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On integral bases of real octic 2-elementary abelian extensions
(実8次 2-基本アーベル拡大体の整数基について)

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Abstract. Let K be an abelian field whose Galois group is 2-elementary abelian over the rationals $\boldsymbol{Q}$. If an octic field K is monogenic and a quadratic subfield with odd discriminant and a quartic subfield of K are linearly disjoint, then K coincides with the field $\boldsymbol{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, namely K is equal to the cyclotomic field $\boldsymbol{Q}(\zeta_{24})$ [MN]. In this article, we explain how to prove that all the real octic fields K are non-monogenic, that is, the rings Z_K of integers in K do not have any power integral basis. Finally, we propose a few problems on the evaluation on the field index of K and the non-essential factor (außerwesentliche Diskriminantenteiler) of K.

§1. Introduction

Let K be an algebraic number field over the rationals $\boldsymbol{Q}$. We denote the ring of integers in K by Z_K. When $Z_K = \boldsymbol{Z}[\alpha]$ for some element α of Z_K, it is said that α generates a power integral basis of the ring Z_K or simply Z_K has a power integral basis. The field K is called monogenic if Z_K has a power integral basis. It is known as a problem of Hasse to characterize whether a field K is monogenic or not[Gy]. In this article, we consider the fields K whose Galois groups are 2-elementary abelian. Since the field K for $[K : \boldsymbol{Q}] \geq 16$

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is non-monogenic, i.e., the ring Z_K of integers in K has no power integral basis by virtue of the decomposition theory of a prime number ([Lemma 1, SN], [MNS], [Wa]) and by the works of K. S. Williams, M.-N. Gras and F. Tanoé for Dirichlet fields K,([Wi], [GT]) it is enough for us to investigate the octic 2-elementary abelian fields. Let k and L be a quadratic subfield of odd discriminant and a quartic subfield of K, respectively. If k and L are linearly disjoint, then such an octic field $K = kL$ is non-monogenic except for the cyclotomic field $\boldsymbol{Q}(\zeta_{24})$ of conductor 24 [MN]. In this paper, we will show an integral basis of the ring Z_K over the ring $\boldsymbol{Z}$ of rational integers in an octic field K [Theorem 1]. Next, being based on the linear equations

$$a_{i1}E_{i1} + a_{i2}E_{i2} + a_{i3}E_{i3} = 0 \quad (1 \le i \le 7)$$

with suitable factors a_{ij} of the field discriminant D_K, where $(a_{ij}, D_i) = 1$ and units E_{ij} as *coefficients* of *valuables* a_{ij} in each quadratic subfield $k_j = \boldsymbol{Q}(\sqrt{D_j})$ [Proposition 2], we can prove that all the real 2-elementary abelian fields K of degree 8 have no power integral basis[Theorem 2].

§2. Integral bases

We determine explicit integral bases of some octic fields K whose Galois groups are 2-elementary abelian. We denote the Galois group

$$\langle \tau, \sigma, \rho \quad | \quad \tau : \sqrt{mn} \mapsto -\sqrt{mn}, \sigma : \sqrt{dn} \mapsto -\sqrt{dn}, \rho : \sqrt{d_1 m_1 n_1 \ell} \mapsto -\sqrt{d_1 m_1 n_1 \ell} \rangle$$

of $K/\boldsymbol{Q}$ by G.

The following lemma and proposition are available to deduce the type of 2-elementary abelian extension fields K which would have power integral bases.

Lemma 1([SN]). *Let ℓ be a prime number and let $F/\boldsymbol{Q}$ be a Galois extension of degree $n = efg$ with ramification index e and the relative degree f with respect to ℓ. If one of the following conditions is satisfied, then Z_F has no power integral basis, i.e., F is non-monogenic ;*

(1) $e\ell^f < n$ *if* $f = 1$;

or

(2) $e\ell^f \le n + e - 1$ *if* $f \ge 2$.

Proposition 1([MN]). *Let $a_1, a_2, \cdots, a_r$ be square free rational integers and F be the field $\boldsymbol{Q}(\sqrt{a_1}, \sqrt{a_2}, \cdots, \sqrt{a_r})$ of degree $2^r, r \ge 4$. Then F is non-monogenic.*

Proof. Without loss of generality, we may assume that there exists at most two generators $\sqrt{a_1}, \sqrt{a_2}$ of F with $a_j \not\equiv 1 \pmod 4 (1 \le j \le 2)$. Then the ramification index e of the prime is at most 2^2. Since the Galois group $G = Gal(F/Q)$ is 2-elementary, the relative degree f of the prime 2 is at most 2, because the inertia subgroup of G is cyclic. In Lemma 1 let ℓ be equal to 2. Then we can deduce $e\ell^f \leq 2^2 \cdot 2^1 < 2^r$ if $f = 1$ and $e\ell^f \leq 2^2 \cdot 2^2 \leq 2^r + e - 1$ if $f = 2$. Thus F is non-monogenic. □

By the proof of Proposition 1, if an octic field K is monogenic, it is sufficient to consider that K contains *two* quadratic subfields of even discriminant and *one* of odd discriminant.

The main theorem is based on the following theorem, which is an extension of a result of the case of quartic fields [M_1, M_2, Wi].

Theorem 1([PMN]). *Let K be an octic field $Q(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell})$ with $d = d_1d_2, m = m_1m_2$, $n = n_1n_2, mn \equiv 3, dn \equiv 2, d_1m_1n_1\ell \equiv 1, d_2 \equiv 2 \pmod 4), d_1, m_1, n_1 \geq 1$ and $dmn\ell$ is square free. Let D_K be the field discriminant of the octic field K. Then we have $D_K = 2^{12}(dmn\ell)^4$ and an integral basis of K is :*

$$\begin{aligned} Z_K = Z\Big[&1, \sqrt{mn}, \sqrt{dn}, \frac{\sqrt{dm}+\sqrt{dn}}{2}, \frac{1+\sqrt{d_1m_1n_1\ell}}{2}, \frac{\sqrt{mn}+\sqrt{d_1m_2n_2\ell}}{2}, \\ &\frac{\sqrt{dn}+\sqrt{d_2m_1n_2\ell}}{2}, \frac{\sqrt{dm}+\sqrt{dn}+e_1\sqrt{d_2m_2n_1\ell}+e_2\sqrt{d_2m_1n_2\ell}}{4}\Big] \end{aligned}$$

where $e_i = \pm 1$ $(i = 1, 2), e_1 \equiv d_1m_1, e_2 \equiv d_1n_1 \pmod 4$.

§3. Non-monogenic field

It is known that in the case of $d_1m_1n_1 = 1$ that is, there exist a quartic subfield L and a quadratic k of K with $(D_L, D_k) = 1$, the fields K are non-monogenic except for the cyclotomic field $Q(\zeta_{24})$ of conductor 24 [MN], where D_F means the discriminant of an algebraic number field F over Q. From now on, we consider the case of $d_1m_1n_1 \geq 1$ and as an application of Theorem 1, we can slightly generalize Proposition 5 in [MN], whose proof was done using the relative different with respect to K over a suitable quadratic subfield. We assume that K is *monogenic.*

Let

$$\begin{aligned} \xi =& b_1\sqrt{mn} + b_2\sqrt{dn} + b_3\frac{\sqrt{dm}+\sqrt{dn}}{2} + b_4\frac{1+\sqrt{d_1m_1n_1\ell}}{2} + b_5\frac{\sqrt{mn}+\sqrt{d_1m_2n_2\ell}}{2} \\ &+ b_6\frac{\sqrt{dn}+\sqrt{d_2m_1n_2\ell}}{2} + b_7\frac{\sqrt{dm}+\sqrt{dn}+e_1\sqrt{d_2m_2n_1\ell}+e_2\sqrt{d_2m_1n_2\ell}}{4} \end{aligned}$$

be a generator of a power integral basis of Z_K. Now we calculate a factor $(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho$

of the discriminant $d_{K/Q}(\xi)=\Delta^2\left[1,\xi,\xi^2,\xi^3,\xi^4,\xi^5,\xi^6,\xi^7\right]$ of a number ξ;

$$
\begin{aligned}
&(\xi-\xi^{\sigma})(\xi-\xi^{\sigma})^{\rho}\\
&=\left\{(2b_2+b_3+b_6+\frac{b_7}{2})\sqrt{dn}+(b_3+\frac{b_7}{2})\sqrt{dm}+(b_6+\frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell}+\frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2}\right\}\\
&\times\left\{(2b_2+b_3+b_6+\frac{b_7}{2})\sqrt{dn}+(b_3+\frac{b_7}{2})\sqrt{dm}-(b_6+\frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell}-\frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2}\right\}\\
&=\left\{(2b_2+b_3+b_6+\frac{b_7}{2})\sqrt{dn}+(b_3+\frac{b_7}{2})\sqrt{dm}\right\}^2-\left\{(b_6+\frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell}+\frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2}\right\}^2\\
&=\left\{(2b_2+b_3+b_6)^2+(2b_2b_7+b_3b_7+b_6b_7)+\frac{b_7{}^2}{4}\right\}dn+(b_3{}^2+b_3b_7+\frac{b_7{}^2}{4})dm\\
&-(b_6{}^2+b_6b_7e_2+\frac{b_7{}^2e_2{}^2}{4})d_2m_1n_2\ell-\frac{b_7{}^2e_1{}^2d_2m_2n_1\ell}{4}\\
&+\left\{(2^2b_2b_3+2b_3{}^2+2b_3b_6+2b_3b_7+2b_2b_7+b_6b_7+\frac{b_7^2}{2})d-(b_6b_7e_1d_2\ell+\frac{b_7{}^2e_2e_1d_2\ell}{2})\right\}\sqrt{mn},
\end{aligned}
$$

namely, this factor is an integer of the quadratic field $k_1=\boldsymbol{Q}(\sqrt{mn})$ of the fixed field by the subgroup $<\sigma,\rho>$ in G. Then we denote it by $\eta_{11}=B+C(\sqrt{mn})$. Thus we obtain

$$
\begin{aligned}
B/d_2&\equiv\left\{b_3{}^2+b_6{}^2+b_3b_7+\frac{b_7{}^2}{4}\right\}d_1n+\left(b_3{}^2+b_3b_7+\frac{b_7{}^2}{4}\right)d_1m\\
&-\left(b_6{}^2+b_6b_7+\frac{b_7{}^2}{4}\right)m_1n_2\ell-\frac{b_7{}^2m_2n_1\ell}{4}\\
&\equiv\frac{b_7^2}{4}\left(d_1(m+n)-(m_1n_2+m_2n_1)\ell\right)\\
&\equiv\frac{\{d_1(m+n)-(d_1n+4k+d_1m+4k)\}}{4}\equiv 0 \pmod 2,
\end{aligned}
$$

by $d_1m_1n_1\ell\equiv 1+4k \pmod 8$ and $m+n\equiv 0 \pmod 4$, since $m_1n_2\ell\cdot 1\equiv d_1m_1^2n_1n_2\ell^2+4m_1n_2\ell k\equiv d_1n+4k \pmod 8$ and $m_2n_1\ell\cdot 1\equiv d_1m_1m_2n_1^2\ell^2+4m_2n_1\ell k\equiv d_1m+4k \pmod 8$.

$$
\begin{aligned}
C/d_2&\equiv\left(b_6b_7+\frac{b_7{}^2}{2}\right)d_1-\left(b_6b_7e_1\ell+\frac{b_7{}^2e_2e_1\ell}{2}\right)\\
&\equiv b_6b_7\,(d_1-e_1\ell)+\frac{b_7^2}{2}\,(d_1-e_2e_1\ell)\equiv 0 \pmod 2
\end{aligned}
$$

by $e_1\equiv d_1m_1$, $e_2\equiv d_1n_1 \pmod 4$, since $d_1-e_2e_1\ell\equiv d_1-d_1^2m_1n_1\ell\equiv d_1(1-d_1m_1n_1\ell)\equiv 0 \pmod 4$. So we can write $\eta_{11}=(\xi-\xi^{\sigma})(\xi-\xi^{\sigma})^{\rho}=2d_2E_1$ for an integer $E_1=B_1+C_1\sqrt{mn}$ in $k_1=\boldsymbol{Q}(\sqrt{mn})$. By the same computation, we obtain $\eta_{12}=(\xi-\xi^{\rho})(\xi-\xi^{\rho})^{\sigma}=\ell E_2$, $\eta_{13}=(\xi-\xi^{\sigma\rho})(\xi-\xi^{\sigma\rho})^{\rho}=d_1E_3$ for units E_j in $k_1 (j=2,3)$. By the assumption that Z_K is generated by ξ, we have

$$d_{K/Q}(\xi)=\pm N_K(\mathfrak{d}(\xi))=\pm D_K,$$

where $\mathfrak{d}(\alpha), N_K(\alpha)$ and $N_K(\mathfrak{a})$ means the different of a number, norm of α and an ideal $\mathfrak{a}$ with respect to $K/\boldsymbol{Q}$, respectively[Wa]. Then, because η_{1j} is a partial factor of $d_{K/Q}(\xi)$, the integers E_j should be units in $k_1 = \boldsymbol{Q}(\sqrt{mn})$. Here the following is our basic identity:

$$(\xi-\xi^{\sigma})(\xi-\xi^{\sigma})^{\rho}-(\xi-\xi^{\rho})(\xi-\xi^{\rho})^{\sigma}-(\xi-\xi^{\sigma\rho})(\xi-\xi^{\sigma\rho})^{\rho}=0$$

for $(\xi-\xi^{\sigma})(\xi-\xi^{\sigma})^{\rho}=\eta_{11}, (\xi-\xi^{\rho})(\xi-\xi^{\rho})^{\sigma}=\eta_{12}$ and $(\xi-\xi^{\sigma\rho})(\xi-\xi^{\sigma\rho})^{\rho}=\eta_{13}$. Then we have the equation

$$2d_2E_1-\ell E_2-d_1E_3=0 \quad in \quad k_1=\boldsymbol{Q}(\sqrt{D_1}), \quad D_1=m_1\cdot 2m_2\cdot n_1\cdot 2n_2,$$

where E_1, E_2 and E_3 are units in k_1 .

In the same way, we obtain seven equations corresponding to each of the seven quadratic subfields k_j of K.

Proposition 2. *If $K=\boldsymbol{Q}(\sqrt{mn},\sqrt{dn},\sqrt{d_1m_1n_1\ell})$ is monogenic, then the following simultaneous equations hold:*

(1) $\ell E_{11}+2d_2E_{12}+d_1E_{13}=0 \quad in \quad k_1=\boldsymbol{Q}(\sqrt{D_1}), \quad D_1=m_1\cdot 2m_2\cdot n_1\cdot 2n_2,$

(2) $\ell E_{21}+2m_2E_{22}+m_1E_{23}=0 \quad in \quad k_2=\boldsymbol{Q}(\sqrt{D_2}), \quad D_2=d_1\cdot 2d_2\cdot n_1\cdot 2n_2,$

(3) $\ell E_{31}+2n_2E_{32}+n_1E_{33}=0 \quad in \quad k_3=\boldsymbol{Q}(\sqrt{D_3}), \quad D_3=d_1\cdot 2d_2\cdot m_1\cdot 2m_2,$

(4) $2d_2E_{41}+2m_2E_{42}+2n_2E_{43}=0 \quad in \quad k_4=\boldsymbol{Q}(\sqrt{D_4}), \quad D_4=d_1\cdot m_1\cdot n_1\cdot \ell,$

(5) $2d_2E_{51}+m_1E_{52}+n_1E_{53}=0 \quad in \quad k_5=\boldsymbol{Q}(\sqrt{D_5}), \quad D_5=d_1\cdot 2m_2\cdot 2n_2\cdot \ell,$

(6) $d_1E_{61}+2m_2E_{62}+n_1E_{63}=0 \quad in \quad k_6=\boldsymbol{Q}(\sqrt{D_6}), \quad D_6=2d_2\cdot m_1\cdot 2n_2\cdot \ell,$

(7) $d_1E_{71}+m_1E_{72}+2n_2E_{73}=0 \quad in \quad k_7=\boldsymbol{Q}(\sqrt{D_7}), \quad D_7=2d_2\cdot 2m_2\cdot n_1\cdot \ell,$

where each E_{ij} is a unit in the corresponding quadratic subfield k_i of K and each D_i the field discriminant of k_i, respectively.

For the case of a real quadratic field, the following lemma holds:

Lemma 2. *Let E_j be a power $\varepsilon_0{}^j=\dfrac{u_j+v_j\sqrt{D}}{2}$ of the fundamental unit $\varepsilon_0=\dfrac{u+v\sqrt{D}}{2}>1$ in a real quadratic field $\boldsymbol{Q}(\sqrt{D})$ with the field discriminant D and $\overline{\alpha}=\alpha^{\gamma}$ for α in $\boldsymbol{Q}(\sqrt{D})$ and $\gamma(\neq I)$ in $Gal(\boldsymbol{Q}(\sqrt{D})/\boldsymbol{Q})$. Let*

$$\begin{cases} a+bE_j+cE_k=0, \\ a+b\overline{E_j}+c\overline{E_k}=0 \end{cases} \qquad (*)$$

for $abc\neq 0$. Denote the matrix

$$\begin{pmatrix} 1 & E_j & E_k \\ 1 & \overline{E_j} & \overline{E_k} \end{pmatrix}$$

attached to the the equation $(*)$ *by* A *and the rank of* A *by* r_D. *Then we have a solution* (a, b, c) *of rational integers :*

$$\begin{cases} a \pm b \pm c = 0 & for \quad r_D = 1, \\ \dfrac{a}{u_k v_j - u_j v_k} = \dfrac{b}{2v_k} = \dfrac{c}{-2v_j} & for \quad r_D = 2 \end{cases}$$

with $E_i = \dfrac{u_i + v_i\sqrt{D}}{2}$.

Proof. This lemma means that the integral solutions should be on the plane for the rank $r_D = 1$ of the coefficient matrix A and on the line i.e. the intersection of two planes for $r_D = 2$, respectively.

First, we consider the case of $r_D = 1$, then for

$$\begin{cases} E_i = \dfrac{u_i + v_i\sqrt{D}}{2}, \\ \overline{E_i} = \dfrac{u_i - v_i\sqrt{D}}{2}, \end{cases}$$

$E_i, \overline{E_i}$ should be a rational number. Then we have $E_j = u_j = \pm 1$ and $E_k = u_k = \pm 1$. Hence $a \pm b \pm c = 0$. Second, we assume $r_D = 2$. Then we have

$$a : b : c = \begin{vmatrix} E_j & E_k \\ \overline{E_j} & \overline{E_k} \end{vmatrix} : \begin{vmatrix} E_k & 1 \\ \overline{E_k} & 1 \end{vmatrix} : \begin{vmatrix} 1 & E_j \\ 1 & \overline{E_j} \end{vmatrix} = u_k v_j - u_j v_k : 2v_k : -2v_j.$$

Hence

$$\frac{a}{u_k v_j - u_j v_k} = \frac{b}{2v_k} = \frac{c}{-2v_j}.$$

□

In the case of *any* octic field $\boldsymbol{Q}(\sqrt{m_1 m_2 n_1 n_2}, \sqrt{d_1 d_2 n_1 n_2}, \sqrt{d_1 m_1 n_1 \ell})$, by the following lemma, we can deduce to evaluate the rank r_D of a quadratic field $\boldsymbol{Q}(\sqrt{D})$ for a few cases with respect to the order of values $d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell$ in the set of seven parameters.

Lemma 3. *Let denote the set* $\{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$ *by* D. *Then it holds that:*

(1) *For one parameter* s *in* D, *there exist only four quadratic subfields* k_j *whose discriminants* D_j *are divisible by* s.

(2) *For two parameters* s, t *in* D, *there exist only two quadratic subfields* k_j *whose discriminants* D_j *are divisible by* st.

(3) *Let* s, t, u *be three parameters in* D, *such that* stu *is a divisor of the field discriminant of* D_j *of* k_j. *Then there exists only one quadratic subfield* k_j *whose discriminant* D_j *is divisible by* stu.

Proof. (1) We can confirm the claim (1) for each of $\binom{\sharp D}{1} = 7$ parameter in D from seven equations in Proposition 2, such that there exist just four fields k_1, k_3, k_4, k_6 whose discriminant is divisible by m_1.

(2) We can do the claim (2) of $\binom{\sharp D}{2} = 21$ pairs of parameters in D by the same way as in (1). For instance, there exist just two fields k_3, k_7 whose discriminants are divisible by d_2m_2.

(3) We assume that $D_i = stua$ and $D_j = stub$. Then we have $D_iD_j = (stu)^2ab$. However, the quadratic subfield $\boldsymbol{Q}(\sqrt{ab})$ does not coincide with any $k_j (1 \leq j \leq 7)$. □

Remark 1. We can confirm that the number of triplets (s, t, u) within the order of parameters in D is equal to $28 = 7 \times 1 \times \binom{4}{3} < \binom{\sharp D}{3} = 35$ such that each of stu is a divisor of the field discriminant D_j of k_j.

Next, we prepare the key lemma for the proof of Theorem 2.

Lemma 4. *For the set $D = \{a, b, c, d, e, f, g\}$ of seven positive rational integers, assume that $a > b \geq c > \max\{d, e, f, g\}$ and $d > f$ or $a > b > c \geq \max\{d, e, f, g\}$ and $d > f$. Then*

(1) *For the field $\boldsymbol{Q}(\sqrt{bcst})$, where $s, t \in D \backslash \{a, b, c\}$ and units E_i in $\boldsymbol{Q}(\sqrt{bcst})$, the rank r_{bcst} of the equations*

$$\begin{cases} a + uE_j + vE_k = 0, \\ a + u\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{u, v\} = D \backslash \{a, b, c, s, t\}$ is equal to 1.

(2) *For the field $\boldsymbol{Q}(\sqrt{astu})$, where $s, t, u \in D \backslash \{a, b, c\}$ and units E_i in $\boldsymbol{Q}(\sqrt{astu})$, the rank r_{astu} of the equations*

$$\begin{cases} b + cE_j + vE_k = 0, \\ b + c\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{v\} = D \backslash \{a, b, c, s, t, u\}$ is equal to 1.

Sketch of Idea. Our idea for the proof of this lemma is as follows. For the quadratic subfield k including the coefficients of the simultaneous equation $(*)$, if the field discriminant D_k is divisible by the biggest parameter(case (1)) or the second and the third ones(case (2)), since the fundamental unit(> 1) of k is relatively big, the ratios for the line in Lemma 2 would not be permitted. Thus the ranks of the coefficient matrix for both cases should be equal to one, respectively, namely any integral solution of $(*)$ lies on the plane[PMN].

□

Finally, we show the following main theorem, which is a generalization of a prototype[PMN].

Theorem 2. *Let* $K = \boldsymbol{Q}(\sqrt{a_1}, \cdots, \sqrt{a_r})$ *be the* 2*-elementary abelian extensions over* $\boldsymbol{Q}$ *whose degree* 2^r *is greater than* 8 *or real octic ones for square free integers* $a_1, \cdots, a_r$. *Then the fields* K *are non-monogenic.*

Sketch of Proof. By Proposition 1, it is enough to consider an octic field K. Let $(2) = \mathfrak{L}_1{}^e \cdots \mathfrak{L}_g{}^e$ be the prime ideal decomposition of a rational prime 2 in K. For the ramification index of 2, if $e \le 1$, then by Lemma 1 and the relative degree f of a prime 2 is at most 2, we have $1 \cdot 2^1 < 8$ or $1 \cdot 2^2 \le 8 + 1 - 1$ for $e = 1$ and $2 \cdot 2^1 \le 8$ or $2 \cdot 2^2 \le 8 + 2 - 1$ for $e = 2$, namely K is non-monogenic. Then in the case of $e \ge 3$, we can deduce that the type of an octic field K is $K = \boldsymbol{Q}(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, where $a_1 = mn \equiv 3, a_2 = dn \equiv 2, a_3 = d_1 m_1 n_1 \ell \equiv 1 \pmod 4$, for $d = d_1 d_2, m = m_1 m_2, n = n_1 n_2$ and $dmn\ell$ is square free. Put $D = \{d_1, 2d_2, m_1, 2m_2, n_1, 2m_2, \ell\}$. We denote *again* by $\{a, b, c, d, e, f, g\}$ any transposition on the seven parameters in D. Without loss of generality, we may assume that $a > b > c \ge \max\{d, e, f, g\}$. Using Lemma 4, it is enough for us to consider the following two cases.

Case (I). The field K includes $k_{j_1} = \boldsymbol{Q}(\sqrt{abct})$ for some $t \in D \backslash \{a, b, c\}$, for instance, $t = d$.

Case (II). The field K does not include the field $\boldsymbol{Q}(\sqrt{abcs})$ for any $s \in D \backslash \{a, b, c\}$.

In the case (I), we can deduce that the four parameters a, b, c, d with $c \ge d$ must lie on suitable two planes and in the case (II), a, b, e, g with $e > g$ do on four planes, respectively. However, the order of the parameters would be destroyed. Then we can prove that any real octic fields K does not have a power integral basis[PNM]. □

Remark 2. Recently, in [PNM] we proved that all the 2-elementary abelian fields K with degree $[K : \boldsymbol{Q}] \ge 8$ are non-monogenic exept for the field $\boldsymbol{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3}) = \boldsymbol{Q}(\zeta_{24})$.

Problem. For a primitive elment ξ in K, let $\mathrm{Ind}(\xi)$, $\tilde{m}(K)$ and $m(K)$ be the index $\sqrt{\left|\frac{d_K(\xi)}{D_K}\right|}$ of an elemnet ξ, the minimum index $\min_{\xi \in K}\{\mathrm{Ind}(\xi)\}$ of K and the field index $\gcd_{\xi \in K}\{\mathrm{Ind}(\xi)\}$ of K, respectively. Let the fields K run through all the real octic fields whose Galois groups are 2-elementary abelian. Then evaluate the values of

$$\inf_K \tilde{m}(K) \quad \text{and} \quad \inf_K m(K),$$

respectively.

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