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Directed sets and inverse limits

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Abstract

We show that the tree property for directed sets is equivalent to the nontriviality of certain inverse limits.

1 Directed sets and cofinal types

First we review the basic facts about cofinal types.

Definition 1.1 Let $\langle D, \leq_D \rangle, \langle E, \leq_E \rangle$ be directed sets. A function $f: E \to D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e\ [f(e') \geq_D d]$$

is called a *convergent function*. If such a function exists we write $D \leq E$ and say *E is cofinally finer than* D. $\leq$ is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g: D \to E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D\ [g(d') \leq_E e \to d' \leq_D d]$$

is called a *Tukey function*.

If there exists a directed set C into which D and E can be embedded cofinally, we say *D is cofinally similar with* E. In this case we write $D \equiv E$. $\equiv$ is an equivalence relation, and the eqivalence classes with respect to $\equiv$ are the *cofinal types*.

Proposition 1.2 *For directed sets D and E, the follwing are equivalent.*

(a) $D \equiv E$.
(b) $D \leq E$ *and* $E \leq D$.

So we can regard $\leq$ as an ordering on the class of all cofinal types.

Definition 1.3 For a directed set D,

$$\mathsf{add}(D) \stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\},$$
$$\mathsf{cof}(D) \stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}.$$

These are the *additivity* and the *cofinality* of a directed set. We restrict ourselves to directed sets D without maximum, so $\mathsf{add}(D)$ is well-defined.

Proposition 1.4 *For a directed set D (without maximum),*

$$\aleph_0 \leq \mathsf{add}(D) \leq \mathsf{cof}(D) \leq |D|.$$

Furthermore, $\mathsf{add}(D)$ *is regular and* $\mathsf{add}(D) \leq \mathsf{cf}(\mathsf{cof}(D))$. *Here* cf *is the cofinality of a cardinal, which is the same as the additivity of it.*

Proposition 1.5 *For directed sets D and E, $D \leq E$ implies*

$$\mathsf{add}(D) \geq \mathsf{add}(E) \quad \textit{and} \quad \mathsf{cof}(D) \leq \mathsf{cof}(E).$$

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

2 The width of a directed set

In the following, κ is always an infinite regular cardinal. If P is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for X a subset of P and $a \in P$. As usual, for cardinals $\kappa \leq \lambda$, $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.

Definition 2.1 The *width* of a directed set D is defined by

$$\mathsf{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a *thin* subset of D' means

$$\forall d \in D[|X_{\leq d}| < \mathsf{add}(D)].$$

The reason to consider this cardinal function is to give a characterization of the tree proprety. See [2, Theorem 7.1].

Example 2.2 The set of singletons $\{\{\alpha\} \mid \alpha < \lambda\}$ is thin in $\mathcal{P}_\kappa\lambda$, so we have $\mathsf{wid}(\mathcal{P}_\kappa\lambda) \geq \lambda^+$. If κ is strongly inaccessible, then $\mathcal{P}_\kappa\lambda$ is thin in itself, which shows $\mathsf{wid}(\mathcal{P}_\kappa\lambda) = (\lambda^{<\kappa})^+$.

Lemma 2.3 *For a directed set D and a cardinal $\lambda \geq \kappa := \mathsf{add}(D)$, the following are equivalent.*

(a) *D has a thin subset of size λ.*
(b) *$D \geq \mathcal{P}_\kappa\lambda$.*
(c) *There exists an order-preserving function $f\colon D \to \mathcal{P}_\kappa\lambda$ with $f[D]$ cofinal in $\mathcal{P}_\kappa\lambda$.*

Corollary 2.4 *The width of a directed set depends only on its cofinal type.*

Lemma 2.5 $\mathsf{add}(D)^+ \leq \mathsf{wid}(D) \leq \mathsf{cof}(D)^+$.

3 The tree property for directed sets

In the following definition, if D is an infinite regular cardinal κ, a 'κ-tree on κ' coincides with the classical 'κ-tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

Definition 3.1 (κ-tree) ([1]) Let D denote a directed set. A triple $\langle T, \leq_T, s\rangle$ is said to be a *κ-tree* on D if the following holds.

1) $\langle T, \leq_T\rangle$ is a partially ordered set.
2) $s\colon T \to D$ is an order preserving surjection.
3) For all $t \in T$, $s\restriction T_{\leq t}\colon T_{\leq t} \xrightarrow{\sim} D_{\leq s(t)}$ (order isomorphism).
4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level* d of T.

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists!\, t' \leq_T t\ [s(t') = d']$.

We write $t{\downarrow}d$ for this unique t'.

If a κ-tree $\langle T, \leq_T, s\rangle$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t\ [s(t') = d']$,

then it is called a *κ-arbor* on D.

Definition 3.2 (tree property) ([1]) Let $\langle D, \leq_D\rangle$ be a directed set and $\langle T, \leq_T, s\rangle$ a κ-tree on D. $f: D \to T$ is said to be a faithful embedding if f is an order embedding and satisfies $s \circ f = \mathrm{id}_D$. If for each κ-tree T on D there is a faithful embedding from D to T, we say that D has the *κ-tree property.* If D has the $\mathsf{add}(D)$-tree property, we say simply D has the *tree property.*

Proposition 3.3 ([1]) *Let D be directed set and let $\kappa = \mathsf{add}(D)$. D has the tree property iff for any κ-arbor on D there is a faithful embedding into it.*

Proposition 3.4 ([1]) *Let D be directed set and let $\theta < \mathsf{add}(D)$. For any θ-tree T on D, the number of faithful embeddings from D into T is less than θ.*

Proposition 3.5 ([1]) *Let D be directed set and let θ be a cardinal.*

(1) *If $\theta < \mathsf{add}(D)$ then D has the θ-tree property.*
(2) *If $\theta > \mathsf{add}(D)$ then D does not have the θ-tree property.*

Thus we are interested in the case $\theta = \mathsf{add}(D)$.

Proposition 3.6 ([2]) *If E has the tree property, $D \leq E$ in the Tukey ordering and $\mathsf{add}(D) = \mathsf{add}(E)$, then D also has the tree property. Thus the tree property is a property about the cofinal type of a directed set.*

Corollary 3.7 ([1]) *If D has the tree property, then $\mathsf{add}(D)$ has the tree property in the classical sense.*

Theorem 3.8 ([1]) *For a strongly inaccessible cardinal κ, the following are equivalent:*

(a) *κ is strongly compact.*
(b) *All directed sets D with $\mathsf{add}(D) = \kappa$ have the tree property.*

Condition (b) *also holds for $\kappa = \aleph_0$.*

4 Inverse limits

Now we give a characterization of the tree property in terms of various inverse systems.

Theorem 4.1 *Let D be a directed set, and let θ be a cardinal. The following are equivalent:*

(a) *D has the θ-tree property.*
(b) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d'\rangle$ of sets satisfying $|A_d| < \theta$ for all $d \in D$, the inverse limit $\varprojlim_{d\in D} A_d$ is nonempty.*
(c) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d'\rangle$ of groups (respectively of abelian groups or free abelian groups), satisfying $|A_d| < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d\in D} A_d$ has a nonzero element.*
(d) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d'\rangle$ of vector spaces, satisfying $\dim(A_d) < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d\in D} A_d$ has a nonzero element.*

Proof (a) $\Rightarrow$ (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d'\rangle$ be an inverse system of nonempty sets, such that $|A_d| < \theta$ for all $d \in D$. Without loss of generality, we may assume that $\langle A_d \mid d \in D\rangle$ is a disjont family. Put $T := \bigcup_{d\in D} A_d$ and define $s: T \to D$ so that $s^{-1}\{d\} = A_d$ for any $d \in D$. For $t, u \in T$ define the ordering $\leq_T$ on T so that

$$t \leq_T u \iff \text{if } t \in A_d, u \in A_{d'} \text{ then } d \leq_D d' \text{ and } f_{dd'}(u) = t.$$

Then $\langle T, \leq_T, s\rangle$ is a θ-tree on D, and $\varprojlim_{d\in D} A_d$ is the set of all faithful embeddings from D into T. Hence (a) implies (b).
(b) $\Rightarrow$ (a) Let $\langle T, \leq_T, s\rangle$ be a given θ-tree on D. Define $f_{dd'}: s^{-1}\{d'\} \to s^{-1}\{d\}$ so that $f_{dd'}(t) = t\downarrow d$.

Then $\langle s^{-1}\{d\}, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ is an inverse system of nonempty sets, and $\varprojlim_{d\in D} s^{-1}\{d\}$ is the set of all faithful embeddings from D into T.

(b) $\Rightarrow$ (c) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ be a given inverse system of groups, and assume that $|A_d| < \theta$ for all $d \in D$ and that there is some $d_0 \in D$ such that $f_{d_0 d} \neq 0$ for all $d \geq d_0$. Put

$$B_d := f_{dd'}[A_{d_0} \setminus \{0\}] \qquad \text{for } d \geq d_0,$$
$$g_{dd'} := f_{dd'} \restriction B_{d'} \qquad \text{for } d' \geq d \geq d_0.$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D_{\geq d_0},\ d \leq d' \rangle$ is an inverse system of nonempty sets. By (b), we can pick some $b \in \varprojlim_{d \geq d_0} B_d$. Since $D_{\geq d_0}$ is cofinal in D and D is directed, we can extend this b to a unique $a \in \left(\varprojlim_{d\in D} A_d\right) \setminus \{0\}$.

(c) $\Rightarrow$ (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ be an inverse system of nonempty sets such that $|A_d| < \theta$ for all $d \in D$. Since (a), and hence (b) is always true for $\theta = \aleph_0$, we may assume $\theta > \aleph_0$. For $d \in D$, let B_d be the free abelian group with generators in A_d, i.e.

$$B_d := \{b \in {}^{A_d}\mathbb{Z} \mid b(x) = 0 \text{ for all but finitely many } x \in A_d\}.$$

Let $\operatorname{supt}(b) := \{x \in \operatorname{dom}(b) \mid b(x) \neq 0\}$. We identify $b \in B_d$ with the expression $n_0 x_0 + \cdots + n_k x_k$, where $\{x_0, \ldots, x_k\} \supseteq \operatorname{supt}(b)$ and $b(x) = \sum_{\substack{i \leq k \\ x_i = x}} n_i$ for $x \in A_d$. Clearly $|B_d| < \theta$. For $d \leq d'$ in D, put

$$\begin{array}{cccc} g_{dd'}: & B_{d'} & \to & B_d \\ & \Psi & & \Psi \\ & n_0 x_0 + \cdots + n_k x_k & \mapsto & n_0 f_{dd'}(x_0) + \cdots + n_k f_{dd'}(x_k). \end{array}$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ is an inverse system of free abelian groups, and $g_{dd'} \neq 0$ for any $d \leq d'$ in D. Thus by (c), there is some $b^* \in \left(\varprojlim_{d\in D} B_d\right) \setminus \{0\}$. Since $b^* \neq 0$, there is some $d_0 \in D$ such that $b^*(d_0) \neq 0$ for all $d \geq d_0$. Put

$$F_d := \operatorname{supt} b^*(d) \cap f_{d_0 d}^{-1}[\operatorname{supt} b^*(d_0)] \qquad \text{for } d \geq d_0,$$
$$h_{dd'} := f_{dd'} \restriction F_{d'} \qquad \text{for } d' \geq d \geq d_0.$$

Note that $h_{dd'}[F_{d'}] = F_d$. Now $\langle F_d, h_{dd'} \mid d' \geq d \geq d_0 \rangle$ is an inverse system of nonempty finite sets. Since any directed set has the $\aleph_0$-tree proprety, $\varprojlim_{d \geq d_0} F_d \neq \emptyset$. Take any $a \in \varprojlim_{d \geq d_0} F_d$. There is a unique $a' \in \varprojlim_{d\in D} A_d$ which extends a.

(b) $\Rightarrow$ (d) This is similar to the proof of (b) $\Rightarrow$ (c).

(d) $\Rightarrow$ (b) This is similar to the proof of (c) $\Rightarrow$ (b). □

Corollary 4.2 *If G is the inverse limit of $\langle G_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ where each G_d is finite (i.e. G is a profinite group), then $G \neq 0$ iff $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$.*

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