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Some spectral properties which imply Bishop's property (β)

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Let $\mathcal{H}$ be a separable complex Hilbert space. We say that $T\in\mathcal{B}(\mathcal{H})$ has the single valued extension property (SVEP) if for every open set $\mathcal{D}\subset\mathbb{C}$ the zero function is the only analytic solution $f:\mathcal{D}\to\mathcal{H}$ of the equation

$$(T-z)f(z)=0 \quad (z\in\mathcal{D}).$$

We say that T has the Bishop's property (β) if for every open subset $\mathcal{D}\subset\mathbb{C}$ and every sequence of analytic functions $f_n:\mathcal{D}\to\mathcal{H}$ such as

$$\|(T-z)f_n(z)\|\to 0 \text{ (as } n\to\infty)$$

uniformly on every compact subset $\mathcal{K}\subset\mathcal{D}$, the sequence f_n converges to 0 uniformly on $\mathcal{K}$ as $n\to\infty$.

If an operator $T\in\mathcal{B}(\mathcal{H})$ satisfies $T^*T\geq TT^*$ then T is called hyponormal. If $T\in\mathcal{B}(\mathcal{H})$ satisfies $(T^*T)^p\geq(TT^*)^p$ then T is called p-hyponormal. If $T\in\mathcal{B}(\mathcal{H})$ satisfies $|T^2|\geq T^*T$ then T is called class A. If $T\in\mathcal{B}(\mathcal{H})$ satisfies $\|T^2x\|\geq\|Tx\|^2$ for every unit vector $x\in\mathcal{H}$ then T is called paranormal. F. Kimura proved that every p-hyponormal operator has the Bishop's property (β). M. Cho and T. Yamazaki proved that every class A operator has the Bishop's property (β). In this talk, we introduce some properties which imply the Bishop's property (β) and show that every paranormal operator has the Bishop's property (β).

Definition 1. We say that $T\in\mathcal{B}(\mathcal{H})$ has the property (I) if

$$\|(T-\lambda)^*x_n\|\to 0 \text{ (as } n\to\infty)$$

for every $\lambda\in\sigma_a(T)$ and sequence of bounded vectors $\{x_n\}$ of $\mathcal{H}$ such as $\|(T-\lambda)x_n\|\to 0$ (as $n\to\infty$).

We say that $T\in\mathcal{B}(\mathcal{H})$ has the property (I′) if

$$\|(T-\lambda)^*x_n\|\to 0 \text{ (as } n\to\infty)$$

for every $\lambda\in\sigma_a(T)\setminus\{0\}$ and sequence of bounded vectors $\{x_n\}$ of $\mathcal{H}$ such as $\|(T-\lambda)x_n\|\to 0$ (as $n\to\infty$).

Fact. (i) Hyponormal, p-hyponormal and log-hyponormal have the property (I).
(ii) W-hyponormal, class A, class $A(s,t)$, p-quaihyponormal and (p,k)-quasihypo -normal have the property (I′). In each of these classes of operators, there is an example of operator which does not have the property (I).

Definition 2. We say that $T \in \mathcal{B}(\mathcal{H})$ has the property (II) if for every $\lambda, \mu \in \sigma_a(T)$ and for every bounded sequences of vectors x_n and y_n such that $\lambda \neq \mu$ and

$$\|(T-\lambda)x_n\| \to 0, \ \|(T-\mu)y_n\| \to 0 \text{ (as } n \to \infty),$$

the sequence $\langle x_n, y_n \rangle$ converges to 0 as $n \to \infty$.

Lemma 3 If T has the property (I′) then T also has the property (II).

Proof. Let λ, $\mu \in \sigma_a(T)$ ($\lambda \neq \mu$) and $\{x_n\}, \{y_n\}$ sequences of bounded vectors in $\mathcal{H}$ such as $\|(T-\lambda)x_n\| \to 0$ and $\|(T-\mu)y_n\| \to 0$ (as $n \to \infty$). We may assume that $\mu \neq 0$, since T has the property (I′) we have $\|(T-\mu)^* y_n\| \to 0$ (as $n \to \infty$). Hence,

$$(\lambda-\mu)\langle x_n, y_n \rangle = \langle (\lambda - T)x_n, y_n \rangle + \langle x_n, (T-\mu)^* y_n \rangle \to 0 \ (n \to \infty).$$

This implies that $\langle x_n, y_n \rangle \to 0$ and the proof is completed.

Let T be an operator which has the property (II). Then $\ker(T-\lambda) \perp \ker(T-\mu)$ for every λ and μ such as ($\lambda \neq \mu$). Hence if $(T-z)f(z) = 0$ on an open subset $\mathcal{D}$ then

$$\|f(z)\|^2 = \lim_{w \to z} \langle f(z), f(w) \rangle = 0,$$

this shows that $f(z) = 0$. We have the following theorem.

Theorem 4. If T has the property (II) then T has the (SVEP).

Example. Let A be an invertible hyponormal operator such that $A^*A - AA^*$ has dense range, which is equivalent to $\ker(A^*A - AA^*) = \{0\}$, and $T = \begin{pmatrix} A & (A^*A-AA^*)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$. Then $T^*(T^*T - TT^*)T = 0$, i.e., T is quasihyponormal and hence it is paranormal. Since $\ker T = \{0\} \oplus \mathcal{H}$, $\ker T^* = \{-A^{-1}(A^*A - AA^*)^{\frac{1}{2}} u \oplus u \ : \ u \in \mathcal{H}\}$, $\ker T$ does not reduce T. This example shows that a paranormal operator does not necessarily have the the property (I′).

As we see the previous example, paranormal does not have the property (I′) in genegral, however, we see that paranormal has the property (II).

Lemma 5. Let a, b, c_n $(n = 1, 2, 3, \cdots) \in \mathbb{C}$ such as $a \neq 0, a \neq b$, $\sup |c_n| < \infty$ and $T_n = \begin{pmatrix} a & c_n \\ 0 & b \end{pmatrix}$ satisfy

$$\liminf_{n\to\infty} \langle ((T_n)^{2*}(T_n)^2 - 2k(T_n)^* T_n + k^2)\, v, v\rangle \geq 0$$

for each $k > 0$ and $v = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2$. Then $\lim_{n\to\infty} c_n = 0$.

Lemma 6. Every paranormal operator has the property (II).

Proof. It suffices to show that if $\|(T-1)x_n\| \to 0$, $\|(T-\mu)y_n\| \to 0$ $(n \to \infty)$, $\|x_n\| = \|y_n\| = 1$ for all n and $\mu \neq 1$ then $\langle x_n, y_n\rangle \to 0$ as $n \to \infty$. Put $y_n = a_n x_n \oplus b_n z_n$, where $a_n, b_n \in \mathbb{C}$ and $z_n \in (x_n)^\perp$ with $\|z_n\| = 1$. We shall show that $a_n (= \langle y_n, x_n\rangle)$ converges to 0 as $n \to \infty$. Since $\|(T-\mu)y_n\|$ converes to 0, we have

$$\|b_n T z_n - (\mu - 1)a_n x_n \oplus \mu b_n z_n\| \to 0. \tag{1}$$

If there exists a subsequence $\{b_{n_k}\}$ which converges to 0 then $|a_{n_k}| \to 1$ and $\mu - 1 = 0$ follows from (1), a contradiction, so there exists $\epsilon > 0$ such that $|b_n| > \epsilon$ for all n. Hence,

$$\|T z_n - (\mu - 1)\frac{a_n}{b_n} x_n \oplus \mu z_n\| \to 0.$$

So, $\|T(px_n \oplus qz_n) - \begin{pmatrix} 1 & c_n x_n \otimes z_n \\ 0 & \mu \end{pmatrix} \begin{pmatrix} px_n \\ qz_n \end{pmatrix}\| \to 0$, where $c_n = (\mu - 1)\dfrac{a_n}{b_n}$, $p, q \in \mathbb{C}$ and $x_n \otimes z_n$ is a rank one operator defined by

$$(x_n \otimes z_n)u = \langle u, z_n\rangle x_n.$$

Also, we have

$$\begin{aligned}
&\|T^2(px_n \oplus qz_n) - \begin{pmatrix} 1 & c_n x_n \otimes z_n \\ 0 & \mu \end{pmatrix}^2 \begin{pmatrix} px_n \\ qz_n \end{pmatrix}\| \\
=&\|T^2(px_n \oplus qz_n) - \begin{pmatrix} 1 & (1+\mu)c_n x_n \otimes z_n \\ 0 & \mu^2 \end{pmatrix} \begin{pmatrix} px_n \\ qz_n \end{pmatrix}\| \to 0.
\end{aligned}$$

Hence

$$\|T^2(px_n \oplus qz_n)\|^2 - \| \begin{pmatrix} 1 & (1+\mu)c_n x_n \otimes z_n \\ 0 & \mu^2 \end{pmatrix} \begin{pmatrix} px_n \\ qz_n \end{pmatrix} \|^2$$

$$= \|T^2(px_n \oplus qz_n)\|^2 - \| \begin{pmatrix} 1 & (1+\mu)c_n \\ 0 & \mu^2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \|^2 \to 0,$$

$$\|T(px_n \oplus qz_n)\|^2 - \| \begin{pmatrix} 1 & c_n x_n \otimes z_n \\ 0 & \mu \end{pmatrix} \begin{pmatrix} px_n \\ qz_n \end{pmatrix} \|^2$$

$$= \|T(px_n \oplus qz_n)\|^2 - \| \begin{pmatrix} 1 & c_n \\ 0 & \mu \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \|^2 \to 0 \text{ for every } \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2.$$

Put $T_n = \begin{pmatrix} 1 & c_n \\ 0 & \mu \end{pmatrix}$ and $v_n = px_n \oplus qz_n$, then

(2)

$$\langle (T^{2*}T^2 - 2kT^*T + k^2)v_n, v_n \rangle - \langle ((T_n)^{2*}(T_n)^2 - 2k(T_n)^*(T_n) + k^2) \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \rangle \to 0$$

for each $k > 0$ and $\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2$.

Paranormality of T implies that $\langle (T^{2*}T^2 - 2kT^*T + k^2)v_n, v_n \rangle \geq 0$, so (2) implies that

$$\liminf_{n\to\infty} \langle ((T_n)^{2*}(T_n)^2 - 2k(T_n)^*(T_n) + k^2) \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \rangle \geq 0$$

for each $k > 0$ and $\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2$. By lemma 5, $\lim_{n\to\infty} c_n = 0$. Hence,

$$|a_n| = \frac{|c_n||b_n|}{|\mu - 1|} \leq \frac{|c_n|}{|\mu - 1|} \to 0.$$

This completes the proof.

For $R > 0$ and $z \in \mathbb{C}$, we denote the open ball with center z and radius R by $B(z; R)$. Let $\mathcal{D} \subset \mathbb{C}$ be an open set, $f : \mathcal{D} \to \mathcal{H}$ analytic function and

(3)
$$f(z) = \sum_{l=0}^{\infty} (z - z_0)^l a_l \ (|z - z_0| < R)$$

be Taylar expansion of f. Here $z_0 \in \mathcal{D}$, $\overline{B(z_0; R)} \subset \mathcal{D}$ and $a_l \in \mathcal{H}$. For each compact set $\mathcal{K}$, define the norm $\| \ \|_{\mathcal{K}}$ by

$$\|f\|_{\mathcal{K}} := \sup_{z\in\mathcal{K}} |f(z)|.$$

Lemma 7. Let $\mathcal{D}$, $z_0 \in \mathcal{D}$, $R > 0$ and $f(z) = \sigma_{l=0}^{\infty}(z - z_0)^l a_l$ ($|z - z_0| < R$) be as above. If f is bounded (i.e., $M = \sup_{z\in\mathcal{D}} |f(z)| < \infty$), then

$$\|a_l\| \leq \frac{M}{R^l}.$$

Lemma 8. Let $\mathcal{D}$ be an open subset of $\mathbb{C}$, $z_0 \in \mathcal{D}$, $R > 0$ such as $\overline{B(z_0;R)} \subset \mathcal{D}$, $f_n : \mathcal{D} \to \mathcal{H}$ a sequence of analytic functions and $f_n(z) = \sum_{l=0}^{\infty}(z - z_0)^l a_l^{(n)}$ ($|z - z_0| < R$) be Taylar expansion of f_n. If f_n is uniformly bounded on $\overline{B(z_0;R)}$ (i.e., $M = \sup_{n\geq 1} \|f_n\| < \infty$), then

$$\|f_n(z) - f_n(z_0)\| \leq \frac{MRr}{R-r} \text{ for all } z \in \overline{B(z_0;r)}, 0 < r < R. \tag{4}$$

A sequence of analytic functions $f_n : \mathcal{D} \to \mathcal{H}$, where $\mathcal{D}$ is a open subset of $\mathbb{C}$, converges uniformly 0 on every compact subset $\mathcal{K}$ of $\mathcal{D}$ if and only if for any $\epsilon > 0$ and any $z_0 \in \mathcal{D}$ there exists $r > 0$ and $N \in \mathbb{N}$ such taht $\overline{B(z;r)} \subset \mathcal{D}$ and $\|f_n\|_{\overline{B(z;r)}} < \epsilon$ for all $n > N$.

Theorem 9. If an operator T has the property (II) then T also has property (β).

Proof. Let $\mathcal{D} \subset \mathbb{C}$ be an open subset and $f_n : \mathcal{D} \to \mathcal{H}$ is a sequence of analytic functions such that

$$\|(T - z)f_n(z)\| \to 0 \text{ for all } z \in \mathcal{D}.$$

We sha ll show that f_n converges uniformly 0 on every compact subset $\mathcal{K}$ of $\mathcal{D}$. By conseidering $g_n = \dfrac{f_n}{1 + \|f_n\|_{\mathcal{K}}}$ instead of f_n, if necessary, we may assume $\sup_n \|f_n\|_{\mathcal{K}} < \infty$ for every compact subset $\mathcal{K}$ of $\mathcal{D}$ without loss of generality.

Let $\epsilon > 0$ be arbitary, $z_0 \in \mathcal{D}$ any point and $R > 0$ such as $\overline{B(z_0;R)} \subset \mathcal{D}$. Put $M = \sup \|f_n\|_{\overline{B(z_0;R)}} < \infty$, then

$$\|f_n(z) - f_n(z_0)\| \leq \frac{MRr}{R-r} \text{ for all } z \in \overline{B(z_0;r)},\ 0 < r < R,$$

by lemma 8. Choose $r > 0$ small enough such that $\dfrac{M^2Rr}{R-r} < \dfrac{\epsilon^2}{8}$, $\dfrac{MRr}{R-r} < \dfrac{\epsilon}{2}$ then for all n and $z \in \overline{B(z_0;r)}$

$$\|f_n(z_0)\|^2 \leq |\langle f_n(z), f(z_0)\rangle| + \frac{M^2Rr}{R-r} \leq \langle f_n(z), f_n(z_0)\rangle| + \frac{\epsilon^2}{8} \tag{5}$$

$$\|f_n(z)\| \leq \|f_n(z_0)\| + \frac{MRr}{R-r} \leq \|f_n(z_0)\| + \frac{\epsilon}{2}. \tag{6}$$

Let $z_1 \in B(z_0; r) \setminus \{z_0\}$ arbitary. Then, by assumption

$$\|(T - z_0)f_n(z_0)\| \to 0 \text{ and } \|(T - z_1)f_n(z_1)\| \to 0,$$

since T has property (II)

$$\langle f_n(z_1), f_n(z_0)\rangle \to 0.$$

Hence there exists a natural number N such that $|\langle f_n(z_1), f(z_0)\rangle| \leq \frac{\epsilon^2}{8}$ for all $n \geq N$. Thus $\|f_n(z_0)\|^2 \leq |\langle f_n(z_1), f_n(z_0)\rangle| + \frac{\epsilon^2}{8} < \frac{\epsilon^2}{8} + \frac{\epsilon^2}{8} = \frac{\epsilon^2}{4}$ by (5) and

$$\|f_n(z)\| \leq \|f_n(z_0)\| + \frac{\epsilon}{2} \leq \epsilon, \quad z \in B(z_0; r)$$

for all $n > N$ by (6). This completes the proof.

Corollary 8. Every paranormal operator has the Bishop's property (β).