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HODGE DECOMPOSITION OF L^r-VECTOR FIELDS ON A BOUNDED DOMAIN AND ITS APPLICATION TO THE NAVIER STOKES EQUATIONS

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ABSTRACT. We present two types of decomposition of L^r-vector fields on a bounded domain in $\mathbb{R}^3$ into the sum of the scalar and vector potentials, and the harmonic vector fields with adequate boundary conditions. These decompositions concern an extension of Friedrichs' inequality into L^r-space and some variational inequalities of vector fields which are tangential or normal to the boundary. As the application of these decompositions, we further present some existence theorems of solutions of nonhomogeneous boundary value problems for the stationary Navier-Stokes equations in a bounded domain with multiply connected boundary.

1. HODGE DECOMPOSITION OF L^r-VECTOR FIELDS ON A BOUNDED DOMAIN

In this section, we present the Hodge (or Helmholtz-Weyl) decomposition theorem of L^r-vector fields on a bounded domain in $\mathbb{R}^3$. In what follows, Ω is assumed to be a bounded domain in $\mathbb{R}^3$ with the C^∞-boundary.

First, let us recall the generalized trace theorem for the normal and tangential components on $\partial\Omega$ of the vector fields in the following spaces:

$$E^r_{div}(\Omega) \equiv \{u \in (L^r(\Omega))^3 \,|\, \operatorname{div} u \in L^r(\Omega)\} \text{ with the norm } \|u\|_{E^r_{div}} = \|u\|_r + \|\operatorname{div} u\|_r,$$

$$E^r_{rot}(\Omega) \equiv \{u \in (L^r(\Omega))^3 \,|\, \operatorname{rot} u \in (L^r(\Omega))^3\} \text{ with the norm } \|u\|_{E^r_{rot}} = \|u\|_r + \|\operatorname{rot} u\|_r,$$

where $\|\cdot\|_r$ denotes the norm in $L^r(\Omega)$ or $(L^r(\Omega))^3$. It is known that there exist bounded operators γ_ν and τ_ν on the $E^r_{div}(\Omega)$ and $E^r_{rot}(\Omega)$ with properties that

$$\gamma_\nu : u \in E^r_{div}(\Omega) \longmapsto \gamma_\nu u \in W^{1-\frac{1}{r'},r'}(\partial\Omega)^*, \quad \gamma_\nu u = u \cdot \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}),$$

$$\tau_\nu : u \in E^r_{rot}(\Omega) \longmapsto \tau_\nu u \in (W^{1-\frac{1}{r'},r'}(\partial\Omega)^*)^3, \quad \tau_\nu u = u \times \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}),$$

where ν is the outward unit normal to $\partial\Omega$ and X^* denotes the dual space of the Banach space X. In fact, these properties are easily derived from the generalized Stokes formulas such that

$$(1) \quad (u, \nabla f) = -(\operatorname{div} u, f) + <\gamma_\nu u, \gamma_0 f>_{\partial\Omega}$$
$$\text{for all } u \in E^r_{div}(\Omega) \text{ and all } f \in W^{1,r'}(\Omega),$$

$$(2) \quad (u, \operatorname{rot}\phi) = (\operatorname{rot} u, \phi) + <\tau_\nu u, \gamma_0 \phi>_{\partial\Omega}$$
$$\text{for all } u \in E^r_{rot}(\Omega) \text{ and all } \phi \in (W^{1,r'}(\Omega))^3,$$

where γ_0 denotes the usual trace operator from $W^{1,r'}(\Omega)$ onto $W^{1-\frac{1}{r'},r'}(\partial\Omega)$; $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$; $<\cdot,\cdot>_{\partial\Omega}$ is the duality pairing between $W^{1-\frac{1}{r'},r'}(\partial\Omega)^*$ and $W^{1-\frac{1}{r'},r'}(\partial\Omega)$ or its vectorial version.

Then, let us define two spaces $X^r(\Omega)$ and $V^r(\Omega)$ for $1 < r < \infty$ by

$$X^r(\Omega) = \{u \in (L^r(\Omega))^3 \mid \operatorname{div} u \in L^r(\Omega),\ \operatorname{rot} u \in (L^r(\Omega))^3,\ \gamma_\nu u = 0\},$$
$$V^r(\Omega) = \{u \in (L^r(\Omega))^3 \mid \operatorname{div} u \in L^r(\Omega),\ \operatorname{rot} u \in (L^r(\Omega))^3,\ \tau_\nu u = 0\},$$

equipped with the norm $\|u\|_{X^r}$ and $\|u\|_{V^r}$ such that

$$\|u\|_{X^r},\ \|u\|_{V^r} \equiv \|\operatorname{div} u\|_r + \|\operatorname{rot} u\|_r + \|u\|_r.$$

In Theorem 1.8 below, we shall see that both $X^r(\Omega)$ and $V^r(\Omega)$ are closed subspaces in $(W^{1,r}(\Omega))^3$, since it holds that

(3) $\|\nabla u\|_r \le C\|u\|_{X^r}$ for all $u \in X^r(\Omega)$, $\|\nabla u\|_r \le C\|u\|_{V^r}$ for all $u \in V^r(\Omega)$

respectively, where $C = C(r, \Omega)$ is a constant depending only on r and Ω. Furthermore, we define the spaces $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ by

$$X^r_\sigma(\Omega) \equiv \{u \in X^r(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega\},\ V^r_\sigma(\Omega) \equiv \{u \in V^r(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega\}.$$

Finally, we denote by $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ the space of L^r- harmonic vector fields on Ω, that is,

$$X^r_{har}(\Omega) \equiv \{u \in X^r_\sigma(\Omega) \mid \operatorname{rot} u = 0 \ \text{ in } \Omega\},$$
$$V^r_{har}(\Omega) \equiv \{u \in V^r_\sigma(\Omega) \mid \operatorname{rot} u = 0 \ \text{ in } \Omega\}.$$

Our main result is now stated as follows.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with the C^∞-boundary. Let $1 < r < \infty$.*

(I) It holds that

$$X^r_{har}(\Omega) = \{h \in (C^\infty(\bar\Omega))^3 \mid div u = 0, rot u = 0 \text{ in } \Omega,\ h\cdot\nu = 0 \text{ in } \partial\Omega\} (\equiv X_{har}(\Omega)),$$
$$V^r_{har}(\Omega) = \{h \in (C^\infty(\bar\Omega))^3 \mid div u = 0, rot u = 0 \text{ in } \Omega,\ h\times\nu = 0 \text{ in } \partial\Omega\} (\equiv V_{har}(\Omega)).$$

Furthermore, dim $X_{har}(\Omega) < \infty$ and dim $V_{har}(\Omega) < \infty$.

(II) For every $u \in (L^r(\Omega))^3$, there exist $p^{\|} \in W^{1,r}(\Omega)$, $w^{\|} \in V^r_\sigma(\Omega)$ and $h^{\|} \in X_{har}(\Omega)$ such that u can be represented as

(4) $$u = h^{\|} + rot w^{\|} + \nabla p^{\|}.$$

Such triplet $\{p^{\|}, w^{\|}, h^{\|}\}$ is subordinate to the estimate

(5) $$\|\nabla p^{\|}\|_r + \|w^{\|}\|_{V^r} + \|h^{\|}\|_r \le C\|u\|_r,$$

with the constant $C = C(r,\Omega)$ independent of u. The above decomposition (5) is unique: if u has another expression

$$u = \widetilde{h^{\|}} + rot\widetilde{w^{\|}} + \nabla\widetilde{p^{\|}},$$

for $\widetilde{p^{\|}} \in W^{1,r}(\Omega)$, $\widetilde{w^{\|}} \in V^r_\sigma(\Omega)$ and $\widetilde{h^{\|}} \in X_{har}(\Omega)$, then we have

(6) $$h^{\|} = \widetilde{h^{\|}},\ rot w^{\|} = rot\widetilde{w^{\|}},\ \nabla p^{\|} = \nabla\widetilde{p^{\|}}.$$

(III) For every $u \in (L^r(\Omega))^3$, there exist $p^\perp \in W^{1,r}_0(\Omega)$, $w^\perp \in X^r_\sigma(\Omega)$ and $h^\perp \in V_{har}(\Omega)$ such that u can be represented as

(7) $$u = h^\perp + rot w^\perp + \nabla p^\perp.$$

Such triplet $\{p^\perp, w^\perp, h^\perp\}$ is subordinate to the estimate

(8) $$\|\nabla p^\perp\|_r + \|w^\perp\|_{V^r} + \|h^\perp\|_r \le C\|u\|_r,$$

with the constant $C = C(r,\Omega)$ independent of u. The decomposition (8) is unique: if u has another expression

$$u = \widetilde{h^\perp} + rot\widetilde{w^\perp} + \nabla\widetilde{p^\perp},$$

for $\widetilde{p^{\perp}} \in W_0^{1,r}(\Omega)$, $\widetilde{w^{\perp}} \in X_\sigma^r(\Omega)$ *and* $\widetilde{h^{\perp}} \in V_{har}(\Omega)$, *then we have*

$$h^{\perp} = \widetilde{h^{\perp}}, \quad rot\, w^{\perp} = rot\, \widetilde{w^{\perp}}, \quad \nabla p^{\perp} = \nabla \widetilde{p^{\perp}}. \tag{9}$$

Remark.

(1) Since $h^{||} + \mathrm{rot} w^{||}$ in the part II above belongs to the space

$$\mathbb{L}_\sigma^r(\Omega) = \{ u \in (L^r(\Omega))^3 \mid \mathrm{div}\ u = 0 \text{ in } \Omega, \gamma_\nu u = 0 \},$$

the decomposition (4) yields the following Helmholtz decomposition for $u \in (L^r(\Omega))^3$:

$$u = v + \nabla p \quad \text{(direct sum)},$$

where $v \in \mathbb{L}_\sigma^r(\Omega)$, $p \in W^{1,r}(\Omega)$ with $1 < r < \infty$. The Helmholtz decomposition was shown for smooth vector fields on Ω when $r = 2$ by Weyl [20]. The case for more general L^r-vector fields on Ω was treated by Fujiwara-Morimoto [7] , Solonnikov [17]and Simader-Sohr [16].

(2) Similar decompositions to (4) and (7) in Theorem 1.1 for L^2-vector fields on Ω were investigated by many authors (see, for example, Friedrichs [6], Morrey [14], Georgescu [8], Foias-Temam [5], and Bendali-Domingues-Gallic [3]).

Outline of the Proof of Theorem 1.1.

We start with the proof of the part (II). Firstly, notice that the scalar and vector potentials $p^{||}$ and $w^{||}$ are determined formally as the solutions of the following boundary value problems:

$$\begin{cases} \Delta p^{||} = \mathrm{div}\ u \ \text{ in } \Omega, \\ \dfrac{\partial p^{||}}{\partial \nu} = u \cdot \nu \ \text{ on } \partial\Omega, \end{cases} \tag{10}$$

and

$$\begin{cases} \mathrm{rot}\ \mathrm{rot}\ w^{||} = \mathrm{rot}\ u \ \text{ in } \Omega, \\ \mathrm{div}\ w^{||} = 0 \ \text{ in } \Omega, \\ w^{||} \times \nu = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{11}$$

Since we just assume that $u \in (L^r(\Omega))^3$, we need to seek the weak solutions of (10) and (11) such that $p^{||} \in W^{1,r}(\Omega)$ and $w^{||} \in V_\sigma^r(\Omega)$ satisfying the following weak forms:

$$(\nabla p^{||}, \nabla \eta) = (u, \nabla \eta) \text{ for } {}^\forall \eta \in W^{1,r'}(\Omega), \tag{12}$$

$$(\mathrm{rot}\ w^{||}, \mathrm{rot}\ \psi) = (u, \mathrm{rot}\ \psi) \text{ for } {}^\forall \psi \in V_\sigma^{r'}(\Omega). \tag{13}$$

The existence of a weak solution $p^{||} \in W^{1,r}(\Omega)/\mathbb{R}$ of (10) was proven in [16]. On the other hand, the existence of a weak solution $w^{||} \in V_\sigma^r(\Omega)$ of (11) relies essentially on the following variational inequality.

Lemma 1.2. *Let* $\{\psi_1, \dots, \psi_L\}$ *be a basis of* $V_{har}(\Omega)$. *Then there is a constant* $C = C(r, \Omega)$ *such that the estimate*

$$\begin{aligned} &\|\nabla w\|_r + \|w\|_r \\ &\le C \sup \left\{ \frac{|(\mathrm{rot}\ w, \mathrm{rot}\ \psi)|}{\|\nabla \psi\|_{r'} + \|\psi\|_{r'}}; \psi \in V_\sigma^{r'}(\Omega),\ \psi \neq 0 \right\} + C \sum_{j=1}^{L} |(w, \psi_j)| \end{aligned} \tag{14}$$

holds for any $w \in V_\sigma^r(\Omega)$.

We assume that $X_{har}(\Omega) = \{0\}$ to clarify the point of the argument for a while. General case shall be treated with slight modification (see [9]). Then, the existence of the weak solution $w^{||} \in V^r_\sigma(\Omega)$ of (11) can be shown by using Lemma 1.2 as follows. Let us consider the operator $F : V^r_\sigma(\Omega) \longrightarrow V^{r'}_\sigma(\Omega)^*$ defined by

$$< Fw, \psi >= (\text{rot } w, \text{rot } \psi) \text{ for } {}^\forall\psi \in V^{r'}_\sigma(\Omega),$$

where $< \cdot, \cdot >$ denotes the duality pairing between $V^{r'}_\sigma(\Omega)^*$ and $V^{r'}_\sigma(\Omega)$. It follows from Lemma 1.2 that the range $R(F)$ of F is closed in $V^{r'}_\sigma(\Omega)^*$. Hence, by virtue of Hahn-Banach Theorem, we can conclude that

$$R(F) = V^{r'}_\sigma(\Omega)^*,$$

which implies the existence of a weak solution $w^{||}$ of (11).

Now, putting $h^{||} = u - \nabla p^{||} - \text{rot } w^{||}$, it is not difficult to see that $h^{||} \in X^r_{har}(\Omega)$ by referring to (10), (11). Further, the estimate (5) and the uniqueness of the decomposition (4) also follow from (10), (11) and the properties of $h^{||}$ stated in the part I.

Finally, we mention that Lemma 1.2 stated above is proven by combining another variational inequality for u with the boundary condition $u \times \nu|_{\partial\Omega} = 0$ in Proposition 1.7 and L^r-Friedrichs' inequality (32) in Theorem 1.8 stated below.

We now turn to the proof of the part III. In this case, we determine formally the scalar and vector potentials $p^\perp$ and $w^\perp$ as the solutions of the following boundary value problems:

$$(15) \qquad \begin{cases} \Delta p^\perp = \text{div } u \text{ in } \Omega, \\ p^\perp = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$(16) \qquad \begin{cases} \text{rot rot } w^\perp = \text{rot } u \text{ in } \Omega, \\ \text{div } w^\perp = 0 \text{ in } \Omega, \\ w^\perp \cdot \nu = 0 \text{ on } \partial\Omega, \\ w^\perp \times \nu = u \times \nu \text{ on } \partial\Omega. \end{cases}$$

By the same reasoning as in the proof of the part I, we shall seek the weak solutions $p^\perp \in W^{1,r}_0(\Omega)$, $w^\perp \in X^r_\sigma(\Omega)$ of (15), (16) which satisfy the following weak forms:

$$(17) \qquad (\nabla p^\perp, \nabla\eta) = (u, \nabla\eta) \text{ for } {}^\forall\eta \in W^{1,r'}_0(\Omega),$$

$$(18) \qquad (\text{rot } w^\perp, \text{rot } \psi) = (u, \text{rot } \psi) \text{ for } {}^\forall\psi \in X^{r'}_\sigma(\Omega).$$

The existence of a weak solution $p^\perp \in W^{1,r}_0(\Omega)/\mathbb{R}$ of (15) was shown in [15] and the existence of a weak solution $w^\perp \in X^r_\sigma(\Omega)$ of (16) can be shown by referring to the following variational inequality.

Lemma 1.3. *Let $\{\phi_1, \ldots, \phi_N\}$ be a basis of $X_{har}(\Omega)$. Then there is a constant $C = C(r, \Omega)$ such that the estimate*

$$(19) \quad \|\nabla w\|_r + \|w\|_r \\ \leq C \sup\left\{ \frac{|(\text{rot } w, \text{rot } \psi)|}{\|\nabla\psi\|_{r'} + \|\psi\|_{r'}}; \psi \in X^{r'}_\sigma(\Omega),\ \psi \neq 0 \right\} + C\sum_{j=1}^N |(w, \phi_j)|$$

holds for any $w \in X^r_\sigma(\Omega)$.

This lemma is derived from the variational inequality for u with the boundary condition $u \cdot \nu|_{\partial\Omega} = 0$ in Proposition 1.7 and the L^r-Friedrichs' inequality (30) in Theorem 1.8. The remainder of the proof is quite similar to that of the part II. So we omit it.

The statement of the part I follows from Theorem 1.8 via standard argument. We now end the outline of proof of Theorem 1.1. □

One of immediate consequences of Theorem 1.1 is

Corollary 1.4. *Let Ω be the same as in Theorem 1.1 and let $1 < r < \infty$. Then we have*

$$(L^r(\Omega))^3 = X_{har}(\Omega) \oplus rot\, V^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega) \ (\textit{direct sum}), \tag{20}$$

$$(L^r(\Omega))^3 = V_{har}(\Omega) \oplus rot\, X^r_\sigma(\Omega) \oplus \nabla W^{1,r}_0(\Omega) \ (\textit{direct sum}). \tag{21}$$

If u is in $(W^{1,r}(\Omega))^3$, it is not difficult to see that the weak solutions $w^{||}$ and $w^{\perp}$ of (11) and (16), in fact, fulfill the following boundary value problems:

$$\begin{cases} -\Delta w^{||} = \text{rot } u \ \text{ in } \Omega, \\ \text{div } w^{||} = 0 \ \text{ on } \partial\Omega, \\ w^{||} \times \nu = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{22}$$

and

$$\begin{cases} -\Delta w^{\perp} = \text{rot } u \ \text{ in } \Omega, \\ w^{\perp} \cdot \nu = 0 \ \text{ on } \partial\Omega, \\ \text{rot } w^{\perp} \times \nu = u \times \nu \ \text{ on } \partial\Omega. \end{cases} \tag{23}$$

It has been checked in [9] that both boundary value problems (22) and (23) take the form of uniformly elliptic operator with the complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [1]. Hence, by virtue of Solonnikov's results in [18] and [19], we readily have the following generalized Biot-Savart's law.

Theorem 1.5. *Let Ω be the same as in Theorem 1.1. Let $1 < r < \infty$.*

(I) Given $u \in (W^{1,r}(\Omega))^3$ with $u \cdot \nu|_{\partial\Omega} = 0$, there exist a harmonic vector field $h^{||} \in X_{har}(\Omega)$ and a 3×3 -matrix valued Green function $G^{||}(x,y)$ defined on $\overline{\Omega} \times \overline{\Omega}$ such that u is represented by

$$u(x) = h^{||}(x) + rot \int_\Omega G^{||}(x,y) rot\, u(y) dy + \nabla \int_\Omega g_N(x,y) div\, u(y) dy, \tag{24}$$
$$\textit{for all } x \in \Omega,$$

where $g_N(x,y)$ is the Neumann function of Neumann boundary value problem of the Poisson equation on Ω. Furthermore, the Green function $G^{||}$ obeys the estimates, for any multi-indices α, β,

$$|\partial^\alpha_x \partial^\beta_y G^{||}(x,y)| \le C|x-y|^{-1-|\alpha|-|\beta|} \ \textit{in } \overline{\Omega} \times \overline{\Omega}, \tag{25}$$

where C is a constant independent of x, y, α, and β.

(II) Given $u \in (W^{1,r}(\Omega))^3$ with $u \times \nu|_{\partial\Omega} = 0$, there exist a harmonic vector field $h^{\perp} \in V_{har}(\Omega)$ and a 3×3 -matrix valued Green function $G^{\perp}(x,y)$ defined on $\overline{\Omega} \times \overline{\Omega}$ such that u is represented by

$$u(x) = h^{\perp}(x) + rot \int_\Omega G^{\perp}(x,y) rot\, u(y) dy + \nabla \int_\Omega g_D(x,y) div\, u(y) dy, \tag{26}$$
$$\textit{for all } x \in \Omega,$$

where $g_D(x,y)$ *is the Green function of Dirichlet boundary value problem of the Poisson equation on* Ω. *Furthermore, the Green function* $G^{\perp}$ *obeys the estimates, for any multi-indices* α, β,

$$|\partial_x^\alpha \partial_y^\beta G^{\perp}(x,y)| \leq C|x-y|^{-1-|\alpha|-|\beta|} \quad in \ \overline{\Omega} \times \overline{\Omega}, \tag{27}$$

where C *is a constant independent of* x, y, α, *and* β.

As the application of Theorem 1.5, we present the following L^∞−gradient bounds for smooth vector fields on Ω which are tangential or normal to the boundary.

Theorem 1.6. *Suppose that* Ω *is a bounded domain in* $\mathbf{R}^3$ *with the* C^∞*-boundary. Let* $1 < r < \infty$ *and let* $u \in (W^{s,r}(\Omega))^3$ *for* $s > 1+\frac{3}{r}$ *with* $u\cdot\nu|_{\partial\Omega} = 0$ *or* $u\times\nu|_{\partial\Omega} = 0$. *Then we have*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\{1+\|u\|_{L^r(\Omega)}+(\|div\, u\|_{bmo}+\|rot\, u\|_{bmo})\log(e+\|u\|_{W^{s,r}(\Omega)})\}, \tag{28}$$

where $C = C(r,\Omega)$ *is a constant independent of* u.

As for the definition of *bmo*-norm on Ω and the proof of Theorem 1.6, see [13].

In the proof of Theorem 1.1, we used essentially the following variational inequalities and Friedrichs' inequalities in L^r-space. The complete proof of those inequalities are given in [9].

Proposition 1.7. (Variational inequalities) *Suppose that* Ω *is the same domain as in Theorem 1.1. Let* $1 < r < \infty$. *Then there is a constant* $C = C(r,\Omega)$ *such that the estimate*

$$\|\nabla u\|_r + \|u\|_r \leq C \sup\Big\{\frac{|(\nabla w, \nabla\phi) + (u,\phi)|}{\|\nabla\phi\|_{r'} + \|\phi\|_{r'}}; \quad \phi \in (C^\infty(\overline{\Omega}))^3,\ \phi\times\nu|_{\partial\Omega} = 0 \ \ (resp.\ \phi\cdot\nu|_{\partial\Omega} = 0)\Big\} \tag{29}$$

holds for any $u \in (W^{1,r}(\Omega))^3$ *with* $u\times\nu|_{\partial\Omega} = 0$ *(resp.* $u\cdot\nu|_{\partial\Omega} = 0$*).*

Theorem 1.8. (L^r-Friedrichs' inequalities) *Suppose that* Ω *is the same domain as in Theorem 1.1. Let* $1 < r < \infty$.

(I) Let $\{\phi_1, \ldots, \phi_N\}$ *be a basis of* $X_{har}(\Omega)$.

(i) It holds that $X^r(\Omega) \subset (W^{1,r}(\Omega))^3$ *with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leq C(\|div\, u\|_r + \|rot\, u\|_r + \sum_{j=1}^N |(u,\phi_j)|) \quad for\ all\ u \in X^r(\Omega), \tag{30}$$

where $C = C(r,\Omega)$.

(ii) Let $s \geq 2$. *Suppose that* $u \in (L^r(\Omega))^3$ *and* $div\, u \in W^{s-1,r}(\Omega)$, $rot\, u \in (W^{s-1,r}(\Omega))^3$ *and* $\gamma_\nu u \in W^{s-\frac{1}{r},r}(\partial\Omega)$. *Then we have* $u \in (W^{s,r}(\Omega))^3$ *with the estimate*

$$\|u\|_{W^{s,r}(\Omega)} \leq C(\|div\, u\|_{W^{s-1,r}(\Omega)} + \|rot\, u\|_{W^{s-1,r}(\Omega)} + \|\gamma_\nu u\|_{W^{s-\frac{1}{r},r}(\partial\Omega)} + \sum_{j=1}^N |(u,\phi_j)|), \tag{31}$$

where $C = C(r,\Omega)$.

(II) Let $\{\psi_1, \ldots, \psi_L\}$ *be a basis of* $V_{har}(\Omega)$.

(i) It holds that $V^r(\Omega) \subset (W^{1,r}(\Omega))^3$ *with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leq C(\|div\, u\|_r + \|rot\, u\|_r + \sum_{j=1}^L |(u,\psi_j)|) \quad for\ all\ u \in V^r(\Omega), \tag{32}$$

where $C = C(r,\Omega)$.

(ii) Let $s \geq 2$. *Suppose that* $u \in (L^r(\Omega))^3$ *and* $div u \in W^{s-1,r}(\Omega)$, $rot u \in (W^{s-1,r}(\Omega))^3$ *and* $\tau_\nu u \in (W^{s-\frac{1}{r},r}(\partial\Omega))^3$. *Then we have* $u \in (W^{s,r}(\Omega))^3$ *with the estimate*

$$(33)\quad \|u\|_{W^{s,r}(\Omega)} \leq C(\|div u\|_{W^{s-1,r}(\Omega)} + \|rot u\|_{W^{s-1,r}(\Omega)} + \|\tau_\nu u\|_{W^{s-\frac{1}{r},r}(\partial\Omega)} + \sum_{j=1}^{L} |(u,\psi_j)|),$$

where $C = C(r,\Omega)$.

Finally, we mention the fact that the bases of the harmonic spaces $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are given explicitly as the solutions of certain elliptic boundary value problems, when the domain Ω satisfies the following

Assumptions:

(i) The boundary $\partial\Omega$ has $L+1$ connected components $\Gamma_0, \Gamma_1, \ldots \Gamma_L$ of $C^\infty-$ surfaces. The $\Gamma_1, \ldots, \Gamma_L$ lie inside Γ_0 and $\Gamma_i \cap \Gamma_j = \phi$ for $i \neq j$.

(ii) There are $N-$ pieces of $C^\infty-$ surfaces $\Sigma_1, \ldots, \Sigma_N$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \neq j$, and such that

$$\dot{\Omega} \equiv \Omega \setminus \Sigma, \quad \Sigma \equiv \cup_{j=1}^{N} \Sigma_j, \quad \text{is a simply connected domain with the Lipschitzian boundary.}$$

Proposition 1.9. *Suppose that* Ω *is a bounded domain in* $\mathbb{R}^3$ *with the* C^∞*-boundary satisfying the assumptions above.*

(I) (A basis of $X_{har}(\Omega)$*) For* $i = 1, \ldots, N$, *there exists a solution* $\varphi^i \in C^\infty(\dot{\Omega})$ *unique up to an additive constant of the boundary value problem such that*

$$(34)\quad \begin{cases} \Delta\varphi^i = 0 \text{ in } \dot{\Omega}, \\ \dfrac{\partial\varphi^i}{\partial\nu} = 0 \text{ on } \partial\Omega, \\ \left[\dfrac{\partial\varphi^i}{\partial\nu_j}\right]_j = 0, \quad [\varphi^i]_j = \delta_{ij}, \\ \qquad j = 1, \ldots, N. \end{cases}$$

Moreover, the vector fields $\{\nabla\varphi^i\}_{i=1}^{N}$, *which are included in* $(C^\infty(\overline{\Omega}))^3$, *form the basis of the space* $X_{har}(\Omega)$.

Here $[f]_j$ *means the jump of the value of* f *on* Σ_j, *which is given by*

$$[f]_j = f|_{\Sigma_j^+} - f|_{\Sigma_j^-},$$

where Σ_j^+, Σ_j^- *are the two sides of* Σ_j *and* ν_j *is the unit normal on* Σ_j *directed from* Σ_j^- *toward* Σ_j^+.

(II) (A basis of $V_{har}(\Omega)$*) For* $i = 1, \ldots, L$, *there exists a unique solution* $\psi^i \in C^\infty(\overline{\Omega})$ *of the boundary value problem such that*

$$(35)\quad \begin{cases} \Delta\psi^i = 0 \text{ in } \Omega, \\ \psi^i|_{\Gamma_0} = 0, \\ \psi^i|_{\Gamma_j} = \delta_{ij}, \ j = 1, \ldots, L. \end{cases}$$

Moreover, the vector fields $\{\nabla\psi^i\}_{i=1}^{L}$, *which are included in* $(C^\infty(\overline{\Omega}))^3$, *form the basis of the space* $V_{har}(\Omega)$.

The proof of Proposition 1.9 is given in [9]. This proposition will be important in the next section.

2. Existence of solutions of nonhomogeneous boundary value problems for the stationary Navier-Stokes equations in a bounded domain

As the application of our decomposition theorems stated in §1 to the Navier-Stokes equations, we present some results on the existence of solutions of nonhomogeneous boundary value problems for the stationary Navier-Stokes equations in a bounded domain.

Let Ω be a bounded domain in $\mathbb{R}^3$ with the smooth boundary satisfying the assumptions which was stated just before Proposition 1.9 in §1. We consider the stationary Navier-Stokes equations under the inhomogeneous boundary condition:

$$(36)\qquad \begin{cases} -\mu\Delta u + (u\cdot\nabla)u + \nabla p = f \text{ in } \Omega, \\ \operatorname{div} u = 0 \text{ in } \Omega, \\ u = a \text{ on } \partial\Omega. \end{cases}$$

Here $u = u(x)$ denotes velocity vector fields on Ω, $p = p(x)$ pressure, f the external force, and a the prescribed boundary data; μ denotes the coefficient of viscosity. As a consequence of the incompressibility condition $\operatorname{div} u = 0$ of (36), the boundary data a should satisfy the following *flux condition*

$$(37)\qquad \text{(FC)} \qquad \int_{\partial\Omega} a\cdot\nu\, dS \left(= \sum_{i=0}^{L}\int_{\Gamma_i} a\cdot\nu\, dS\right) = 0.$$

Leray has shown in [12] that the problem (36) has at least one solution for any $\mu > 0$, under the *restricted flux condition*

$$(38)\qquad \text{(RFC)} \qquad \int_{\Gamma_i} a\cdot\nu dS = 0 \quad \text{for } i = 0, 1, \ldots, L.$$

However, the question asking the existence of solution of (36) with a satisfying only the flux condition (37) has been still open.

We are going to study this problem by applying our decomposition theorem stated in the preceding section. More precisely, we utilize the following variant of the decomposition (7) in Theorem 1.1 (III).

Theorem 2.1. *For any $u \in (W^{1,2}(\Omega))^3$, there are $p \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, $w \in X_\sigma^2(\Omega) \cap (W^{2,2}(\Omega))^3$, and $h \in V_{har}(\Omega)$ such that*

$$(39)\qquad u = h + \operatorname{rot} w + \nabla p.$$

Such triplet $\{p,\ w,\ h\}$ is subordinate to the estimate

$$(40)\qquad \|\nabla p\|_{W^{1,2}(\Omega)} + \|w\|_{W^{2,2}(\Omega)} + \|h\|_{W^{1,2}(\Omega)} \leq C\|u\|_{W^{1,2}(\Omega)},$$

where C is a constant depending only on Ω. Furthermore, the decomposition (39) is unique in the sense stated in Theorem 1.1.

Observing the fact that the scalar potential p and the vector potential w are determined as the solutions of (15) and (23), this theorem can be easily proven by combining the argument shown in outline of proof of Theorem 1.1 with the regularity theory for elliptic boundary value problems.

Before stating the crucial proposition, we observe an elementary fact on the basis $\{\nabla\psi^i\}_{i=1}^L$ presented in Proposition 1.9 (II).

Lemma 2.2. *Let $\{\nabla\psi^i\}_{i=1}^L$ be the basis of $V_{har}(\Omega)$ in Proposition 1.9 (II). Then, it holds that*

$$(41)\qquad \int_\Omega \nabla\psi^i\cdot\nabla\psi^j\, dx = \int_{\Gamma_j}\frac{\partial\psi^i}{\partial\nu}\, dS = \int_{\Gamma_i}\frac{\partial\psi^j}{\partial\nu}\, dS \quad \textit{for } i,j = 1,\ldots,L.$$

Proof. By integration by parts, we see from (35) that

$$\int_\Omega \nabla\psi^i \cdot \nabla\psi^j \, dx = \int_{\partial\Omega} \frac{\partial\psi^i}{\partial\nu}\psi^j \, dS - \int_\Omega \Delta\psi^i\psi^j \, dx$$
$$= \int_{\Gamma_j} \frac{\partial\psi^i}{\partial\nu} dS,$$

which implies the first equality in (41). Whereas, the symmetry of the integrand on the left hand side in the above yields promptly the second equality in (41). □

Put

$$e_{ij} = \int_{\Gamma_j} \frac{\partial\psi^i}{\partial\nu} dS. \tag{42}$$

Notice that Lemma 2.2 shows that $e_{ij} = e_{ji}$.

Based on Theorem 2.1, we now show the following extension theorem of the boundary data a which satisfies only the flux condition (37).

Proposition 2.3. *For any $a \in (W^{\frac{1}{2},2}(\partial\Omega))^3$ satisfying the flux condition (37), there exists an extension A of a into Ω such that $A = h + \text{rot } w$, $h \in V_{har}(\Omega)$, $w \in X^2_\sigma(\Omega) \cap (W^{2,2}(\Omega))^3$ and*

$$\gamma_0(h + \text{rot } w) = a, \tag{43}$$

where the γ_0 is the usual trace operator of Ω.

Furthermore, it holds that

(i) the vector potential w obeys the estimates

$$\|w\|_{W^{2,2}(\Omega)} \le c\|a\|_{W^{\frac{1}{2},2}(\partial\Omega)}, \tag{44}$$

where c is a constant depending only on Ω,

(ii) the harmonic part h is given explicitly by

$$h = \sum_{k=1}^{L}(\sum_{i=k}^{L} c_{ik}(\sum_{j=1}^{i} c_{ij} \int_{\Gamma_j} a \cdot \nu dS))\nabla\psi^k. \tag{45}$$

Here c_{ij}, $i = 1, \dots, L$, $j = 1, \dots, i$, are defined by

$$c_{ij} = \frac{\widetilde{e_{ij}}}{\sqrt{d_{i-1}d_i}}. \tag{46}$$

where $\widetilde{e_{11}} = 1$, $d_0 = 1$, and $\widetilde{e_{ij}}$ for $i > 1$ is the (i, j)-cofactor of Gram matrix of $\{\nabla\psi^j\}_{j=1}^i$:

$$E_i = \begin{pmatrix} e_{11} & \dots & e_{1i} \\ \vdots & \dots & \vdots \\ e_{i1} & \dots & e_{ii} \end{pmatrix},$$

and $d_i = \det E_i$, $i \ge 1$.

Remark. Notice that the topological characterization of the domain Ω appears explicitly through the harmonic part h above.

We are now in a position to introduce the definition of a weak solution of (36). Firstly, take an extension A of a into Ω such that

$$A \in (W^{1,2}(\Omega))^3, \ \text{div } A = 0 \text{ in } \Omega, \ \gamma_0(A) = a, \ \|A\|_{W^{1,2}(\Omega)} \le c\|a\|_{W^{\frac{1}{2},2}(\partial\Omega)}. \tag{47}$$

Note that the extension A of a given in Proposition 2.3 certainly fulfills the conditions (47). Then, putting $v = u - A$, one sees that the boundary value problem (36) is converted into

$$(48) \qquad \begin{cases} -\mu\Delta v + (v\cdot\nabla)v + (v\cdot\nabla)A + (A\cdot\nabla)v + \nabla p = \hat{f} \text{ in } \Omega, \\ \operatorname{div} v = 0 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases}$$

where

$$\hat{f} = f + \mu\Delta A - (A\cdot\nabla)A.$$

We introduce here the following function spaces: Let $\mathbb{H}^1_{0,\sigma}(\Omega)$ be the completion of $C^\infty_{0,\sigma}(\Omega)$ with respect to the Dirichlet norm $\|\nabla\cdot\|_{L^2(\Omega)}$. Here $C^\infty_{0,\sigma}(\Omega) = \{u \in (C^\infty_0(\Omega))^3 \mid \operatorname{div} u = 0 \text{ in } \Omega\}$. We denote by $\mathbb{H}^1_{0,\sigma}(\Omega)^*$ the dual space of $\mathbb{H}^1_{0,\sigma}(\Omega)$. The inner product and the norm in $L^2(\Omega)$ or $(L^2(\Omega))^3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Hereafter we assume that $f \in \mathbb{H}^1_{0,\sigma}(\Omega)^*$. Since $A \in (W^{1,2}(\Omega))^3$, one easily sees that $\hat{f} \in \mathbb{H}^1_{0,\sigma}(\Omega)^*$. Then we call v is a weak solution of (48), provided that $v \in \mathbb{H}^1_{0,\sigma}(\Omega)$ satisfies the following integral identity:

$$(49) \qquad \mu(\nabla v, \nabla\phi) - (v, (v\cdot\nabla)\phi) - (A, (v\cdot\nabla)\phi) - (v, (A\cdot\nabla)\phi) = <\hat{f}, \phi>$$

for every $\phi \in \mathbb{H}^1_{0,\sigma}(\Omega)$, where $<\cdot,\cdot>$ denotes the duality pairing between $\mathbb{H}^1_{0,\sigma}(\Omega)^*$ and $\mathbb{H}^1_{0,\sigma}(\Omega)$. Furthermore, if v is a weak solution of (48) with an A satisfying (47), we then call $u = v + A$ a weak solution of (36).

Now we state our main theorem in this section.

Theorem 2.4. *Suppose that $f \in \mathbb{H}^1_{0,\sigma}(\Omega)^*$, $a \in (W^{\frac{1}{2},2}(\partial\Omega))^3$ and $\int_{\partial\Omega} a\cdot\nu dS = 0$. Then, if the estimate*

$$(50) \qquad \sup_{w\in\chi(\Omega),\nabla w\neq 0} \frac{(h,(w\cdot\nabla)w)}{\|\nabla w\|^2} < \mu$$

holds, there exists at least one weak solution u of (36).

Here h is the harmonic part of the extension A of the boundary data a given by Proposition 2.3 (ii), and

$$\chi(\Omega) = \{w \in \mathbb{H}^1_{0,\sigma}(\Omega) \mid {}^\exists q \in L^2(\Omega) \text{ s.t. } \nabla q = -(w\cdot\nabla)w \text{ in } \Omega\}.$$

Furthermore, it should be noticed here that the estimate

$$\begin{aligned}(h,(w\cdot\nabla)w) &\le \|h\|_{L^3(\Omega)}\|w\|_{L^6(\Omega)}\|\nabla w\|_{L^2(\Omega)} \\ &\le C_{s,3}\|h\|_{L^3(\Omega)}\|\nabla w\|^2_{L^2(\Omega)}\end{aligned}$$

holds for every $w \in \mathbb{H}^1_{0,\sigma}(\Omega)(\supset \chi(\Omega))$. Here $C_{s,3}$ is the optimal constant of the Sobolev's inequality for a bounded domain $\Omega \subset \mathbb{R}^3$ such that

$$\|w\|_{L^6(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}.$$

Hence we readily derive from Theorem 2.4 the following corollary, which states that the smallness of the harmonic part h of the extension A of the boundary data a compared to the coefficient of viscosity μ ensures the existence of a weak solution of (36).

Corollary 2.5. *Let f and a be the same data as in Theorem 2.4. Then, if the estimate*

$$(51) \qquad C_{s,3}\|h\|_{L^3(\Omega)} < \mu$$

holds, there exists at least one weak solution of (36).

We give the proof of Proposition 2.3 which is divided into three steps.

Proof of Proposition 2.3.

Step 1. Since $a \in (W^{\frac{1}{2},2}(\partial\Omega))^3$ and $\int_{\partial\Omega} a \cdot \nu \, dS = 0$, the existence of an extension A of a satisfying the conditions (47) follows from the well-known results (for example, see [11]).

Step 2. For the extension A obtained in the preceding step, we apply Theorem 2.1 to obtain the decomposition such that

$$A = h + \text{rot } w + \nabla p,$$

where $h \in V_{har}(\Omega)$, $w \in X^2_\sigma(\Omega) \cap (W^{2,2}(\Omega))^3$ and $p \in W^{1,2}_0(\Omega)$. However, since

$$0 = \text{div } A = \text{div } h + \text{div } (\text{rot } w) + \text{div } (\nabla p) = \Delta p \quad \text{in } \Omega$$

and

$$\gamma_0 p = 0,$$

we conclude that $p \equiv 0$ in Ω. Therefore, we see that

$$A = h + \text{rot } w.$$

The estimate (44) is the direct consequence of (40) in Theorem 2.1. and the estimate in (47).

Step 3. By virtue of the orthogonalization of Schmidt and Proposition 1.9 (II), we obtain an *orthonormal* basis $\{w^i\}_{i=1}^L$ of $V_{har}(\Omega)$ given by

$$w^i(x) = \sum_{j=1}^{i} c_{ij} \nabla \psi^j,$$

where c_{ij} are the same constants as in (46). Then, we have

$$\begin{aligned} h &= \sum_{i=1}^{L} (A, w^i) w^i \\ &= \sum_{i=1}^{L} ((A, \sum_{j=1}^{i} c_{ij} \nabla\psi^j) \sum_{k=1}^{i} c_{ik} \nabla\psi^k) \\ &= \sum_{i=1}^{L} (\sum_{j,k=1}^{i} c_{ij} c_{ik} (A, \nabla\psi^j) \nabla\psi^k) \\ &= \sum_{k=1}^{L} (\sum_{i=k}^{L} c_{ik} (\sum_{j=1}^{i} c_{ij} (A, \nabla\psi^j))) \nabla\psi^k. \end{aligned} \tag{52}$$

On the other hand, from (35) and (47) we can see

$$\begin{aligned} (A, \nabla\psi^j) &= \int_{\partial\Omega} (a \cdot \nu) \psi^j \, dS - \int_\Omega (\text{div } A) \psi^j \, dx \\ &= \int_{\Gamma_j} a \cdot \nu \, dS. \end{aligned} \tag{53}$$

Hence, it follows from (52), (53) that

$$h = \sum_{k=1}^{L} (\sum_{i=k}^{L} c_{ik} (\sum_{j=1}^{i} c_{ij} \int_{\Gamma_j} a \cdot \nu dS)) \nabla\psi^k,$$

which is the equality (45). Now we complete the proof of Proposition 2.3. □

We finally give

Proof of Theorem 2.4.

Let A be the same extension of a as in Proposition 2.3. We are going to seek a weak solution v of (48), that is, $v \in \mathbb{H}^1_{0,\sigma}(\Omega)$ satisfying the identity (49) for every

$\phi \in \mathbb{H}^1_{0,\sigma}(\Omega)$. The existence of this weak solution is shown with the aid of the Leray-Schauder theorem, provided that the following uniform bound of the Dirichlet norm of all possible weak solutions of (48) with every $\tilde{\mu}$ ($\geq \mu$) holds:

$$\sup_{v \in S(\tilde{\mu}), \tilde{\mu} \in [\mu, \infty)} \|\nabla v\| < \infty, \tag{54}$$

where

$$S(\tilde{\mu}) = \{ v \in \mathbb{H}^1_{0,\sigma}(\Omega) \,|\, v \text{ is a weak solution of (48) with } \tilde{\mu} \text{ in place of } \mu \}.$$

However, for μ sufficiently large, we can show the existence theorem of a weak solution of (48) as follows (see, the proof of Theorem 2.1 in [2]).

Lemma 2.6. *There exist constants $\bar{\mu} = \bar{\mu}(A)$ and $\bar{M} = \bar{M}(f)$ such that, for every $v \in S(\tilde{\mu})$ with $\tilde{\mu} \in [\bar{\mu}, \infty)$, the estimate*

$$\|\nabla v\| \leq \frac{\bar{M}}{\tilde{\mu}} \tag{55}$$

holds.

So, we have only to show the uniform estimate such that

$$\sup_{v \in S(\tilde{\mu}), \tilde{\mu} \in [\mu, \bar{\mu}]} \|\nabla v\| < \infty, \tag{56}$$

in place of (54).

The proof of the uniform estimate (56) is carried out by contradiction. Let us assume that (56) does not hold. Then there exists a sequence $\{\mu_j\}_{j=1}^{\infty} \subset [\mu, \bar{\mu}]$ converging to some $\mu_0 \in [\mu, \bar{\mu}]$, for which the norm $N_j = \|\nabla v^j\|$ of the corresponding solution $v^j \in S(\mu_j)$ tends to infinity. Put $w^j = N_j^{-1} v^j$. Since $\|\nabla w^j\| = 1$ for all j, we can extract a subsequence from $\{v^j\}_{j=1}^{\infty}$ which converges weakly in $\mathbb{H}^1_{0,\sigma}(\Omega)$ and strongly in $(L^6(\Omega))^3$ to some element of $w \in \mathbb{H}^1_{0,\sigma}(\Omega) \cap (L^6(\Omega))^3$. We can assume, without loss of generality, that the whole sequence $\{w^j\}_{j=1}^{\infty}$ converges to w in the sense above mentioned. Since $v^j \in S(\mu_j)$, it holds that

$$\mu_j(\nabla v^j, \nabla \phi) - (v^j, (v^j \cdot \nabla)\phi) - (A, (v^j \cdot \nabla)\phi) - (v^j, (A \cdot \nabla)\phi) = < \hat{f}, \phi >, \tag{57}$$

for every $\phi \in \mathbb{H}^1_{0,\sigma}(\Omega)$. Taking $\phi = N_j^{-1} w^j = N_j^{-2} v^j$ in (57), we have

$$\mu_j - (A, (w^j \cdot \nabla)w^j) = < \hat{f}, w^j > N_j^{-1}.$$

Passing to the limit $j \to \infty$ in the above equality, it is not difficult to see

$$1 - \frac{1}{\mu_0}(A, (w \cdot \nabla)w) = 0, \quad \mu_0 \in [\mu, \bar{\mu}]. \tag{58}$$

On the other hand, multiplying both sides of (57) by N_j^{-2}, then taking the limit $j \to \infty$ in the resulting equality, we obtain

$$-((w \cdot \nabla)w, \phi) = (w, (w \cdot \nabla)\phi) = 0$$

for every $\phi \in \mathbb{H}^1_{0,\sigma}(\Omega)$. Hence, $w \in \mathbb{H}^1_{0,\sigma}(\Omega)$ is a weak solution of the stationary Euler equation: $^{\exists} q \in L^2(\Omega)$ such that

$$(w \cdot \nabla)w + \nabla q = 0, \quad \text{div } w = 0 \quad \text{in } \Omega. \tag{59}$$

We remark that the function q in (59), in fact, satisfies that $q|_{\Gamma_i} = c_i (const.)$ for $i = 0, 1, \ldots, L$ (see Lemma 2 in [4]).

Therefore, remembering that the extension A is given by the form $A = h + \text{rot } w$ as in Proposition 2.3, we find from (58), (59)

$$\begin{aligned} 1 &= \frac{1}{\mu_0}(A, (w \cdot \nabla)w) \\ &= -\frac{1}{\mu_0}(A, \nabla q) \\ &= -\frac{1}{\mu_0}(h + \text{rot } w, \nabla q) \\ &= -\frac{1}{\mu_0}(h, \nabla q) - \frac{1}{\mu_0}(\text{rot } w, \nabla q). \end{aligned}$$

Whereas,

$$\begin{aligned} (\text{rot } w, \nabla q) &= \int_{\partial\Omega} \nu \times w \cdot \nabla q \, dS + (w, \text{rot } (\nabla q)) \\ &= -\int_{\partial\Omega} \nu \times \nabla q \cdot w \, dS \\ &= 0, \end{aligned}$$

because $q|_{\Gamma_i} = c_i(\mathit{const.})$, $i = 0, 1, \ldots, L$, and $\nu \times \nabla$ is a tangential differentiation on $\partial\Omega$. Consequently, we finally reach

$$\begin{aligned} 1 &= -\frac{1}{\mu_0}(h, \nabla q) \\ &= \frac{1}{\mu_0}(h, (w \cdot \nabla)w). \end{aligned}$$

Hence, from the assumption (50) it follows that

$$1 = \frac{1}{\mu_0}(h, (w \cdot \nabla)w) < \frac{\mu}{\mu_0}\|\nabla w\|^2 \leq \|\nabla w\|^2,$$

which contradicts with that $\|\nabla w\| \leq 1$. We complete the proof of Theorem 2.4. □

We can show the similar statement to Theorem 2.4 when Ω is a 2-D bounded domain. We will give such generalizations of Theorem 2.4, and some concrete results concerning the existence of weak solutions of (36) when Ω is restricted to an annulus or a concentric circle and so on, in the forthcoming paper [10].

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