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# Truth－table reductions and minimum sizes of forcing conditions（preliminary draft） 

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#### Abstract

This note is a refinement of our former note［KSY05］＂Logarithmic truth－ table reductions and minimum sizes of forcing conditions（preliminary draft）＂ Sūrikaiseki－kenkyūsho Kōkyūroku 1442 （2005），42－47．The current note ex－ tends and corrects［KSY05］．In our former works，for a given concept of reduc－ tion，we study the following hypothesis：＂For a random oracle $A$ ，with proba－ bility one，the degree of the one－query tautologies with respect to $A$ is strictly higher than the degree of $A$ ．＂In our former works，the following three results


[^0]are shown: (1) the hypothesis for polynomial-time Turing reduction is equivalent to the assertion that the probabilistic complexity class $R$ is not equal to NP, (2) the hypothesis for polynomial-time truth-table reduction implies that P is not NP, (3) [KSY05] the hypothesis holds for $(\log n)^{O(1)}$-question truth-table-reduction (without polynomial-time bound). In this note, we show that if $\varepsilon$ is an enough small positive number, then we can substitute $\varepsilon \ell$ for $(\log n)^{O(1)}$ in the statement of (3), where $\ell$ denotes the total number of occurrences of symbols in a relativized formula. We also show the hypothesis holds for monotone truth-table reduction.

## 1 Preface

In our former works [Su98, Su99, Su00, Su01, Su02, Su05, KSY05], by extending the work of Ambos-Spies [Am86] and related works, we consider the relationships with the canonical product measure of Cantor space and complexity of one-query tautologies. A formula $F$ of the relativized propositional calculus is called a onequery forumla if $F$ has exactly one occurrence of a query symbol. For example,

$$
\left(q_{0} \Leftrightarrow \xi^{3}\left(q_{1}, q_{2}, q_{3}\right)\right) \Rightarrow\left(q_{1} \Rightarrow q_{0}\right)
$$

is a one-query formula, where $q_{0}, q_{1}, q_{2}, q_{3}$ are usual propositional variables. We assume that each propositional variable takes the value 0 or 1 ( 0 denotes false and 1 denotes true). And, $\xi^{3}$ in the above formula is a query symbol. For a given oracle $A$, a function $A^{3}$ is defined as follows, where $\lambda$ is the empty string, and the query symbol $\xi^{3}$ is interpreted as the function $A^{3}$.

$$
\begin{array}{lcc}
A^{3}(000)=A(\lambda), & A^{3}(001)=A(0), & A^{3}(010)=A(1), \\
A^{3}(100)=A(01), & A^{3}(011)=A(00) \\
\end{array}
$$

Thus, more informally, the following holds for each $j=0,1, \cdots, 2^{3}-1$, where the order of strings is defined as the canonical length-lexicographic order.

$$
A^{3}(\text { the }(j+1) \text { st } 3 \text {-bit string })=A(\text { the }(j+1) \text { st string }) .
$$

For each $n$, an $n$-ary Boolean function $A^{n}$ is defined in the same way, and an interpretation of the query symbol $\xi^{n}$ is defined in the same way. For an oracle $A$, the concept of a tautology with respect to $A$ is defined in a natural way. If a one-query formula $F$ is a tautology with respect to $A$, then we say $F$ is a one-query tautology with respect to $A$. The set of all one-query tautologies with respect to $A$ is denoted by 1 TAUT ${ }^{A}$.

In [Su02], for a given concept $\leq_{\alpha}$ of reduction, we study the following hypothesis, where 1 TAUT $^{X}$ denotes the set of all one-query tautologies with respect to an oracle $X$.

One-query-jump hypothesis for $\leq_{\alpha}$ : The class $\left\{X: 1 \operatorname{TAUT}^{X} \leq_{\alpha} X\right\}$ has measure zero.

For a given reduction $\leq_{\alpha}$, we denote the corresponding one-query-jump hypothesis by $\left[\leq_{\alpha}\right]$.

In [Su98], it is shown that the one query-jump hypothesis for p -T reduction is equivalent to " $R \neq N P$."

And, in [Su02], it is shown that the one query-jump hypothesis for p -tt reduction implies "P $\neq$ NP."

In [Su05], we show that the one query-jump hypothesis for p -btt reduction holds, where p-btt denotes polynomial-time bounded-truth-table reduction. The anonymous referee of [Su05] noticed that the one query-jump hypothesis holds for bounded-truth-table reduction without polynomial-time bound, and Kumabe independently noticed the same result. The referee's proof, which may be found in [Su05], uses some concepts of resource-bounded generic oracles in [AM97]. Kumabe's proof is more simple.

In [KSY05] we show that the one query-jump hypothesis holds for $(\log n)^{O(1)}$ _ question tt-reduction (without polynomial-time bound).

A Boolean formula is called monotone if every propositonal conncetive in it is either disjunction or conjunction, and it does not have an occurrences of negation symbol. A tt-reduction is called a monotone tt-reduction if its truth table is monotone for every input. In $\S 3$, we show that the one query-jump hypothesis holds for monotone tt-reduction (without polynomial-time bound). In §4, we show the following. If $\varepsilon$ is an enough small positive number then the one query-jump hypothesis holds for $\varepsilon \ell$-question tt-reduction (without polynomial-time bound), where $\ell$ denotes the total number of occurrences of symbols in a relativized formula. In $\S 5$, we apply the result of $\S 4$ to minimum sizes of forcing conditions.

Corrigendum to our former note Theorem 4 in our former note [KSY05, p.45] has an error in its proof.

## 2 Notation

Most of our notation is the same as that of [Su02], [Su05] and [KSY05]. Almost all undefined notions may be found in these papers.
$\omega$ stands for $\{0,1,2,3 \cdots\}$, while $\mathbb{N}$ stands for $\{1,2,3 \cdots\}$. In some textbooks, the complexity class $R$ is denoted by RP. For the detail of the class $R$, see for example [BDG88].

The definition of polynomial-time truth-table reduction and its variant may be found in [LLS75].
monotone tt-reduction
If $A$ is tt-reducible to $B$ via $f$ and, if for any input $x$, propositional connectives used in the truth table (i.e., the $\varphi_{x}$ of $f(x)=\left(\varphi_{x}, s_{x, 1}, \cdots, s_{x, k}\right)$ ) is conjunction and
disjunction only, and negation is not used, then we say " $A$ is monotone tt-reducible to $B$ via $f$ ". If $A$ is monotone tt-reducible to $B$ via some function, then we say " $A$ is monotone tt-reducible to $B^{\prime \prime}$.
$\ell(F)$, length of a formula
In this note, a given relativised formula $F$, the symbol $\ell(F)$ denotes the total number of occurrences of propositional variables ( $q_{0}, q_{1}, q_{2}, \cdots$ ), propositonal connectives ( $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow)$, query symbols $\left(\xi^{1}, \xi^{2}, \xi^{3}, \cdots\right)$ and punctuation marks (commas, parentheses). In the case of a given string $x$ is not (the binary code of) a relativized formula, the symbol $\ell(x)$ denotes the binary length of $x$.
$\varepsilon \ell$-question tt-reduction
Suppose that $\varepsilon$ is a positive real number. If $A$ is tt-reducible to $B$ via $f$ and, if for any input $x$ it holds that

$$
k \leq \varepsilon \ell(x)
$$

where $k$ is the norm of $f$ at $x$, then we say " $A$ is $\varepsilon \ell$-question tt-reducible to $B$ via $f$ ". If $A$ is $\varepsilon \ell$-question tt-reducible to $B$ via some function, then we say " $A$ is $\varepsilon \ell$-question tt-reducible to $B^{\prime \prime}$.

## 3 Monotone truth table redcution

Theorem 1 The Lebesgue measure of the set

$$
\left\{X: \text { 1TAUT }^{X} \text { is monotone tt-reducible to } X\right\}
$$

is zero. In other words, one-query jump hypothesis holds for monotone tt-reduction (without polynomial-time bound).

## 4 The case where norm is linear of length of a formula

Theorem 2 (Main Theorem) Let $\varepsilon$ be a positive real number and suppose that $\varepsilon$ is enough small. Then the Lebesgue measure of the following class is zero.

$$
\left\{X: 1 \mathrm{TAUT}^{X} \leq_{\varepsilon \ell-\mathrm{tt}} X\right\}
$$

In other words, the one-query-jump hypothesis holds for $\varepsilon \ell$-question tt-reduction (without polynomial-time bound).

## 5 Lower bounds for forcing complexity

Theorem 3 Let $\varepsilon$ be a positive real number and suppose that $\varepsilon$ is enough small. Let $\mathcal{D}_{\mathrm{e} \ell}$ be the class of all oracles $D$ such that there exists a positive integer $c$ (c may
depend on $D$ ) of the following property. For any $F \in 1^{T A U T}{ }^{D}$ such that $\ell(F) \geq c$, there exists a forcing condition $S$ such that $S$ is a subfunction of $D, S$ forces $F$ to be a tautology and such that $|\operatorname{dom} S| \leq \varepsilon \ell(F)$, where the left-hand side denotes the cardinality of $\operatorname{dom} S$. Then $\mathcal{D}_{\varepsilon \ell}$ has measure zero.

## References

[Am86] Ambos-Spies, K.: Randomness, relativizations, and polynomial reducibilities. In: Structure in Complexity Theory, Lect. Notes Comput. Sci. 223 (A. L. Selman, Eds.), pp.23-34, Springer, Berlin, 1986.
[AM97] Ambos-Spies, K., Mayordomo, E.: Resource-bounded measure and randomness. In: Complexity, logic, and recursion theory, Lecture Notes in Pure and Applied Mathematics 187 (A. Sorbi, Eds.), pp.1-47, Marcel Dekker, New York, 1997.
[BDG88] Balcázar, J. L., Díaz, J., Gabarró, J.: Structural complexity I. Springer, Berlin, 1988.
[BG81] Bennett, C. H., Gill, J.: Relative to a random oracle $A, \mathrm{P}^{A} \neq \mathrm{NP}^{A} \neq$ co-NP ${ }^{A}$ with probability 1. SIAM J. Comput., 10 (1981), pp. 96-113.
[Do92] Dowd, M.: Generic oracles, uniform machines, and codes. Information and Computation, 96 (1992), pp. 65-76.
[KSY05] Kumabe, M., Suzuki, T. and Yamazaki, T.: Logarithmic truth-table reductions and minimum sizes of forcing conditions (preliminary draft). Sūrikaisekikenkyūsho Kōkyūroku, 1442 (2005), pp. 42-47.
[LLS75] Ladner, R. E., Lynch, N. A., Selman, A. L.: A comparison of polynomial time reducibilities. Theoret. Comput. Sci., 1 (1975), pp.103-123.
[Su98] Suzuki, T.: Recognizing tautology by a deterministic algorithm whose while-loop's execution time is bounded by forcing. Kobe Journal of Mathematics, 15 (1998), pp. 91-102.
[Su99] Suzuki, T.: Computational complexity of Boolean formulas with query symbols. Doctoral dissertation (1999), Institute of Mathematics, University of Tsukuba, Tsukuba-City, Japan.
[Su00] Suzuki, T.: Complexity of the $r$-query tautologies in the presence of a generic oracle. Notre Dame J. Formal Logic, 41 (2000), pp. 142-151.
[Su01] Suzuki, T.: Forcing complexity: minimum sizes of forcing conditions. Notre Dame J. Formal Logic, 42 (2001), pp. 117-120.
[Su02] Suzuki, T.: Degrees of Dowd-type generic oracles. Inform. and Comput., 176 (2002), pp. 66-87.
[Su05] Suzuki, T.: Bounded truth table does not reduce the one-query tautologies to a random oracle. Archive for Mathematical Logic, 44 (2005), pp. 751-762.


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