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SUBMODULES OF HILBERT C^*-MODULES AND THEIR ORTHOGONAL COMPLEMENTED SUBSPACES

by

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1. Introduction

Let A be a C^*-algebra and let X be a Hilbert A-module with an A-valued inner product $\langle \cdot\ ,\ \cdot \rangle$. For any closed subspace Y of X, we denote by $Y^\perp$ the orthogonally complemented subspace of Y in X, i.e.,

$$Y^\perp = \{\, x \in X \mid \langle x\ ,\ y\rangle = 0 \text{ for all } y \in Y \,\}.$$

We say that a closed A-submodule Y of a Hilbert A-module X is *orthogonally complemented* in X if X coincides with $Y \oplus Y^\perp$, and that a closed A-submodule Y of a Hilbert A-module X is *orthogonally closed* in X if $(Y^\perp)^\perp = Y$. If Y is orthogonally complemented in X, then it is orthogonally closed in X. But the converse is not necessarily true. As is well known, every closed subspace of a Hilbert space is orthogonally complemented. This fact is a reason why it is easier to work on Hilbert spaces than on Banach spaces. Thus we have reached a question of when every closed submodule Y of a Hilbert C^*-module X is orthogonally closed or orthogonally complemented in X. The purpose of this article is to introduce complete answers (containing the author's unpublished results) to the above question, which have been obtained by Magajna [11], Schweitzer [12] and the author [4], [6]. Here it would be significant to remark that although the Hilbert C^*-modules to be considered in this article are supposed to be full, the assumption to be full is not essential in the subject (or the question) mentioned above. In the latter half part of §2, we mention the author's density theorem which says that a submodule Y of a Hilbert C^*-module X is dense in $(Y^\perp)^\perp$ in some topology.

In §3, as an easy application of results in §2 and the author's results on C^*-crossed products ([2], [5]), we discuss the orthogonal complementedness of closed submodules in crossed products of Hilbert C^*-modules.

2. Recent Developments and Density Theorem

Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *right Hilbert A-module*, we mean a right A-module X equipped with an A-valued pairing $\langle \cdot\ ,\ \cdot\rangle$, called an A-valued inner product, satisfying the following conditions:

(1) $\langle \cdot\, , \,\cdot \rangle$ is sesquilinear. (We make the convention that $\langle \cdot\, , \,\cdot \rangle$ is conjugate-linear in the first variable and is linear in the second variable.)

(2) $\langle x\, , \,y \rangle = \langle y\, , \,x \rangle^*$ for all $x, y \in X$.

(3) $\langle x\, , \,ya \rangle = \langle x\, , \,y \rangle a$ for all $a \in A$ and $x, y \in X$.

(4) $\langle x\, , \,x \rangle \geqq 0$ for all $x \in X$, and $\langle x\, , \,x \rangle = 0$ implies that $x = 0$.

(5) X is a Banach space with respect to the norm $\|x\| = \|\langle x\, , \,x \rangle\|^{\frac{1}{2}}$.

Furthermore, X is said to be *full* if X satisfies an additional condition:

(6) the closed linear span of $\{\langle x\, , \,y \rangle \mid x, y \in X\}$ coincides with A.

Let A be a C^*-algebra. Left Hilbert A-modules are defined similarly, except that we require that A should act on the left of X, that the A-valued inner product $\langle \cdot\, , \,\cdot \rangle$ should be linear in the first variable, and that $\langle ax\, , \,y \rangle = a\langle x\, , \,y \rangle$ for all $a \in A$ and $x, y \in X$.

Let A and B be C^*-algebras. We denote by ${}_A\langle \cdot\, , \,\cdot \rangle$ the A-valued inner product on the left Hilbert A-module and by $\langle \cdot\, , \,\cdot \rangle_B$ the B-valued inner product on the right Hilbert B-module, respectively. By an *$A-B$-imprimitivity bimodule* X, we mean a full left Hilbert A-module and full right Hilbert B-module X satisfying

(7) ${}_A\langle x\, , \,y \rangle \cdot z = x \cdot \langle y\, , \,z \rangle_B$ for all $x, y, z \in X$.

Here we remark that it follows from the above condntion (7) that the following condition holds:

(8) ${}_A\langle xb\, , \,y \rangle = {}_A\langle x\, , \,yb^* \rangle$ and $\langle ax,\, y \rangle_B = \langle x\, , \,a^*y \rangle_B$ for all $a \in A$, $b \in B$ and $x, y \in X$.

Now we consider the question of when every closed submodule Y of a Hilbert C^*-module X is orthogonally closed or orthogonally complemented in X. Note that there are known cases where a single closed submodule Y of a Hilbert C^*-module X becomes orthogonally closed (or orthogonally complemented) in X. For example, if T is an adjointable linear operator with closed range on X, each of the kernel of T and the range of T is orthogonally complemented. But all closed submodules of a Hilbert C^*-module are not necessarily orthogonally closed (hence complemented) in general, as the following examples show.

Example 2.1. Let $A = C([0,1])$ be the C^*-algebra of all continuous functions on the closed interval $[0,1]$.

(1) Put $X = A$ as a Hilbert A-module with the A-valued inner product $\langle \cdot\, , \,\cdot \rangle$ defined by $\langle f\, , \,g \rangle = f^*g$. Consider a closed submodule $Y = \{f \in X \mid f(0) = 0\}$. Then $Y^\perp = \{0\}$. Hence $Y \oplus Y^\perp \neq X$.

(2) Put $J = \{f \in A \mid f(0) = 0\}$ and let $X = A \oplus J$ as a Hilbert A-module with the A-valued inner product $\langle\langle \cdot\, , \,\cdot \rangle\rangle$ defined by

$$\langle\langle f_1 \oplus f_2\, , \,g_1 \oplus g_2 \rangle\rangle = \langle f_1\, , \,g_1 \rangle + \langle f_2\, , \,g_2 \rangle.$$

Consider $Y = \{f \oplus f \mid f \in J\}$. Then $Y^{\perp} = \{g \oplus (-g) \mid g \in J\}$. Hence we see that

$$Y \oplus Y^{\perp} = \{(f+g) \oplus (f-g) \mid f, g \in J\} = J \oplus J \neq X (= A \oplus J).$$

We denote by $\widehat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes of nonzero irreducible representations of A equipped with the Jacobson topology. We note that $\widehat{A}$ is a locally compact space, not necessarily a T_0-space.

The first answer to our question above was given by Magajna [11]. Here recall that a C^*-algebra A is called *dual* if it is a type I C^*-algebra with discrete spectrum, or equivalently if A is isomorphic to a C^*-subalgebra of the C^*-algebra $\mathcal{C}(\mathcal{H})$ of all compact linear operators on some Hilbert space $\mathcal{H}$.

Theorem 2.2 ([11, Theorem 1]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Then every closed right B-submodule of X is orthogonally complemented if and only if B is isomorphic to a dual C^*-algebra.*

Soon after the above result was shown, Schweitzer [12] elaborated on Magajna's theorem, that is, he showed the following theorem:

Theorem 2.3 ([12, Theorem 1]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. If every closed right B-submodule of X is orthogonally closed, then there are families $\{\mathcal{H}_i\}_{i\in I}$, $\{\mathcal{K}_i\}_{i\in I}$ of Hilbert spaces such that $A \cong \sum_{i\in I}^{\oplus} \mathcal{C}(\mathcal{H}_i)$, $B \cong \sum_{i\in I}^{\oplus} \mathcal{C}(\mathcal{K}_i)$ and $X \cong \sum_{i\in I}^{\oplus} \mathcal{C}(\mathcal{K}_i, \mathcal{H}_i)$, where the symbol " $\cong$ " means isomorphic.*

Remark that it is trivial that the converse holds in Theorem 2.3. As a corollary, furthermore we immediately have the following:

Corollary 2.4 ([12]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Then every closed right B-submodule of X is orthogonally closed if and only if every closed right B-submodule of X is orthogonally complemented in X.*

Let X be an $A-B$-imprimitivity bimodule. For a closed $A-B$-subbimodule Y of X, it is not difficult to prove that for $x \in X$,

$${}_A\langle x\,,\,y\rangle = 0 \text{ for all } y \in Y \iff \langle x\,,\,y\rangle_B = 0 \text{ for all } y \in Y.$$

Thus we see that

$$Y^{\perp} = \{\, x \in X \mid {}_A\langle x\,,\,y\rangle = \langle x\,,\,y\rangle_B = 0 \text{ for all } y \in Y\,\}.$$

Theorem 2.5 ([4, Theorem 2.3]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Consider the following conditions:*

(1) *The spectrum $\widehat{A}$ of A is discrete in the Jacobson topology.*

(2) *The spectrum $\widehat{B}$ of B is discrete in the Jacobson topology.*

(3) *Every closed $A-B$-subbimodule of X is comlemented in X.*

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *If either $\widehat{A}$ or $\widehat{B}$ is a T_1-space, then conditions* (1) – (3) *are equivalent.*

In the above theorem, the implication (3) $\Longrightarrow$ (2) is not true in general. Hence the assumption that either $\widehat{A}$ or $\widehat{B}$ be a T_1-space is necessary to show the implication (3) $\Longrightarrow$ (2).

Recall that the primitive spectrum Prim(A) of a C^*-algebra A is the topological space, consisting of all primitive ideals of A, endowed with the Jacobson topology.

Theorem 2.6 ([4, Theorem 2.6]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Consider the following conditions:*

(1) *The primitive spectrum* Prim(A) *of A is discrete in the Jacobson topology.*

(2) *The primitive spectrum* Prim(B) *of B is discrete in the Jacobson topology.*

(3) *Every closed $A-B$-subbimodule of X is comlemented in X.*

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *If either* Prim(A) *or* Prim(B) *is a T_1-space, then conditions* (1) – (3) *are equivalent.*

Note that a separable C^*-algebra is dual if and only if $\widehat{A}$ is discrete ([3]). But even though a *nonseparable* C^*-algebra A has discrete spectrum $\widehat{A}$, A is not necessariry dual.

Theorem 2.7 ([6, Theorem 2.3]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Then every closed $A-B$-submodule of X is orthogonally closed in X if and only if every closed $A-B$-submodule of X is orthogonally complemented in X.*

Corollary 2.8 ([6, Corollay 2.4]). *Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Consider the following conditions* (1) – (4) :

(1) *The spectrum $\widehat{A}$ of A is discrete in the Jacobson topology.*

(2) *The spectrum $\widehat{B}$ of B is discrete in the Jacobson topology.*

(3) *Every closed $A-B$-submodule of X is complemented in X.*

(4) *Every closed $A-B$-submodule of X is orthogonally closed in X.*

Then we have (1) $\iff$ (2) $\Longrightarrow$ (3) $\iff$ (4). *If either* $\widehat{A}$ *or* $\widehat{B}$ *is a* T_1*-space, then conditions* (1) – (4) *are equivalent.*

Remark 2.9. Let A and B be C^*-algebras and let X be an $A-B$-imprimitivity bimodule. Consider the following conditions (1) - (4).

(1) Every closed B-submodule of X is orthogonally closed in X.

(2) Every closed B-submodule of X is orthogonally comlemented in X.

(3) Every closed $A-B$-submodule of X is orthogonally closed in X.

(4) Every closed $A-B$-submodule of X is orthogonally comlemented in X.

Then we have (1) $\iff$ (2) $\Longrightarrow$ (3) $\iff$ (4). The implication (2) $\Longleftarrow$ (3) does not hold in general. For example, consider a UHF C^*-algebra B which is not of type I. Then $\widehat{B}$ is not T_1-space, hence not discrete. Since B is simple, X is automatically full. Then X has no nontrivial $A-B$-subbimodules, hence condition (4) above holds. Since B is not a dual C^*-algebra, condition (2) does not hold.

Recall here that if H is a Hilbert space, every subspace K of H is dense in $K^{\perp\perp}$ in the norm topology. Thus we have a problem of how such a result is generalized for Hilbert C^*-modules. In the rest of this section, we mention the density theorem that any submodule Y of a Hilbert C^*-module is dense in $Y^{\perp\perp}$ with respect to some topology. Its proof will appear in [10]. For this, we must intoroduce such a topology on Hilbert C^*-modules.

Definition 2.10. Let A be a C^*-algebra and let X be a Hilbert A-module. The $\sigma(X, S(A))$-topology on X is the topology generated by a basis consisting of the sets

$$\mathcal{O} = \{x \in X \mid |\varphi_i(\langle x - x_0 \,,\, y\rangle)| < \varepsilon \text{ for } i = 1, 2, \cdots, n\}$$

for all choices of $x_0 \in X, \varphi_1, \varphi_2, \cdots, \varphi_n \in S(A)$ and $\varepsilon > 0$. For a subset Y of X, we denote by $\overline{Y}$ the closure of Y in the $\sigma(X, S(A))$-topology.

Theorem 2.11. **(Density)** *Let A be a C^*-algebra and let X be a Hilbert A-module. Then every A-submodule Y is dense in $Y^{\perp\perp}$ in the $\sigma(X, S(A))$-topology, that is,*

$$\overline{Y} = Y^{\perp\perp}.$$

If we apply our density theorem above to Hilbert spaces H, we easily obtain the well known result that every subspace K of H is dense in $K^{\perp\perp}$ in the norm topology.

Corollary 2.12. *Let A be a C^*-algebra and let X be a Hilbert A-module. For any A-submodule Y, we have $Y \oplus Y^{\perp}$ is dense in X in the $\sigma(X, S(A))$-topology.*

The following result characterizes the closedness of submodules in terms of the $\sigma(X, S(A))$-topology.

Corollary 2.13. *Let A be a C^*-algebra and let X be a Hilbert A-module. An A-submodule Y is closed in the $\sigma(X, S(A))$-topology, that is, $Y = \overline{Y}$, if and only if there exists a submodule Y_0 of X such that $Y = Y_0^{\perp}$.*

The following result characterizes dual C^*-algberas in terms of the $\sigma(X, S(A))$-topology.

Corollary 2.14. *Let A be a C^*-algebra and let X be a Hilbert A-module. Then every closed A-submodule Y of X is closed in X in the $\sigma(X, S(A))$-topology, that is, $Y = \overline{Y}$ if and only if A is a dual C^*-algebra.*

3. Applications to Crossed Products of Hilbert C^*-modules

Let (A, G, α) be a C^*-dynamical system. By a *C^*-dynamical system,* we mean a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G with left invariant Haar measure ds and a group homomorphism α from G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $K(A, G)$ the linear space of all continuous functions from G into A with compact support and by $L^1(A, G)$ the completion of $K(A, G)$ by the L^1-norm. Note that $L^1(A, G))$ admits the Banach*-algebra structure. Then the C^*-crossed product $A \times_\alpha G$ of A by G is the enveloping C^*-algebra of $L^1(A, G)$.

Recall that for any covariant representation $(\pi, u, \mathcal{H})$, the representation $(\pi \times u, \mathcal{H})$ of $A \times_\alpha G$ is defined by

$$(\pi \times u)(x) = \int_G \pi(x(t))u_t dt, \quad x \in L^1(A, G).$$

For a given representation $(\pi_A, \mathcal{H})$ of A, we always denote by $\widetilde{\pi_A}$ the representation of A on the Hilbert space $L^2(\mathcal{H}_A, G)$ defined by

$$(\widetilde{\pi_A}(a)\xi)(t) = \pi_A(\alpha_{t^{-1}}(a))\xi(t)$$

for $a \in A, \xi \in L^2(\mathcal{H}_A, G)$, where $L^2(\mathcal{H}_A, G)$ is the Hilbert space of all square integrable functions from G into $\mathcal{H}_A$. Define a unitary representation λ^A on $L^2(\mathcal{H}_A, G)$ by

$$(\lambda^A{}_s\xi)(t) = \xi(s^{-1}t).$$

Then $(\widetilde{\pi_A}, \lambda^A, L^2(\mathcal{H}_A, G))$ is a covariant representation of A. If π_A is faithful, then $(\widetilde{\pi_A} \times \lambda^A)(A \times_\alpha G)$ is called the reduced C^*-crossed product of A by G and we denote it by $A \times_{\alpha,r} G$.

Let (A, G, α) and (B, G, β) be C^*-dynamical systems and let X be a left A-Hilbert module (resp. a right B-Hilbert module). Suppose that there exists an *α-compatible action* (resp. a *β-compatible action*) η of G on X, that is, a group homomorphism from G into the group of invertible linear transformations on X such that

(E1) $\eta_t(a \cdot x) = \alpha_t(a)\eta_t(x)$ (resp. $\eta_t(x \cdot b) = \eta_t(x)\beta_t(b)$);

(E2) ${}_A\langle\eta_t(x), \eta_t(y)\rangle = \alpha_t({}_A\langle x, y\rangle)$ (resp. $\langle\eta_t(x), \eta_t(y)\rangle_B = \beta_t(\langle x, y\rangle_B)$)

for each $t \in G, a \in A, b \in B, x, y \in X$; and such that $t \to \eta_t(x)$ is continuous from G into X for each $x \in X$ in norm.

Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Let η be an (α, β)-compatible action of G on a left A- and right B-Hilbert module X, that is, η is an α-compatible and β-compatible action of G. Then there exists a left $(A \times_\alpha G)$- and right $(B \times_\beta G)$- Hilbert module $X \times_\eta G$ containing a dense subspace $K(X, G)$ such that

$$(f \cdot x)(s) = \int_G f(t)\eta_t(x(t^{-1}s))dt,$$

$$(x \cdot g)(s) = \int_G x(t)\beta_t(g(t^{-1}s))dt,$$

$${}_{A\times_\alpha G}\langle x\ ,\ y\rangle(s) = \int_G {}_A\langle x(st^{-1})\ ,\ \eta_s(y(t^{-1}))\rangle dt,$$

$$\langle x, y\rangle_{B\times_\beta G}(s) = \int_G \beta_{t^{-1}}(\langle x(t)\ ,\ y(ts)\rangle_B)dt$$

for $f \in K(A, G), x, y \in K(X, G)$, and $g \in K(B, G)$. We call $X \times_\eta G$ the *(full) crossed product* of X by G. Here $K(X, G)$ (resp. $K(A, G)$ and $K(B, G)$) denotes the set of continuous functions from G into X (resp. A and B) with compact support.

Let A and B be C^*-algebras and let X be an A-B-Hilbert bimodule. We say that a *representation* of X as an A-B-Hilbert bimodule is a triple (π_A, π_X, π_B) consisting of nondegenerate representations π_A and π_B of A and B on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, together with a linear map $\pi_X : X \to \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that

(R1) $\pi_X(ax) = \pi_A(a)\pi_X(x)$,

(R2) $\pi_X(xb) = \pi_X(x)\pi_B(b)$,

(R3) $\pi_A({}_A\langle x, y\rangle) = \pi_X(x)\pi_X(y)^*$ for $x, y \in X$, and

(R4) $\pi_B(\langle x, y\rangle_B) = \pi_X(x)^*\pi_X(y)$

for all $a \in A, x, y \in X$, and $b \in B$, where $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ denotes the set of all bounded linear operators from $\mathcal{H}_B$ into $\mathcal{H}_A$.

Now we suppose that X is a right Hilbert A-module with an A-inner product $\langle\cdot\ ,\ \cdot\rangle$. We define a linear operator $\Theta_{x,y}$ on X by

$$\Theta_{x,y}(z) = x \cdot \langle y\ ,\ z\rangle$$

for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\Theta_{x,y} \mid x\, y \in X\}$. Then X is a full left $\mathcal{K}(X)$-Hilbert module with respect to the natural left action defined by $t \cdot x = t(x)$ for $t \in \mathcal{K}(X)$ and $x \in X$ and the inner product ${}_{\mathcal{K}(X)}\langle x\ ,\ y\rangle \equiv \Theta_{x,y}$. Thus X is a $\mathcal{K}(X)$-A-Hilbert bimodule.

Definition 3.1. Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Let X be a left A-Hilbert module and a right B-Hilbert module with an (α, β)-compatible action η of G. Suppose that $(\pi_A, u, \mathcal{H}_A)$ and $(\pi_B, v, \mathcal{H}_B)$ are covariant representations of A and B, respectively. Then we say that a representation $(\pi_A, \pi_X, \pi_B, u, v)$ of X into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ is *covariant* if

$$\pi_X(\eta_t(x)) = u_t \pi_X(x) v_t^* \quad \text{for all} \quad x \in X, t \in G.$$

Then we can define the representation $\pi_X \times v$ of $X \times_\eta G$ into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ by

$$(\pi_X \times v)(x) = \int_G \pi_X(x(s)) v_s ds$$

for $x \in K(X, G)$. Thus we obtain the representation $(\pi_A \times u,\ \pi_X \times v,\ \pi_B \times v)$ of $X \times_\eta G$ into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$.

Define the representation $\widetilde{\pi_X}$ of X into $\mathcal{B}(L^2(\mathcal{H}_B, G), L^2(\mathcal{H}_A, G))$ by

$$(\widetilde{\pi_X}(x)\xi)(t) = \pi_X(\eta_{t^{-1}}(x))\xi(t)$$

for all $x \in X$, $t \in G$ and $\xi \in L^2(\mathcal{H}_B, G)$. Then $(\widetilde{\pi_A}, \widetilde{\pi_X}, \widetilde{\pi_B}, \lambda^A, \lambda^B)$ is a covariant representation of X into $\mathcal{B}(L^2(\mathcal{H}_B, G), L^2(\mathcal{H}_A, G))$, that is, we have

$$(\widetilde{\pi_X}(\eta_s(x))\xi)(t) = ((\lambda^A{}_s \widetilde{\pi_X}(x) \lambda^B{}_s^*)\xi)(t)$$

for $s, t \in G$ and $\xi \in L^2(\mathcal{H}_B, G)$. The following definition of a reduced crossed product of a Hilbert C^*-module was introduced by the author [9].

Definition 3.2. Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Let η be an (α, β)-compatible action of G on a left A- and right B-Hilbert module X. Consider a representation (π_A, π_X, π_B) of X, where $(\pi_A, \mathcal{H}_A)$ and $(\pi_B, \mathcal{H}_B)$ are faithful representations of A and B, respectively. Then π_X is automatically faithful. Consider the representation $\widetilde{\pi_X} \times \lambda^B$ of $X \times_\eta G$ into $\mathcal{B}(L^2(\mathcal{H}_B, G), L^2(\mathcal{H}_A, G))$. Then we say that $(\widetilde{\pi_X} \times \lambda^B)(X \times_\eta G)$ is the *reduced crossed product* of X by G, and we denote it by $X \times_{\eta,r} G$. It is easy to verify that $X \times_{\eta,r} G$ is a left $(A \times_{\alpha,r} G)$-Hilbert module and a right $(B \times_{\beta,r} G)$-Hilbert module. We remark that $X \times_{\eta,r} G$ does not depend on the choice of a pair of faithful representations π_A and π_B of A and B.

Proposition 3.3([9, Proposition 2.13]). *Let (A, G, α) be a C^*-dynamical system and let X be a right A-Hilbert module. Suppose that there exists an α-compatible action η of G on X. If G is amenable, then $X \times_\eta G$ is isomorphic to $X \times_{\eta,r} G$.*

The reader is referred to [9] for crossed products of Hilbert C^*-modules and their representations mentioned in the above.

Given a C^*-dynamical system (A, G, α), α induces the natural action of G on $\widehat{A}$ which is defined by

$$(t, [\pi]) \in G \times \widehat{A} \to [\pi \circ \alpha_{t^{-1}}] \in \widehat{A}.$$

This map makes G into a topological transformation group acting on $\widehat{A}$. Let $S_{[\pi]}$ be the stability group at $[\pi]$, which is defined by $S_{[\pi]} = \{t \in G | [\pi \circ \alpha_{t^{-1}}] = [\pi]\}$. If all stability groups are trivial, i.e., $S_{[\pi]}$ consists only of the identity of G at every $[\pi] \in \widehat{A}$, it is said that G acts *freely* on $\widehat{A}$.

Theorem 3.4. *Let (A, G, α) be a C^*-dynamical system and let X be a full right A-Hilbert module with an α-compatible action η of G. Suppose that G acts freely on $\widehat{A}$. Then the following condtions* (1) – (3) *are equivalent:*

(1) *For any closed right A-submodule Y of X, we have $X = Y \oplus Y^{\perp}$.*

(2) *G is discrete, and for any closed $(A \times_\alpha G)$-submodule $\widetilde{Y}$ of $X \times_\eta G$, we have $X \times_\eta G = \widetilde{Y} \oplus (\widetilde{Y})^{\perp}$*

(3) *G is discrete, and for any closed right $(A \times_{\alpha,r} G)$-submodule $\widetilde{Y}$ of $X \times_{\eta,r} G$, we have $X \times_{\eta,r} G = \widetilde{Y} \oplus (\widetilde{Y})^{\perp}$*

Proof. (1) $\Longrightarrow$ (2). Since X is a full right A-Hilbert module, it is a $\mathcal{K}(X) - A$-imprimitivity bimodule. Hence it folllows from Theorem 2.2 that A is a dual C^*-algebra. Since G acts freely on $\widehat{A}$, G is discrete and $A \times_\alpha G$ is a dual C^*-algebra ([2, Theorem], or [5]). This shows codition (2) by Theorem 2.2, since $X \times_\eta G$ is a full right $(A \times_\alpha G)$-Hilbert bmodule.

(2) $\Longrightarrow$ (3). This is trivial.

(3) $\Longrightarrow$ (1). Condition (3) implies that $A \times_{\alpha,r} G$ is a dual C^*-algebra. Since G is discrete, A is embedded into $A \times_{\alpha,r} G$ as a C^*-algebra. Since every C^*-subalgebra of a dual C^*-algebra is dual, so is also A. Hence condition (1) follows from Thoerem 2.2. □

For a C^*-dynamical system (A, G, α), we say that α is *pointwise unitary* if for every irreducible representation (π, H_π) of A, there exists a strongly continuous unitary representation u of G on the Hilbert space H_π such that

$$\pi(\alpha_t(x)) = u_t \pi(x) u_t^*$$

for all $x \in A$ and $t \in G$.

Theorem 3.5. *Let (A, G, α) be a C^*-dynamical system and let X be a full right A-Hilbert module with an α-compatible action η of G. Suppose that G is a compact group. Consider the following condtions.*

(1) *For any closed right A-submodule Y of X, we have $X = Y \oplus Y^{\perp}$.*

(2) *For any closed $(A \times_\alpha G)$-submodule $\widetilde{Y}$ of $X \times_\eta G$, we have $X \times_\eta G = \widetilde{Y} \oplus (\widetilde{Y})^{\perp}$.*

Then we have (1) $\Longrightarrow$ (2). *Furthermore we suppose that* G *is* (*compact*) *abelian. If* A *is of type I and* α *is pointwise unitary, we have* (2) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2). It follows from the proof of Theorem 3.4 above that A is a dual C^*-algebra. On the other hand, by Imai-Takai's duality we see that there exists a coaction δ of G on $A \times_\alpha G$ such that $(A \times_\alpha G) \times_\delta G \cong A \otimes \mathcal{C}(L^2(G))$ (see the paragrapf following Remark 3.6 for the detail of crossed products by coactions). Since G is compact, $A \times_\alpha G$ is embedded into $(A \times_\alpha G) \times_\delta G$ as a C^*-algebra. Since $A \otimes \mathcal{C}(L^2(G))$ is a dual C^*-algebra, so is $A \times_\alpha G$. This shows condition (2) by Theorem 2.2.

(2) $\Longrightarrow$ (1). Suppose that G is abelian. Since A is of type I and α is pointwise unitary, it follows from the proof of (ii) $\Longrightarrow$ (i) in [5, Theorem 3.2] that the action of $\widehat{G}$ induced by the dual action $\widehat{\alpha}$ of $\widehat{G}$ acts freely on the spectrum $(A \times_\alpha G)\widehat{}$. Since $\widehat{G}$ is discrete, $(A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}$ is a dual C^*-algebra. Hence $A \otimes \mathcal{C}(L^2(G))$ is also a dual C^*-algebra, which shows that A is dual. Thus condition (1) follows from Thoerem 2.2. □

Remark 3.6. Let (A, G, α) be a C^*-dynamical system. Suppose that B is an α-invariant C^*-subalgebra of A. Then $B \times_\alpha G$ is not necessarily embedded into $A \times_\alpha G$ as a C^*-algebra (see [1]). Let (A, G, α) and (B, G, β) be C^*-dynamical systems and let X be an $A-B$-imprimitivity bimodule with an (α, β)-compatible action η of G. If Y is an η-invariant closed $A-B$-submodule of X, then $Y \times_\eta G$ is embedded into $X \times_\eta G$ as an $(A \times_\alpha G) - (B \times_\beta G)$-bimodule. This is not suprising. For, the closed $A-B$-submodules in the case of Hilbert C^*-modules correspond to the closed ideals in the case of C^*-algebras. The author will discuss the further details of this subject elsewhere.

Here we briefly review the definition of the crossed products by coactions. Let G be a locally compact group with left invariant Haar measure ds. We denote by λ the left regular representation of G on $L^2(G)$. We define the representation $\tilde{\lambda}$ of $L^1(G)$ on $L^2(G)$ by

$$\tilde{\lambda}(f) = \int_G f(s)\lambda_s ds$$

for $f \in L^1(G)$. Then the reduced group C^*-algebra $C_r^*(G)$ of G is defined as the norm closure of $\tilde{\lambda}(L^1(G))$ in the set of all bounded linear operators on $L^2(G)$. We write λ_f for $\tilde{\lambda}(f)$ above.

Let A be a C^*-algebra and denote by $M(A \otimes_{\min} C_r^*(G))$ the multiplier algebra of the injective C^*-tensor product $A \otimes_{\min} C_r^*(G)$. We then define the C^*-subalgebra $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ of $M(A \otimes_{\min} C_r^*(G))$ by

$$\widetilde{M}(A \otimes_{\min} C_r^*(G)) =$$
$$\{m \in M(A \otimes_{\min} C_r^*(G)) \mid m(1 \otimes x), (1 \otimes x)m \in A \otimes_{\min} C_r^*(G) \text{ for all } x \in C_r^*(G)\}.$$

We denote by W_G the unitary operator on $L^2(G \times G)$ defined by

$$(W_G\xi)(s,t) = \xi(s, s^{-1}t) \text{ for } \xi \in L^2(G \times G) \text{ and } s, t \in G.$$

Define the homomorphism δ_G from $C_r^*(G)$ into $\widetilde{M}(C_r^*(G) \otimes_{\min} C_r^*(G))$ by

$$\delta_G(\lambda_f) = W_G(\lambda_f \otimes 1)W_G{}^* \quad \text{for} \quad f \in L^1(G).$$

We say that an injective homomorphism δ from A into $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ is a *coaction* of a locally compact group G on A if δ satisfies:

(C1) there is an approximate identity $\{e_i\}$ for A such that $\delta(e_i) \to 1$ strictly in $\widetilde{M}(A \otimes_{\min} C_r^*(G))$;

(C2) $(\delta \otimes \mathrm{id})(\delta(a)) = (\mathrm{id} \otimes \delta_G)(\delta(a))$ for all $a \in A$, where we always denote by *id* the identity map on each considered set.

Furthermore, the coaction δ is said to be *nondegenerate* if it satisfies the additional condition:

(C3) for every nonzero $\varphi \in A^*$, there exists $\psi \in C_r^*(G)^*$ such that $(\varphi \otimes \psi) \circ \delta \neq 0$.

This is equivalent to the condition that the closed linear span of $\delta(A)(1_A \otimes C_r^*(G))$ be equal to $A \otimes_{\min} C_r^*(G)$, where 1_A is the identity of the multiplier algebra $M(A)$ for A. (In (C2) and (C3), we implicitly extended δ to $M(A \otimes_{\min} C_r^*(G))$, which is ensured by condition (C1).) We always denote by the same symbol δ the extension of δ to $M(A \otimes_{\min} C_r^*(G))$.

Let δ be a coaction of a locally compact group G on A and let $C_0(G)$ be the set of all continuous functions on G vanishing at infinity. We denote by M_f the multiplication operator on $L^2(G)$ given by $f \in C_0(G)$ which is defined by

$$(M_f\xi)(t) = f(t)\xi(t)$$

for all $\xi \in L^2(G)$. Then the *crossed product* $A \times_\delta G$ of A by δ is the C^*-subalgebra of $M(A \otimes \mathcal{C}(L^2(G)))$ generated by the set $\{\delta(a)(1 \otimes M_f) | a \in A, f \in C_0(G)\}$, where $\mathcal{C}(L^2(G))$ always denotes the C^*-algebra of all compact linear operators on $L^2(G)$.

Let X be a right A-Hilbert module. We define a linear operator $\Theta_{x,y}$ on X by

$$\Theta_{x,y}(z) = x \cdot \langle y \, , \, z \rangle$$

for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\Theta_{x,y} \mid x, y \in X\}$. Then X is a full left $\mathcal{K}(X)$-Hilbert module with respect to the natural left action defined by $t \cdot x = t(x)$ for $t \in \mathcal{K}(X)$ and $x \in X$ and the inner product ${}_{\mathcal{K}(X)}\langle x \, , \, y \rangle \equiv \Theta_{x,y}$. Thus X is a $\mathcal{K}(X)$-A-Hilbert module.

We denote by $M(X)$ the set of all multipliers of a right Hilbert A-module X. We refer to $M(X)$ as the *multiplier bimodule* of X, and note that $M(X)$ is an $M(\mathcal{K}(X)) - M(A)$-Hilbert module, where $M(\mathcal{K}(X))$ and $M(A)$ are the multiplier algebras for $\mathcal{K}(X)$ and A, respectively.

Let $\delta_A : A \to \widetilde{M}(A \otimes_{\min} C_r^*(G))$ be a coaction of a locally compact group G on the C^*-algebra A and let $\delta_B : B \to \widetilde{M}(B \otimes_{\min} C_r^*(G))$ be a coaction of G on the C^*-algebra B. Suppose that X is a $B - A$-Hilbert bimodule. We say that a linear map $\delta_X : X \to M(X \otimes C_r^*(G))$ is a δ_A*-compatible coaction* (resp. a

δ_B*-compatible coaction*) of the locally compact group G on X if δ_X satisfies the following conditions:

(D1) $(1_B \otimes z)\delta_X(x)$ lies in $X \otimes C_r^*(G)$ for all $x \in X$ and $z \in C_r^*(G)$;
(resp. (D1)′ $\delta_X(x)(1_A \otimes z)$ lies in $X \otimes C_r^*(G)$ for all $x \in X$ and $z \in C_r^*(G)$;)

(D2) $\delta_X(b \cdot x) = \delta_B(b) \cdot \delta_X(x)$ for all $x \in X$ and $b \in B$;
(resp. (D2)′ $\delta_X(x \cdot a) = \delta_X(x) \cdot \delta_A(a)$ for all $x \in X$ and $a \in A$;)

(D3) $\delta_B({}_B\langle x\ ,\ y\rangle) = {}_{M(B\otimes_{\min} C_r^*(G))}\langle \delta_X(x)\ ,\ \delta_X(y)\rangle$;
(resp. (D3)′ $\delta_A(\langle x\ ,\ y\rangle_A) = \langle \delta_X(x)\ ,\ \delta_X(y)\rangle_{M(A\otimes_{\min} C_r^*(G))}$;)

(D4) $(\delta_X \otimes \mathrm{id}) \circ \delta_X = (\mathrm{id} \otimes \delta_G) \circ \delta_X$.
(In (D1) and (D2) (resp. in (D1)′ and (D2)′), we implicitly extended the module action on the $(B \otimes_{\min} C_r^*(G)) - (A \otimes_{\min} C_r^*(G))$-Hilbert module $X \otimes C_r^*(G)$ to actions of the multiplier algebras on the multiplier bimodule; in (D3) (resp. (D3)′) we extended the inner products to $M(X \otimes C_r^*(G))$; and in (D4), we used the strictly continuous extensions of $\delta_X \otimes \mathrm{id}$ and $\mathrm{id} \otimes \delta_G$ to make sense of the compositions.)

Furthermore, we say that δ_X is *nondegenerate* if δ_X satisfies the following additional conditions:

(D5) the closed linear span of $(1_B \otimes C_r^*(G))\delta_X(X)$ is equal to $X \otimes C_r^*(G)$;
(D5)′ the closed linear span of $\delta_X(X)(1_A \otimes C_r^*(G))$ is equal to $X \otimes C_r^*(G)$.

Suppose that a C^*-algebra A is concretely represented on some Hilbert space $\mathcal{H}_A$. Let δ_A be a coaction of G on A and let X be a right A-Hilbert module, and we suppose that $\mathcal{K}(X)$ is concretely represented on some Hilbert space $\mathcal{H}_\mathcal{K}$. Given a δ_A-compatible coaction δ_X of G on X, the *crossed product* $X \times_{\delta_X} G$ of X by δ_X is the right $(A \times_{\delta_A} G)$-Hilbert closed submodule of $M(X \otimes C_r^*(G)) \subset \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G))$ generated by the set $\{\delta_X(x)(1_A \otimes M_f) | x \in X, f \in C_0(G)\}$. Then the inner product on $X \times_{\delta_X} G$ is given in terms of the usual operator adjoint $* : \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G)) \to \mathcal{B}(L^2(\mathcal{H}_\mathcal{K}, G), L^2(\mathcal{H}_A, G))$ by

$$\langle x\ ,\ y\rangle_{A\times_{\delta_A} G} = x^* y \quad \text{for } x, y \in X \times_{\delta_X} G.$$

Suppose that δ_A is nondegenerate. If, for every irreducible representation π of A, there exists a unitary $W \in M(\pi(A) \otimes_{\min} C_r^*(G))$ such that

$$(\pi \otimes id)(\delta_A(a)) = W(\pi(a) \otimes 1)W^*$$

for all $a \in A$, then δ_A is called *pointwise unitary.*

Theorem 3.7. *Let A be a C^*-algebra with a nondegenerate coaction δ_A of G and let X be a full right A-Hilbert module with an δ_A-compatible coaction δ_X of G. Consider the following condtions.*

(1) *G is discrete and for any closed submodule Y of X, we have $X = Y \oplus Y^\perp$.*

(2) *For any closed $(A \times_{\delta_A} G)$-submodule $\widetilde{Y}$ of $X \times_{\delta_X} G$, we have $X \times_{\delta_X} G = \widetilde{Y} \oplus (\widetilde{Y})^\perp$.*

Then we have (1) $\Longrightarrow$ (2). *If* α *is pointwise unitary and if* $\widehat{A}$ *is a Hausdorff sapce, we have* (2) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2). It follows from the proof of Theorem 3.4 that A is a dual C^*-algebra. Then we see that $A \times_{\delta_A} G$ is a dual C^*-algebra (see [2, Theorem 3.1]). Thus we obtain condition (2) by Theorem 2.2.

(2) $\Longrightarrow$ (1). Condition (2) and Theorem 2.2 imply that $A \times_{\delta_A} G$ is a dual C^*-algebra. Then condition (1) follows from [2, Thereom 3.1]. □

Remark 3.8. Let A be a C^*-algebra with a coaction δ_A of a compact group G and let X be a full right A-Hilbert module with an δ_A-compatible coaction δ_X of G. If every closed $(A \times_{\delta_A} G)$-submodule $\widetilde{Y}$ of $X \times_{\delta_X} G$ is orthogonally complemented, then so is every closed submodule Y of X in X (see [2, Proposition 3.3]).

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