Re}\left\{\alpha F^{(k)}(z) + \beta F^{(k+1)}(z)\right\} \\
&\quad = \mathrm{Re}\left\{\alpha(1-t)f^{(k)}(z) + \alpha t g^{(k)}(z) + \beta(1-t)zf^{(k+1)}(z) + \beta t z g^{(k+1)}(z)\right\} \\
&\quad = (1-t)\mathrm{Re}\left\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\right\} + t\mathrm{Re}\left\{\alpha g^{(k)}(z) + \beta z g^{(k+1)}(z)\right\} \\
&\quad > (1-t)\gamma + t\gamma = \gamma.
\end{aligned}$$

Therefore $F(z) \in \mathcal{B}_k(\alpha,\beta,\gamma)$, that is, $\mathcal{B}_k(\alpha,\beta,\gamma)$ is convex.

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In the present paper, we consider some properties of functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$.

2 Properties of the class $\mathcal{B}_1(\alpha,\beta,\gamma)$ and $\mathcal{B}_2(\alpha,\beta,\gamma)$

We begin with the statement and the proof of the following result.
For cases $k=1$, we obtain

Theorem 1. *A function $f(z)\in\mathcal{A}$ is in the class of $\mathcal{B}_1(\alpha,\beta,\gamma)$ if and only if*

$$f(z)=z+2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{1}{n((n-1)\beta+\alpha)}x^{n-1}z^n\right)d\mu(x) \tag{2.1}$$

where $\mu(x)$ is the probability measure on $X=\{x\in\mathbb{C}:|x|=1\}$.

Proof. For $f(z)\in\mathcal{A}$, we define

$$p(z)=\frac{\alpha f'(z)+\beta zf''(z)-\gamma}{\alpha-\gamma}. \tag{2.2}$$

Then $p(z)$ is Carathéodory function . Therefore we can write

$$\frac{\alpha f'(z)+\beta zf''(z)-\gamma}{\alpha-\gamma}=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x) \qquad \text{(see [1]).} \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned} z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f'(z)+zf''(z)\right) &= \frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(\alpha-\gamma)\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x)\right\} \\ &= \frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(\alpha-\gamma)\int_{|x|=1}(1+xz)(1+xz+x^2z^2+\ldots)d\mu(x)\right\}. \end{aligned} \tag{2.4}$$

Integrating the both sides of (2.4), we know that

$$\int_0^z\zeta^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f'(\zeta)+\zeta f''(\zeta)\right)d\zeta=\frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z\left(\alpha\zeta^{\frac{\alpha}{\beta}-1}+2(\alpha-\gamma)\sum_{n=1}^{\infty}x^n\zeta^{n+\frac{\alpha}{\beta}-1}\right)d\zeta\right\}d\mu(x),$$

that is, that

$$\begin{aligned} z^{\frac{\alpha}{\beta}}f'(z) &= \frac{1}{\beta}\int_{|x|=1}\left\{\beta z^{\frac{\alpha}{\beta}}+2(\alpha-\gamma)\left(\sum_{n=1}^{\infty}\frac{\beta}{n\beta+\alpha}x^nz^{n+\frac{\alpha}{\beta}}\right)\right\}d\mu(x) \\ &= z^{\frac{\alpha}{\beta}}+2(\alpha-\gamma)z^{\frac{\alpha}{\beta}}\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^nz^n\right)d\mu(x). \end{aligned}$$

Thus, we have

$$f'(z)=1+2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^nz^n\right)d\mu(x). \tag{2.5}$$

An integration of both sides in (2.5) gives us that

$$\int_0^z f'(\zeta)d\zeta = \int_0^z \left\{1 + 2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^n\zeta^n\right)d\mu(x)\right\}d\zeta,$$

or

$$\begin{aligned} f(z) &= z + 2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n\beta+\alpha)}x^n z^{n+1}\right)d\mu(x) \\ &= z + 2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{1}{n\left((n-1)\beta+\alpha\right)}x^{n-1}z^n\right)d\mu(x). \end{aligned}$$

This completes the proof of Theorem 1. □

Corollary 1. *The extreme points of $\mathcal{B}_1(\alpha,\beta,\gamma)$ are*

$$f_x(z) = z + 2(\alpha-\gamma)\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}z^n \qquad (|x|=1).$$

In view of Theorem 1, we have the following corollary for a_n.

Corollary 2. *If $f(z)\in\mathcal{A}$ is in the class $\mathcal{B}_1(\alpha,\beta,\gamma)$, then*

$$|a_n| \leqq \frac{2(\alpha-\gamma)}{n\left((n-1)\beta+\alpha\right)} \qquad (n=2,3,4,\ldots).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + 2(\alpha-\gamma)\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}z^n \qquad (|x|=1).$$

Further, the following distortion inequality follows from Theorem 1.

Corollary 3. *If $f(z)\in\mathcal{A}$ is in the class $\mathcal{B}_1(\alpha,\beta,\gamma)$, then*

$$|f(z)| \leqq |z| + 2(\alpha-\gamma)\left(\sum_{n=2}^{\infty}\frac{|z|^n}{n\left((n-1)\beta+\alpha\right)}\right) \qquad (z\in\mathbb{U}).$$

Remark 2. If $\beta>0$ and $\dfrac{\alpha}{\beta}=j$ $(j=2,3,4,\ldots)$ in Corollary 3, then we see that

$$\begin{aligned} \sum_{n=2}^{\infty}\frac{|z|^n}{n\left((n-1)\beta+\alpha\right)} &\leqq \frac{|z|^2}{\beta}\sum_{n=2}^{\infty}\frac{1}{n(n+j-1)} \\ &= \frac{|z|^2}{\beta(j-1)}\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+j-1}\right) \end{aligned}$$

$$= \frac{|z|^2}{\beta(j-1)} \sum_{n=2}^{j} \frac{1}{n} < \frac{\log(j)}{\beta(j-1)} |z|^2.$$

Therefore, we have that

$$|f(z)| < |z| + \frac{2(\alpha-\gamma)\log(j)}{\beta(j-1)} |z|^2$$
$$< 1 + \frac{2(\alpha-\gamma)\log(j)}{\beta(j-1)}.$$

Next, for cases $k = 2$ we show

Theorem 2. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{B}_2(\alpha,\beta,\gamma)$ if and only if*

$$f(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)\left((n-2)\beta + \alpha\right)} z^n \right) d\mu(x)$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

Proof. For $f(z) \in \mathcal{A}$, we define

$$p(z) = \frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma}.$$

Then $p(z)$ is Carathéodory function. Hence, we can write

$$\frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x). \tag{2.6}$$

In view of (2.6), we have that

$$\begin{aligned} z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta} f''(z) + z f'''(z)\right) &= \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1} \left\{ \gamma + (2\alpha a_2 - \gamma) \int_{|x|=1} \left(1 + 2\sum_{n=1}^{\infty} x^n z^n\right) d\mu(x) \right\} \\ &= \frac{1}{\beta} \int_{|x|=1} \left(2\alpha a_2 z^{\frac{\alpha}{\beta}-1} + 2(2\alpha a_2 - \gamma) \sum_{n=1}^{\infty} x^n z^{n+\frac{\alpha}{\beta}-1} \right) d\mu(x). \end{aligned} \tag{2.7}$$

Integrating the both sides of (2.7), we have that

$$\int_0^z \zeta^{\frac{\alpha}{\beta}-1} \left(\frac{\alpha}{\beta} f''(\zeta) + \zeta f'''(\zeta) \right) d\zeta$$
$$= \frac{1}{\beta} \int_{|x|=1} \left\{ \int_0^z \left(2\alpha a_2 \zeta^{\frac{\alpha}{\beta}-1} + 2(2\alpha a_2 - \gamma) \left(\sum_{n=1}^{\infty} x^n \zeta^{n+\frac{\alpha}{\beta}-1} \right) \right) d\zeta \right\} d\mu(x),$$

that is, that

$$z^{\frac{\alpha}{\beta}} f''(z) = \frac{1}{\beta} \int_{|x|=1} \left\{ 2\beta a_2 z^{\frac{\alpha}{\beta}} + 2(2\alpha a_2 - \gamma) \left(\sum_{n=1}^{\infty} \frac{\beta}{n\beta + \alpha} x^n z^{n+\frac{\alpha}{\beta}} \right) \right\} d\mu(x).$$

This implies that

$$f''(z)=\int_{|x|=1}\left\{2a_2+2(2\alpha a_2-\gamma)\left(\sum_{n=1}^{\infty}\frac{x^n}{n\beta+\alpha}z^n\right)\right\}d\mu(x). \tag{2.8}$$

An integration of both sides in (2.8) gives us that

$$\int_0^z f''(\zeta)d\zeta=\int_0^z\left\{2a_2+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{x^n}{n\beta+\alpha}\zeta^n\right)d\mu(x)\right\}d\zeta$$

or

$$f'(z)-1=2a_2z+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{x^n}{(n+1)(n\beta+\alpha)}z^{n+1}\right)d\mu(x).$$

Therefore, we know that

$$f'(z)=1+2a_2z+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}z^n\right)d\mu(x). \tag{2.9}$$

Applying the same method for (2.9), we see that

$$\int_0^z f'(\zeta)d\zeta=\int_0^z\left\{1+2a_2\zeta+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}\zeta^n\right)d\mu(x)\right\}d\zeta.$$

Thus, we obtain that

$$\begin{aligned} f(z) &= z+a_2z^2+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{x^{n-1}}{(n+1)n\left((n-1)\beta+\alpha\right)}z^{n+1}\right)d\mu(x)\\ &= z+a_2z^2+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=3}^{\infty}\frac{x^{n-2}}{n(n-1)\left((n-2)\beta+\alpha\right)}z^n\right)d\mu(x) \end{aligned}$$

This completes the proof of Theorem 2. □

Corollary 4. *The extreme points of $\mathcal{B}_2(\alpha,\beta,\gamma)$ are*

$$f_x(z)=z+a_2z^2+2(2\alpha a_2-\gamma)\left(\sum_{n=3}^{\infty}\frac{x^{n-2}}{n(n-1)\left((n-2)\beta+\alpha\right)}z^n\right)\qquad(|x|=1).$$

In view of Theorem 2, we have the following corollary for a_n.

Corollary 5. *If $f(z)\in\mathcal{A}$ is in the class $\mathcal{B}_2(\alpha,\beta,\gamma)$, then*

$$|a_n|\leqq\frac{2(2\alpha a_2-\gamma)}{n(n-1)\left((n-2)\beta+\alpha\right)}\qquad(n=3,4,5,\dots).$$

Equality holds for the function $f(z)$ given by

$$f(z)=z+a_2z^2+2(2\alpha a_2-\gamma)\left(\sum_{n=3}^{\infty}\frac{x^{n-2}}{n(n-1)\left((n-2)\beta+\alpha\right)}z^n\right)\qquad(|x|=1).$$

Further, the following distortion inequality follows from Theorem 2.

Corollary 6. *If $f(z) \in \mathcal{A}$ is in the class $\mathcal{B}_2(\alpha,\beta,\gamma)$, then*

$$|f(z)| \leqq |z| + |a_2||z|^2 + 2(2\alpha a_2 - \gamma)\left(\sum_{n=3}^{\infty} \frac{|z|^n}{n(n-1)\left((n-2)\beta+\alpha\right)}\right) \qquad (z \in \mathbb{U}).$$

3 Properties of the class $\mathcal{B}_k(\alpha,\beta,\gamma)$

For cases k is any natural number, we have

Theorem 3. *A function $f(z) \in \mathcal{A}$ belongs to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$ if and only if*

$$f(z) = z + a_2 z^2 + \cdots + a_k z^k + 2(k!\alpha a_k - \gamma)\int_{|x|=1}\left(\sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right) d\mu(x)$$

for $k = 1, 2, 3, \ldots$, where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

Proof. For $f(z) \in \mathcal{A}$, we define

$$p(z) = \frac{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z) - \gamma}{k!\alpha a_k - \gamma}.$$

Since $p(z)$ is Carathéodory function, we can write that

$$\frac{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z) - \gamma}{k!\alpha a_k - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x). \tag{3.1}$$

This means that

$$\begin{aligned} z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta} f^{(k)}(z) + z f^{(k+1)}(z)\right) &= \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1}\left\{\gamma + (k!\alpha a_k - \gamma)\int_{|x|=1}\left(1 + 2\sum_{n=1}^{\infty} x^n z^n\right) d\mu(x)\right\} \\ &= \frac{1}{\beta}\int_{|x|=1}\left(k!\alpha a_k z^{\frac{\alpha}{\beta}-1} + 2(k!\alpha a_k - \gamma)\sum_{n=1}^{\infty} x^n z^{n+\frac{\alpha}{\beta}-1}\right) d\mu(x). \end{aligned} \tag{3.2}$$

Integrating the both sides of (3.2), we obtain that

$$\begin{aligned} &\int_0^z \zeta^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta} f^{(k)}(\zeta) + \zeta f^{(k+1)}(\zeta)\right) d\zeta \\ &\quad = \frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z \left(k!\alpha a_k \zeta^{\frac{\alpha}{\beta}-1} + 2(k!\alpha a_k - \gamma)\left(\sum_{n=1}^{\infty} x^n \zeta^{n+\frac{\alpha}{\beta}-1}\right)\right) d\zeta\right\} d\mu(x), \end{aligned}$$

that is, that

$$z^{\frac{\alpha}{\beta}} f^{(k)}(z) = \frac{1}{\beta} \int_{|x|=1} \left\{ k!\beta a_k z^{\frac{\alpha}{\beta}} + 2(k!\alpha a_k - \gamma) \left(\sum_{n=1}^{\infty} \frac{\beta}{n\beta + \alpha} x^n z^{\frac{\alpha}{\beta}-1} \right) \right\} d\mu(x).$$

This is equivalent to

$$f^{(k)}(z) = \int_{|x|=1} \left\{ k!a_k + 2(k!\alpha a_k - \gamma) \left(\sum_{n=1}^{\infty} \frac{x^n}{n\beta + \alpha} z^n \right) \right\} d\mu(x). \tag{3.3}$$

Now, since $f(0) = 0$, $f'(0) = 1$, and $f^{(m)}(0) = m!a_m$ $(m = 2, 3, 4, \dots)$, we see that

$$\begin{aligned} \int_0^z f^{(m)}(\zeta) d\zeta &= f^{(m-1)}(z) - f^{(m-1)}(0) \\ &= f^{(m-1)}(z) - (m-1)!a_{m-1}. \end{aligned}$$

Furthermore, we know that

$$\int_0^z \int_0^{\zeta_m} \cdots \int_0^{\zeta_2} m!a_m d\zeta_1 d\zeta_2 \dots d\zeta_m = a_m z^m,$$

and

$$\sum_{n=1}^{\infty} \frac{x^n z^{n+k}}{(n+k)(n+k-1)\dots(n+1)(n\beta+\alpha)} = \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)\left((n-k)\beta+\alpha\right)}.$$

Therefore, integrating k times the both sides in (3.3), we obtain that

$$\begin{aligned} &\int_0^z \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} f^{(k)}(\zeta_1) d\zeta_1 d\zeta_2 \dots d\zeta_k \\ &= \int_0^z \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} \left\{ k!a_k + 2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta + \alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k, \end{aligned}$$

that is, that

$$\begin{aligned} f(z) &= f(0) + \int_0^z f'(0) d\zeta_1 + \int_0^z \int_0^{\zeta_2} f''(0) d\zeta_1 d\zeta_2 + \int_0^z \int_0^{\zeta_3} \int_0^{\zeta_2} f'''(0) d\zeta_1 d\zeta_2 d\zeta_3 + \dots \\ &+ \int_0^z \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} \left\{ k!a_k + 2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta + \alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k. \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \dots + a_k z^k \\ &+ 2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)\left((n-k)\beta+\alpha\right)} \right) d\mu(x). \end{aligned}$$

The proof of Theorem 3 is complete. □

Corollary 7. *The extreme points of* $\mathcal{B}_k(\alpha,\beta,\gamma)$ *are*
$f_x(z)=z+a_2z^2+a_3z^3+\cdots+a_kz^k$

$$+2(k!\alpha a_k-\gamma)\left(\sum_{n=k+1}^{\infty}\frac{x^{n-k}z^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right)\qquad(|x|=1).$$

In view of Theorem 3, we see that

Corollary 8. *If* $f(z)$ *belongs to the class* $\mathcal{B}_k(\alpha,\beta,\gamma)$*, then*

$$|a_n|\leqq\frac{2(k!\alpha a_k-\gamma)}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\qquad(n=k+1,k+2,k+3,\ldots).$$

Equality holds for the function $f(z)$ *given by*

$f(z)=z+a_2z^2+a_3z^3+\cdots+a_kz^k$

$$+2(k!\alpha a_k-\gamma)\left(\sum_{n=k+1}^{\infty}\frac{x^{n-k}z^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right)\qquad(|x|=1).$$

Further, the following distortion inequality follows from Theorem 3.

Corollary 9. *If* $f(z)$ *belongs to the class* $\mathcal{B}_k(\alpha,\beta,\gamma)$*, then*

$|f(z)|\leqq|z|+|a_2||z|^2+|a_3||z|^3+\cdots+|a_k||z|^k$

$$+2(k!\alpha a_k-\gamma)\left(\sum_{n=k+1}^{\infty}\frac{|z|^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right)\qquad(z\in\mathbb{U}).$$

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